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Equilibrium Price as a Coincidence Point of Two Mappings

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Abstract—The existence of an equilibrium price vector in a nonlinear market model is analyzed. In the model, the demand and supply functions are obtained by maximizing the producer utility and profit, respectively. Sufficient conditions for the existence of an equilibrium price vector and its stability with respect to small perturbations in the model are given. The results are consequences of theorems on the existence and stability of coincidence points in the theory of α-covering mappings.

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INTRODUCTION

Existence conditions for an equilibrium play an important role in the study of models of economic processes. By the equilibrium of an economic system with several interdependent participants, we mean such a state in which none of them wants to change their state.

The participants of an economic system are usually divided into producers and consumers. The system includes a given list of products with unified prices. Every producer chooses and implements a technological process of transforming certain products into other ones, trying to maximize their profit. Every consumer chooses and buys the most desired set of products within their income. For the economic system thus organized to be viable, its state must be balanced in terms of material product flows; i.e., the aggregate production (supply) of each product must be no less than its aggregate consumption (demand) or, more restrictively, the former must be exactly equal to the latter. In this case, the state of the system is called an economic equilibrium and the prices at which the equilibrium occurs are called equilibrium prices.

Proofs of the existence of equilibria and the examination of their properties have made up a stage in the development of mathematical economic theory. However, for nonlinear models that describe actual processes more accurately than linear ones, the available mathematical apparatus is insufficient. Based on the results of [1, 2], which concern the existence of coincidence points of mappings in metric spaces, we can substantially expand the available set of research tools and obtain sufficient conditions for the existence of an equilibrium in nonlinear models.

The goal of this work is to derive sufficient conditions for the existence of an equilibrium in particular versions of the demand–supply model. Formally, the problem in question is formulated as follows.

Consider metric spaces X and Y with metrics ρ_X and ρ_Y, respectively. Let $B_X(x, r)$ denote the closed ball of radius r centered at the point x in X. Similar notation is introduced for Y.

Definition 1 (see [1]). Given $\alpha > 0$, a mapping $S: X \longrightarrow Y$ is said to be α-covering if

$$S(B_X(x, r)) \supseteq B_Y(S(x), \alpha r) \quad \forall r \geq 0, \quad \forall x \in X.$$

The following result was obtained in [1].

Coincidence point theorem (see [1]). *Let X be a complete space, and let $S, D: X \longrightarrow Y$ be arbitrary mappings, of which the first is continuous and α-covering, while the second satisfies the Lipschitz condition with a Lipschitz constant $\beta < \alpha$. Then, for arbitrary $x_0 \in X$, there exists $\xi = \xi(x_0) \in X$ such that*

$$S(\xi) = D(\xi), \quad \rho_X(\xi, x_0) \leq \frac{\rho_Y(S(x_0), D(x_0))}{\alpha - \beta}. \tag{1}$$

The solution ξ of Eq. (1) may not be unique. This solution ξ is called a coincidence point of the mappings S and D.

A straightforward consequence of the above theorem (see [1]) is Milyutin's theorem on perturbations of a covering mapping.

Perturbation theorem. *Let X be a complete metric space, Y be a normed space, and $S: X \longrightarrow Y$ be a continuous α-covering mapping. Then, for any mapping $D: X \longrightarrow Y$ satisfying the Lipschitz condition with a Lipschitz constant $\beta < \alpha$, $S + D$ is an $(\alpha - \beta)$-covering mapping.*

The equilibrium price vector in the demand–supply model is a coincidence point of the demand and supply mappings. Using a local version of the coincidence point theorem, namely, Theorem 1 from [2], we analyze the existence of an equilibrium in the demand–supply model.

1. CONSUMER BEHAVIOR MODEL AND THE DEMAND FUNCTION

The idea behind the consumer behavior model is that, for given prices and an available income, consumers seek to maximally satisfy their needs, i.e., to maximize utility. The utility maximization problem for a consumer can be formulated as follows.

There is a consumer with a certain income $I > 0$, and there are $n \in \mathbb{N}$ goods with the jth good having the price $p_j > 0, j = \overline{1, n}$. We are given an open set $G \subset \mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) : x_j > 0, j = \overline{1, n}\}$ of subsets of goods $x = (x_1, \ldots, x_n)$ that can be bought by the consumer. When buying a collection of goods x, the consumer purchases x_1 units of the first good, x_2 units of the second good, ..., x_n units of the nth good. Suppose that we are given a utility function $u : G \longrightarrow \mathbb{R}$ whose values $u(x)$ reflect the consumer's preference for a collection of goods $x \in G$; i.e., if $u(x) > u(\bar{x})$, then the consumer prefers x to $\bar{x}$. Out of all the collections $x \in G$ whose cost $\Sigma_{j=1}^n p_j x_j$ does not exceed I, the consumer buys that one for which the utility function u has the maximal value.

In view of the economic interpretation of the problem, we additionally assume that the utility function u and the set G are such that the utility function maximum is reached only when the income constraint $\Sigma_{j=1}^n p_j x_j \le I$ holds as an exact equality.

Thus, the consumer's choice is reduced to finding a conditional extremum of the utility function:

$$u(x_1, x_2, \ldots, x_n) \longrightarrow \max,$$
$$\sum_{j=1}^n p_j x_j = I, \quad (x_1, x_2, \ldots, x_n) \in G. \tag{2}$$

Assume that the utility function u is such that problem (2) has a unique solution. This solution is known as demand, while the dependence of the demand on the price p is known as the optimal demand function, or the demand function.

Let us find the demand function for Stone's utility function (consumer's preference function). Given numbers $a_j > 0$ and $\alpha_j \in (0, 1), j = \overline{1, n}$, let $G = \{x \in \mathbb{R}^n_+ : x_j > a_j, j = \overline{1, n}\}$, and let the function $u: G \longrightarrow \mathbb{R}$ be define as

$$u(x) = \prod_{j=1}^n (x_j - a_j)^{\alpha_j}.$$

The parameters determining u have the following economic interpretations. The number a_j is the minimum necessary amount of the jth good that is purchased in any case and is not chosen, while the coefficients α_j characterize the relative "value" of the goods for the consumer. Assume that $\Sigma_{j=1}^n p_j a_j < I$. Under these assumptions, the maximum of the utility function is reached only if the income constraint $\Sigma_{j=1}^n p_j x_j \le I$ holds as an exact equality.

Thus, the Stone model has the form

$$u(x) = \prod_{j=1}^{n} (x_j - a_j)^{\alpha_j} \longrightarrow \max,$$
$$\sum_{j=1}^{n} p_j x_j = I, \quad x_j > a_j, \quad j = \overline{1, n}. \tag{3}$$

This maximization problem is solved by applying the Lagrange principle. The Lagrange function for problem (3) is given by

$$L(x_1, x_2, \ldots, x_n, \lambda) = \prod_{j=1}^{n} (x_j - a_j)^{\alpha_j} + \lambda\left(I - \sum_{j=1}^{n} p_j x_j\right).$$

Since the constraints in problem (3) are regular, necessary optimality conditions are that there exists $\lambda \geq 0$ such that

$$\frac{\alpha_j \prod_{i=1}^{n} (x_i - a_i)^{\alpha_i}}{x_j - a_j} - \lambda p_j = 0, \quad j = \overline{1, n},$$
$$\sum_{i=1}^{n} p_i x_i = I, \quad x_j > a_j, \quad j = \overline{1, n}. \tag{4}$$

Since the utility function u is concave and the constraints in problem (3) are linear, conditions (4) are sufficient conditions for a maximum. Thus, a pair (x^*, λ^*) is a solution of system (4) if and only if x^* solves problem (3).

The Lagrange multiplier λ^* is known as the marginal utility of income. The first n equalities in (4) mean that the utility is maximized when the goods to be bought are chosen so that the ratios of their marginal utility to their price are identical for all the goods. In other words, in the optimal collection of goods, the marginal utilities of the selected goods are proportional to their prices; i.e., $\partial u/\partial x_j(x^*) = \lambda^* p_j, j = \overline{1, n}$.

System (4) is equivalent to the system

$$x_j = a_j + \frac{\alpha_j \prod_{i=1}^{n} (x_i - a_i)^{\alpha_i}}{\lambda p_j}, \quad j = \overline{1, n},$$
$$\sum_{i=1}^{n} p_i x_i = I, \quad x_j > a_j, \quad j = \overline{1, n}.$$

Multiplying the first line of the system by λp_j and summing the result over j, we have

$$\lambda \sum_{j=1}^{n} p_j x_j = \lambda \sum_{j=1}^{n} p_j a_j + \sum_{j=1}^{n} \alpha_j \prod_{i=1}^{n} (x_i - a_i)^{\alpha_i}.$$

Substituting $\Sigma_{j=1}^{n} p_j x_j = I$ yields

$$\prod_{i=1}^{n} (x_i - a_i)^{\alpha_i} = \lambda \frac{I - \sum_{j=1}^{n} p_j a_j}{\sum_{j=1}^{n} \alpha_j}.$$

Therefore, the solution of system (4) and, hence, of problem (2) is given by

$$x_i^* = a_i + \frac{\alpha_i\left(I - \sum_{j=1}^{n} p_j a_j\right)}{p_i \sum_{j=1}^{n} \alpha_j}, \quad i = \overline{1, n}.$$

These formulas suggest that the utility is maximized if the consumer first buys the minimum necessary amount a_i of each good and then spends the remaining sum on buying each good in an amount proportional to its "weight" α_i.

Thus, for any $i = \overline{1, n}$, the demand function $D_i : \mathbb{R}_+^n \longrightarrow \mathbb{R}$ for the ith good has the form

$$D_i(p_1, p_2, \ldots, p_n) = a_i + \frac{\alpha_i\left(I - \sum_{j=1}^{n} p_j a_j\right)}{p_i \sum_{j=1}^{n} \alpha_j}, \quad p = (p_1, \ldots, p_n) \in \mathbb{R}_+^n. \tag{5}$$

Note that the functions D_i are defined for $p \in \mathbb{R}_+^n$ such that $\Sigma_{j=1}^{n} p_j a_j < I$. For the other $p \in \mathbb{R}_+^n$, the functions D_i are defined by formula (5).

2. PRODUCER BEHAVIOR MODEL AND THE SUPPLY FUNCTION

Consider the following model of the producer's behavior. There are n different goods, of which the first $m \le n$ are produced with the consumption of all n goods. The other $n - m$ goods are imported into the system from the outside. Given are the price of the jth good $p_j > 0$ and the amount of money $b_i > 0$ spent on buying resources (inputs) required for producing the ith product, where $i = \overline{1, m}$ and $j = \overline{1, n}$. Let $x_{ij} > 0$ denote the amount of the jth product spent on the production of the ith product, $i = \overline{1, m}$, $j = \overline{1, n}$. Given is a profit function $\pi : \mathbb{R}_+^{m \times n} \longrightarrow \mathbb{R}_+$ whose value $\pi(x)$ is the producer's profit at a given resource consumption $x = (x_{11}, x_{12}, \ldots, x_{1n}, \ldots, x_{m1}, x_{m2}, \ldots, x_{mn})$ and at a fixed price $p = (p_1, \ldots, p_n) \in \mathbb{R}_+^n$. Out of all possible input collections $x \in \mathbb{R}_+^{m \times n}$ such that the production cost of the ith good does not exceed b_i for any $i = \overline{1, m}$, the producer chooses that one at which the profit is maximal.

Thus, the producer's choice is reduced to finding a conditional extremum of the profit function:

$$\begin{gathered} \pi(x) \longrightarrow \max, \\ \sum_{j=1}^{n} p_j x_{ij} = b_i, \quad x_{ij} > 0, \quad i = \overline{1, m}, \quad j = \overline{1, n}. \end{gathered} \tag{6}$$

In the production model under study, the profit function is calculated in terms of the production functions $f_i : \mathbb{R}_+^n \longrightarrow \mathbb{R}$, $i = \overline{1, m}$, by applying the formula

$$\pi(x) = \sum_{i=1}^{m} p_i f_i(x_{i1}, x_{i2}, \ldots, x_{in}) \quad \forall x \in \mathbb{R}_+^{m \times n}.$$

The production function $f_i(x_{i1}, x_{i2}, \ldots, x_{in})$ quantifies the amount of the ith good the producer creates by consuming the input collection $(x_{i1}, x_{i2}, \ldots, x_{in})$.

Let $x^* = (x^*_{11}, x^*_{12}, \dots, x^*_{1n}, \dots, x^*_{m1}, x^*_{m2}, \dots, x^*_{mn}) \in \mathbb{R}^{m\times n}_+$ be a solution of problem (6). The total sale amount of the ith good at a given price vector $p \in \mathbb{R}^n_+$ is calculated by the formula

$$S_i(p) = f_i(x^*_{i1}, x^*_{i2}, \dots, x^*_{in}) - \sum_{s=1}^{m} x^*_{si}, \quad i = \overline{1, m}.$$

The quantity $S(p) = (S_1(p), \dots, S_m(p))$ is known as supply, and its dependence on the price vector p is known as the supply function.

Given numbers $C_i > 0$ and $\beta_{ij} > 0$, $i = \overline{1, m}$, $j = \overline{1, n}$ such that $\Sigma_{j=1}^{n}\beta_{ij} < 1$, $i = \overline{1, m}$, let the production functions be given by the formula

$$f_i(x_{i1}, x_{i2}, \dots, x_{in}) = C_i\prod_{j=1}^{n} x_{ij}^{\beta_{ij}}, \quad i = \overline{1, m}.$$

Here, C_i are neutral technical change parameters and $\beta_{ij} > 0$ are the input elasticity coefficients. Note that these production functions have nonincreasing returns to scale (see [3]), which is observed in actual economies.

Thus, the producer's profit function is given by the formula

$$\pi(x) = \sum_{i=1}^{m}\left(C_i p_i \prod_{j=1}^{n} x_{ij}^{\beta_{ij}}\right)$$

for any $x = (x_{11}, x_{12}, \dots, x_{1n}, \dots, x_{m1}, x_{m2}, \dots, x_{mn}) \in \mathbb{R}^{m\times n}_+$. Therefore, problem (6) is reduced to

$$\begin{gathered}\sum_{i=1}^{m}\left(C_i p_i \prod_{j=1}^{n} x_{ij}^{\beta_{ij}}\right) \longrightarrow \max, \\ \sum_{j=1}^{n} p_j x_{ij} = b_i, \quad x_{ij} > 0, \quad i = \overline{1, m}, \quad j = \overline{1, n}.\end{gathered} \tag{7}$$

This maximization problem is solved by applying the Lagrange principle. The Lagrange function for problem (7) has the form

$$\begin{aligned}&L(x_{11}, x_{12}, \dots, x_{1n}, \dots, x_{m1}, x_{m2}, \dots, x_{mn}, \lambda_1, \dots, \lambda_m)\\ &= \sum_{i=1}^{m}\left(C_i p_i \prod_{j=1}^{n} x_{ij}^{\beta_{ij}}\right) + \sum_{i=1}^{m}\lambda_i\left(b_i - \sum_{j=1}^{n} p_j x_{ij}\right).\end{aligned}$$

Since the constraints in problem (7) are regular, necessary optimality conditions are that there exists $\lambda \geq 0$ such that

$$\begin{gathered}C_i p_i \beta_{ij}\prod_{k=1}^{n} x_{ik}^{\beta_{ik}} = \lambda_i p_j x_{ij}, \quad i = \overline{1, m}, \quad j = \overline{1, n}; \\ \sum_{j=1}^{n} p_j x_{ij} = b_i, \quad i = \overline{1, m}; \\ x_{ij} > 0, \quad \lambda_i \geq 0, \quad i = \overline{1, m}, \quad j = \overline{1, n}.\end{gathered} \tag{8}$$

Since $\Sigma_{j=1}^{n}\beta_{ij} < 1$ for any $i = \overline{1, m}$, all the production functions f_i are concave. Therefore, the necessary optimality conditions (8) for problem (7) are sufficient as well.

Conditions (8) imply that, in an optimal combination of inputs, the limiting efficiencies of the inputs are proportional to their prices. Moreover, the Lagrange multipliers λ^*_i corresponding to the optimal solution characterize the limiting efficiency of finance.

Summing up the first line of system (8) over j and substituting the resulting equality into the second line, we have

$$\lambda_i = \frac{1}{b_i}c_i p_i\left(\prod_{k=1}^{n} x_{ik}^{\beta_{ik}}\right)\left(\sum_{j=1}^{n}\beta_{ij}\right), \quad i = \overline{1, m}.$$

Substituting this expression into the first line of system (8) yields its solution

$$x_{ij} = \frac{b_i\beta_{ij}}{p_j\sum_{k=1}^{n}\beta_{ik}}, \quad i = \overline{1, m}, \quad j = \overline{1, n}.$$

According to the above argument, the found vector $x = (x_{11}, \ldots, x_{1n}, \ldots, x_{m1}, \ldots, x_{mn})$ is a solution of system (7).

Thus, the supply function $S_i : \mathbb{R}_+^n \longrightarrow \mathbb{R}$ of the ith good is given by the formulas

$$S_i(p) = K_i\prod_{j=1}^{n} p_j^{-\beta_{ij}} - L_i p_i^{-1}, \quad i = \overline{1, m}, \tag{9}$$

where

$$K_i = \frac{C_i b_i^{\sum_{j=1}^{n}\beta_{ij}}\prod_{j=1}^{n}\beta_{ij}^{\beta_{ij}}}{\left(\sum_{k=1}^{n}\beta_{ki}\right)^{-\sum_{k=1}^{n}\beta_{ki}}}, \quad L_i = \sum_{s=1}^{m}\frac{b_s\beta_{si}}{\sum_{j=1}^{n}\beta_{sj}}, \quad i = \overline{1, m}. \tag{10}$$

3. MAIN RESULTS: EXISTENCE AND STABILITY OF AN EQUILIBRIUM IN THE DEMAND–SUPPLY MODEL

Given positive integers m and n such that $m \le n$, let $I > 0$ be a real number; $a = (a_1, \ldots, a_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n$ and $C = (C_1, \ldots, C_m) \in \mathbb{R}_+^n$ be vectors; and $\mathcal{B}$ be an $n \times m$ matrix with components $\beta_{ij} > 0$ ($i = \overline{1, m}$, $j = \overline{1, n}$) such that

$$\sum_{j=1}^{n}\beta_{ij} < 1 \quad \forall i = \overline{1, m}. \tag{11}$$

Additionally, let $c_1 = (c_{11}, \ldots, c_{n1})$, $c_2 = (c_{12}, \ldots, c_{n2}) \in \mathbb{R}_+^n$ be given vectors such that $c_{j1} < c_{j2}$, $j = \overline{1, n}$. For arbitrary vectors $x = (x_1, \ldots, x_n)$, $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in \mathbb{R}^n$, we define $\langle x, \bar{x}\rangle = \Sigma_{j=1}^{n} x_j\bar{x}_j$. Assume that

$$\langle c_2, a\rangle < I. \tag{12}$$

A mathematical market model is a collection

$$\sigma = (I, a, \alpha, C, \mathcal{B}, c_1, c_2) \in \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^m \times \mathbb{R}_+^{m\times n} \times \mathbb{R}_+^n \times \mathbb{R}_+^n$$

satisfying conditions (11), (12), and $c_{j1} < c_{j2}$ for $j = \overline{1, n}$. The set of collections $\sigma = (I, a, \alpha, C, \mathcal{B}, c_1, c_2)$ satisfying inequalities (11), (12), and $c_{j2} > c_{j1}$ for $j = \overline{1, n}$ is denoted by Σ. Obviously, $\Sigma \subset \mathbb{R}_+^{1+4n+m+mn}$ is an open set.

Given a collection of parameters $(I, a, \alpha, C, \mathcal{B})$, the demand function $D : \mathbb{R}_+^n \longrightarrow \mathbb{R}^m$, $D(\cdot) = (D_1(\cdot), \ldots, D_m(\cdot))$ and the supply function $S : \mathbb{R}_+^n \longrightarrow \mathbb{R}^m$, $S(\cdot) = (S_1(\cdot), \ldots, S_m(\cdot))$ are uniquely determined by formulas (5) and (9), respectively. Here, the numbers K_i and L_i are given by (10). The components of the vectors c_1 and c_2 determine the natural constraints on the price p_j of the jth good; i.e., we assume that $c_{j1} \le p_j \le c_{j2}$ for each $j = \overline{1, n}$.

In this model, it is assumed that there are n goods in the market, of which m are sold by the producer, while the other $n - m$ are imported from the outside. The vector $D(p)$ is called the *aggregate demand corresponding to the price vector* p, and the vector $S(p)$ is called the *aggregate supply corresponding to the price vector* p. Given p, if there is an index i such that $D_i(p) > S_i(p)$, then a shortage occurs in the market; if there is i such that $D_i(p) < S_i(p)$, then there is a surplus. Such prices cannot be considered satisfactory, since the purchasers are affected in the former case, and the marketers, in the latter. For this reason, the equality $S(p) = D(p)$ is considered the best variant for the economy.

Define

$$P = \{p \in \mathbb{R}^n_+ : \langle a, p\rangle < I\}.$$

Definition 2. A vector $p \in P$ is called an equilibrium price vector in a model σ if $S(p) = D(p)$.

Now we state the main results. Define

$$\bar{\alpha}(\sigma) = \min_{i = \overline{1, m}} \left| \frac{L_i(c_{i2} - c_{i1})}{2c_{i2}^2} \right| - \max_{i = \overline{1, m}} \left(K_i \left(\prod_{j=1}^{n} c_{j1}^{-\beta_{ij}} \right) \left(\sum_{j=1}^{n} \beta_{ij} \frac{c_{j2} - c_{j1}}{2c_{j1}} \right) \right),$$

$$\bar{\beta}(\sigma) = \frac{\max_{i = \overline{1, m}} \left[\frac{\alpha_i}{c_{i1}^2}(I - \langle c_1, a\rangle + c_{i1}a_i)(c_{i2} - c_{i1}) + \frac{\alpha_i}{c_{i1}}(\langle a, c_2 - c_1\rangle - a_i(c_{i2} - c_{i1})) \right]}{2\sum_{k=1}^{n} \alpha_k},$$

$$\bar{\gamma}(\sigma) = \max_{i = \overline{1, m}} \left| a_i + \frac{\alpha_i(2I - \langle a, c_2 + c_1\rangle)}{(c_{i2} + c_{i1})\sum_{j=1}^{n} \alpha_j} + \frac{2L_i}{c_{i2} + c_{i1}} - K_i \prod_{j=1}^{n} \left(\frac{c_{j2} + c_{j1}}{2} \right)^{-\beta_{ij}} \right|.$$

Theorem 1. *Assume that a model* $\sigma \in \Sigma$ *satisfies conditions* (11) *and* (12), *and, additionally, let*

(i) $\bar{\alpha}(\sigma) > \bar{\beta}(\sigma)$;

(ii) $\bar{\gamma}(\sigma) < \bar{\alpha}(\sigma) - \bar{\beta}(\sigma)$.

Then there exists an equilibrium price vector $p = (p_1, \ldots, p_n)$ *in the model such that* $c_{j1} < p_j < c_{j2}$, $j = \overline{1, n}$.

The proofs of this and the following theorems are given in Section 4.

Let us show that the equilibrium price vector, which exists by Theorem 1, is stable with respect to small perturbations in the model. Specifically, let a sequence of models $\{\sigma^N\}_{N=1}^{\infty}$, where $\sigma^N = (I^N, a^N, \alpha^N, C^N, \mathcal{B}^N, c_1^N, c_2^N)$, and a model $\sigma = (I, a, \alpha, C, \mathcal{B}, c_1, c_2)$ be given. The sequence of models $\{\sigma^N\}_{N=1}^{\infty}$ is said to converge to the model σ if as $N \longrightarrow \infty$

$$I^N \longrightarrow I, \quad a^N \longrightarrow a, \quad a^N \longrightarrow a, \quad C^N \longrightarrow C, \quad \mathcal{B}^N \longrightarrow \mathcal{B}, \quad c_1^N \longrightarrow c_1, \quad c_2^N \longrightarrow c_2.$$

Define

$$P^N = \{p \in \mathbb{R}^n_+ : \langle a^N, p\rangle < I^N\}.$$

Theorem 2. *Assume that a model* σ *satisfies all the assumptions of Theorem* 1 *and a sequence of models* $\{\sigma^N\}$ *converges to* σ. *Then, for any equilibrium price vector* $p \in P$ *in* σ *satisfying the inequality* $c_{j1} < p_j < c_{j2}$ *for any* $j = \overline{1, n}$, *there exists a positive integer* $\bar{N}$ *and a sequence* $\{p^N\}_{N=\bar{N}}^{\infty} \subset \mathbb{R}^n_+$ *such that*

(i) *for any* $N > \bar{N}$, p^N *is an equilibrium price vector in the model* σ^N;

(ii) $p^N \longrightarrow p$ *as* $N \longrightarrow \infty$.

4. PROOFS OF THE MAIN RESULTS

First, we describe the necessary constructions. Let (X, ρ_X) and (Y, ρ_Y) be metric spaces, $S : X \longrightarrow Y$ be a given mapping, and $M \subset X$ be a given nonempty subset.

Definition 3 (see [2]). Given $\alpha > 0$, S is called an α-covering mapping on the set M if, for any $x \in M$ and $r \geq 0$ such that $B_X(x, r) \subseteq M$, we have

$$B_Y(S(x), \alpha r) \subseteq S(B_X(x, r)).$$

The supremum of all $\alpha > 0$ such that S is an α-covering mapping on M is denoted by $\mathrm{cov}(S|M)$. If $M = X$, this supremum is denoted by $\mathrm{cov}(S)$.

Definition 4 (see [2]). Given $\alpha > 0$, S is called a locally α-covering mapping at a point $x \in X$ if, for any $\varepsilon > 0$, there exists a positive $r < \varepsilon$ such that

$$B_Y(S(x), \alpha r) \subseteq S(B_X(x, r)).$$

The supremum of all $\alpha > 0$ such that S is a locally α-covering mapping at the point $x \in X$ is denoted by $\mathrm{cov}(S|x)$.

Below, we need the following result, which was obtained in [2].

Theorem 3 (see Theorem 1 in [2]). *Assume that X is a complete space and $x_0 \in X$, $\alpha > 0$, and $R > 0$ are given. Let the mapping $S : X \longrightarrow Y$ be closed and α-covering on $B_X(x_0, R)$. Then, for any nonnegative $\beta < \alpha$ and any mapping $D : B_X(x_0, R) \longrightarrow Y$ satisfying the Lipschitz condition with the constant β such that*

$$\rho_Y(S(x_0), D(x_0)) \leq (\alpha - \beta)R,$$

the mappings S and D have a coincidence point $\xi \in X$ (i.e., $S(\xi) = D(\xi)$) such that

$$\rho_X(x_0, \xi) \leq \frac{\rho_Y(S(x_0), D(x_0))}{\alpha - \beta}.$$

The following assertion is a local version of Theorem 2 in [4] on the stability of coincidence points. Let $S_n, D_n : X \longrightarrow Y$, $n = 1, 2, \ldots$, be given mappings.

Theorem 4. *Assume that X is a complete space; $x_0 \in X$ is a given coincidence point of mappings S and D (i.e., $S(x_0) = D(x_0)$); and $\alpha > 0$, $\beta \geq 0$, and $R > 0$ are given numbers with $\alpha > \beta$. Let the following conditions also hold:*

(i) *For any positive integer n, the mapping S_n is closed and α-covering on $B_X(x_0, R)$.*

(ii) *For any positive integer n, the mapping D_n satisfies the Lipschitz condition with the constant β on $B_X(x_0, R)$.*

(iii) $\rho_Y(S_n(x_0), S(x_0)) \longrightarrow 0$, $\rho_Y(D_n(x_0), D(x_0)) \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Then there exists an index $\bar{n} > 0$ and a sequence $\{x_n\} \subset X$ such that

$$S_n(x_n) = D_n(x_n) \quad \forall n > \bar{n},$$

$$x_n \longrightarrow x_0 \quad as \quad n \longrightarrow \infty.$$

Moreover,

$$\rho_X(x_n, x_0) \leq \frac{\rho_Y(S_n(x_0), S(x_0)) + \rho_Y(D_n(x_0), D(x_0))}{\alpha - \beta} \quad \forall n > \bar{n}.$$

Proof. First, we note that the triangle inequality implies that

$$\rho_Y(S_n(x_0), D_n(x_0)) \leq \rho_Y(S_n(x_0), S(x_0)) + \rho_Y(S(x_0), D(x_0))$$
$$+ \rho_Y(D(x_0), D_n(x_0)) = \rho_Y(S_n(x_0), S(x_0)) + \rho_Y(D(x_0), D_n(x_0)).$$

By virtue of (iii), there exists a number $\bar{n} > 0$ such that

$$\rho_Y(S_n(x_0), S(x_0)) \leq 2^{-1}(\alpha - \beta)R, \quad \rho_Y(D_n(x_0), D(x_0)) \leq 2^{-1}(\alpha - \beta)R$$

for any $n > \bar{n}$. Therefore,

$$\rho_Y(S_n(x_0), D_n(x_0)) \leq 2^{-1}(\alpha - \beta)R + 2^{-1}(\alpha - \beta)R = (\alpha - \beta)R$$

for any $n > \bar{n}$. According to Theorem 3, there exists a point $x_n \in X$ such that $S_n(x_n) = D_n(x_n)$ and

$$\rho_X(x_n, x_0) \le \frac{\rho_Y(S_n(x_0), D_n(x_0))}{\alpha - \beta} \le \frac{\rho_Y(S_n(x_0), S(x_0)) + \rho_Y(D(x_0), D_n(x_0))}{\alpha - \beta}$$

for any $n > \bar{n}$.

Recall some facts associated with the above concepts. Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and let $A : X \longrightarrow Y$ be a linear continuous operator. Then $(\operatorname{cov}(A))^{-1}$ coincides with the Banach constant for the operator A.

Let $S : X \longrightarrow Y$ be a strictly differentiable mapping at the point $x_0 \in X$. Then

$$\operatorname{cov}(S|x_0) = \operatorname{cov}\left(\frac{\partial S}{\partial x}(x_0)\right). \tag{13}$$

Let $M \subset X$ be a closed ball of nonzero radius. Then

$$\operatorname{cov}(S|M) = \inf_{x \in \operatorname{int} M} \operatorname{cov}(S|x).$$

This fact is a straightforward consequence of Theorem 4 in [2].

Let $D : M \longrightarrow Y$ be a mapping that is continuous on M and continuously differentiable on $\operatorname{int} M$. Denote by $\operatorname{lip}(D|M)$ the infimum of all $\beta \ge 0$ such that D satisfies on M the Lipschitz condition with the constant β. Then

$$\operatorname{lip}(D|M) = \sup_{p \in \operatorname{int} M}\left\|\frac{\partial D}{\partial p}(p)\right\|.$$

Here and below, the norm of an arbitrary linear operator $A : X \longrightarrow Y$ is defined by the formula $\|A\| = \sup_{\|x\|_X \le 1} \|Ax\|_Y$.

Proof of Theorem 1. The spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are equipped with the norms

$$\|x\|_{\mathbb{R}^n} = 2\max_{j = \overline{1, n}} \frac{|x_j|}{c_{j2} - c_{j1}} \quad \forall x = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

$$\|y\|_{\mathbb{R}^m} = \max_{i = \overline{1, m}} |y_i| \quad \forall y = (y_1, \dots, y_m) \in \mathbb{R}^m.$$

Let $X = \mathbb{R}^n_+$ and $Y = \mathbb{R}^m_+$. Consider metric spaces (X, ρ_X) and (Y, ρ_Y), where the metric ρ_X is defined by the norm $\|\cdot\|_{\mathbb{R}^n}$, and the metric ρ_Y, by the norm $\|\cdot\|_{\mathbb{R}^m}$.

Let

$$\tilde{c} = \frac{c_1 + c_2}{2}, \quad M = B_X(\tilde{c}, 1).$$

Obviously, $M = [c_{11}, c_{12}] \times \ldots \times [c_{n1}, c_{n2}]$. Note that the metric space X is not complete, but the completeness of the ball $B_X(\tilde{c}, 1)$ is sufficient for the subsequent argument.

Our goal is to apply Theorem 3. To this end, we first compute $\operatorname{cov}(S|M)$. For $p \in \operatorname{int} M$, we have

$$\frac{\partial S}{\partial p}(p) = \mathcal{L}(p) - \mathcal{K}(p),$$

where $\mathcal{L}(p), \mathcal{K}(p) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are linear operators defined by the matrices

$$\mathcal{L}(p) = \begin{pmatrix} L_1 p_1^{-2} & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & L_m p_m^{-2} & \dots & 0 \end{pmatrix},$$

$$\mathcal{K}(p) = \begin{pmatrix} K_1\beta_{11}p_1^{-1}\prod_{j=1}^{n}p_j^{-\beta_{1j}} & \dots & K_1\beta_{1n}p_n^{-1}\prod_{j=1}^{n}p_j^{-\beta_{1j}} \\ \dots & \dots & \dots \\ K_m\beta_{m1}p_1^{-1}\prod_{j=1}^{n}p_j^{-\beta_{mj}} & \dots & K_m\beta_{mn}p_n^{-1}\prod_{j=1}^{n}p_j^{-\beta_{mj}} \end{pmatrix}.$$

Therefore, by the perturbation theorem,

$$\operatorname{cov}\left(\frac{\partial S}{\partial p}(p)\right) \geq \operatorname{cov}(\mathcal{L}(p)) - \|\mathcal{K}(p)\|.$$

Moreover,

$$\operatorname{cov}(\mathcal{L}(p)) = \min_{i=\overline{1,m}}\left|\frac{L_i(c_{i2}-c_{i1})}{2p_i^2}\right| \geq \min_{i=\overline{1,m}}\left|\frac{L_i(c_{i2}-c_{i1})}{2c_{i2}^2}\right|,$$

$$\|\mathcal{K}(p)\| = \max_{\|x\|_{\mathbb{R}^n}=1}\|\mathcal{K}(p)x\|_{\mathbb{R}^m} = \max_{\|x\|_{\mathbb{R}^n}=1}\left(\max_{i=\overline{1,n}}\left|\sum_{j=1}^{n}K_i\beta_{ij}p_j^{-1}x_j\prod_{k=1}^{n}p_k^{-\beta_{ik}}\right|\right)$$

$$\leq \max_{\|x\|_{\mathbb{R}^n}=1}\max_{i=\overline{1,m}}\left(K_i\left(\prod_{k=1}^{n}p_k^{-\beta_{ik}}\right)\left(\sum_{j=1}^{n}\beta_{ij}p_j^{-1}|x_j|\right)\right)$$

$$\leq \max_{\|x\|_{\mathbb{R}^n}=1}\max_{i=\overline{1,m}}\left(K_i\left(\prod_{k=1}^{n}p_k^{-\beta_{ik}}\right)\left(\sum_{j=1}^{n}\beta_{ij}\frac{c_{j2}-c_{j1}}{2p_j}\right)\right)$$

$$\leq \max_{i=\overline{1,m}}\left(K_i\left(\prod_{j=1}^{n}c_{j1}^{-\beta_{ij}}\right)\left(\sum_{j=1}^{n}\beta_{ij}\frac{c_{j2}-c_{j1}}{2c_{j1}}\right)\right).$$

Consequently,

$$\operatorname{cov}\left(\frac{\partial S}{\partial p}(p)\right) \geq \min_{i=\overline{1,m}}\left|\frac{L_i(c_{i2}-c_{i1})}{2c_{i2}^2}\right| - \max_{i=\overline{1,m}}\left(K_i\left(\prod_{j=1}^{n}c_{j1}^{-\beta_{ij}}\right)\left(\sum_{j=1}^{n}\beta_{ij}\frac{c_{j2}-c_{j1}}{2c_{j1}}\right)\right).$$

It follows from (13) and Theorem 4 in [2] that

$$\operatorname{cov}(S|M) = \inf_{p\in \operatorname{int}M}\operatorname{cov}(S|p) \geq \bar{\alpha}(\sigma).$$

Now we estimate the Lipschitz constant of the mapping D. For any $p \in \operatorname{int}M$,

$$\frac{\partial D_i}{\partial p_j}(p) = \begin{cases} -\alpha_i a_j\left(p_i\sum_{k=1}^{n}\alpha_k\right)^{-1} & \text{if} \quad i \neq j; \\ -\alpha_i\left(I - \sum_{k=\overline{1,n},\,k\neq i}p_k a_k\right)\left(p_i^2\sum_{k=1}^{n}\alpha_k\right)^{-1} & \text{if} \quad i = j. \end{cases}$$

Therefore, repeating the argument used in estimating the norm of $\mathcal{K}(p)$, we have

$$\left\|\frac{\partial D}{\partial p}(p)\right\| \le \frac{\max\limits_{i=\overline{1,m}}\left[\frac{\alpha_i}{p_i^2}\left(I-\sum\limits_{k=\overline{1,n},k\ne i}p_ka_k\right)(c_{i2}-c_{i1})+\frac{\alpha_i}{p_i}\sum\limits_{j=\overline{1,n},j\ne i}a_j(c_{j2}-c_{j1})\right]}{2\sum\limits_{k=1}^{n}\alpha_k}$$

$$\le \frac{\max\limits_{i=\overline{1,m}}\left[\frac{\alpha_i}{c_{i1}^2}\left(I-\sum\limits_{k=\overline{1,n},k\ne i}c_{k1}a_k\right)(c_{i2}-c_{i1})+\frac{\alpha_i}{c_{i1}}\sum\limits_{j=\overline{1,n},j\ne i}a_j(c_{j2}-c_{j1})\right]}{2\sum\limits_{k=1}^{n}\alpha_k} = \bar{\beta}(\sigma)$$

for any $p \in \operatorname{int} M$. Therefore, $\operatorname{lip}(D|M) \le \bar{\beta}(\sigma)$.

Combining assumptions (i) and (ii) of the theorem with the inequalities $\operatorname{cov}(S|M) \ge \bar{\alpha}(\sigma)$ and $\operatorname{lip}(D|M) \le \bar{\beta}(\sigma)$, we conclude that there exist positive numbers α and β such that $\bar{\beta}(\sigma) < \beta < \alpha < \bar{\alpha}(\sigma)$ and $\bar{\gamma}(\sigma) < \alpha - \beta$, S is α-covering on M, and D is β-Lipschitz continuous on M. Since $\rho_Y(S(\tilde{c}), D(\tilde{c})) = \bar{\gamma}(\sigma)$, it follows from (ii) that $\rho_Y(S(\tilde{c}), D(\tilde{c})) \le (\alpha - \beta)$. Thus, according to Theorem 3, there exists a vector $p \in X$ such that $S(p) = D(p)$ and

$$\rho_X(p, \tilde{c}) \le \frac{1}{\alpha-\beta}\rho_Y(S(\tilde{c}), D(\tilde{c})).$$

This inequality implies that $p \in \operatorname{int} M$, since $M = B_X(\tilde{c}, 1)$ and $\rho_Y(S(\tilde{c}), D(\tilde{c})) = \bar{\gamma}(\sigma) < (\alpha - \beta)$. Therefore, $c_{j1} < p_j < c_{j2}$ for any $j = \overline{1, n}$. Finally, the inclusion $M \subset P$ and, hence, $p \in P$ follow from assumption (12).

Proof of Theorem 2. Our goal is to apply Theorem 4. For this purpose, we need the following auxiliary constructions.

Given an arbitrary $\varepsilon > 0$, let

$$v_1^{\varepsilon} = \tilde{c} - (1-\varepsilon)(\tilde{c} - c_1), \quad v_2^{\varepsilon} = \tilde{c} + (1-\varepsilon)(c_2 - \tilde{c}), \quad M^{\varepsilon} = [v_{11}^{\varepsilon}, v_{12}^{\varepsilon}] \times \ldots \times [v_{n1}^{\varepsilon}, v_{n2}^{\varepsilon}].$$

Obviously, for any positive $\varepsilon < 1$, we have $v_{j1}^{\varepsilon} < v_{j2}^{\varepsilon}$ ($j = \overline{1, n}$) and $M^{\varepsilon} \subset P$.

Let $p \in P$ be an equilibrium price vector in the model σ that satisfies the inequality $c_{j1} < p_j < c_{j2}$ for any $j = \overline{1, n}$. Then $p \in \operatorname{int} M$ and, therefore, there exists $\varepsilon_1 > 0$ such that $p \in M^{\varepsilon}$ for any positive $\varepsilon < \varepsilon_1$.

The assumptions of the theorem imply that, for any $\varepsilon > 0$, there exists an index $N_1(\varepsilon) > 0$ such that $M^{\varepsilon} \subset P^N$ for any $N > N_1(\varepsilon)$.

Let α and β be arbitrary positive numbers such that $\bar{\beta}(\sigma) < \beta < \alpha < \bar{\alpha}(\sigma)$. Since the functions $\bar{\alpha}(\cdot)$ and $\bar{\beta}(\cdot)$ are continuous, there exists a positive $\varepsilon_2 < 1$ such that $\bar{\beta}(I, a, \alpha, C, \mathcal{B}, v_1^{\varepsilon}, v_2^{\varepsilon}) < \beta$ and $\alpha < \bar{\alpha}(I, a, \alpha, C, \mathcal{B}, v_1^{\varepsilon}, v_2^{\varepsilon})$ for any $\varepsilon < \varepsilon_2$. Moreover, for any $\varepsilon > 0$, there exists an index $N_2(\varepsilon) > 0$ such that $\bar{\beta}(I^N, a^N, \alpha^N, C^N, \mathcal{B}^N, v_1^{\varepsilon}, v_2^{\varepsilon}) < \beta$ and $\alpha < \bar{\alpha}(I^N, a^N, \alpha^N, C^N, \mathcal{B}^N, v_1^{\varepsilon}, v_2^{\varepsilon})$ for any $N > N_2(\varepsilon)$.

We set $\varepsilon = 2^{-1}\min\{\varepsilon_1, \varepsilon_2\}$, $N_3 = \max\{N_1(\varepsilon), N_2(\varepsilon)\} + 1$, $X = \mathbb{R}_+^n$, and $Y = \mathbb{R}_+^m$. For convenience, $\mathbb{R}^n$ and $\mathbb{R}^m$ are equipped with other norms:

$$\|x\|_1 = 2\max_{j=\overline{1,n}}\frac{|x_j|}{v_{j2}^{\varepsilon} - v_{j1}^{\varepsilon}} \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

$$\|y\|_2 = \max_{i=\overline{1,m}}|y_i| \quad \forall y = (y_1, \ldots, y_m) \in \mathbb{R}^m.$$

Consider the metric spaces (X, ρ_X) and (Y, ρ_Y), where the metric ρ_X is defined by the norm $\|\cdot\|_1$, while the metric ρ_Y is defined by the norm $\|\cdot\|_2$.

Let $S^N, D^N: X \longrightarrow Y$ be the respective demand and supply functions in the model $\sigma^N = (I^N, a^N, \alpha^N, C^N, \mathcal{B}^N, c_1^N, c_2^N)$. Repeating the argument used in the proof of Theorem 1, we find that, for any $N > N_3$, the mappings S^N are $\bar{\alpha}\,(I^N, a^N, \alpha^N, C^N, \mathcal{B}^N, v_1^\varepsilon, v_2^\varepsilon)$-covering on M^ε and, hence, α-covering on M^ε. Similarly, we find that the mappings D^N on M^ε satisfy the Lipschitz condition with a constant $\beta < \alpha$.

Choose an arbitrary $R > 0$ such that $B_X(p, R) \subset M^\varepsilon$. From formulas (5), (9), and (10), it follows that $S^N(p) \longrightarrow S(p)$ and $D^N(p) \longrightarrow D(p)$ as $N \longrightarrow \infty$. Therefore, according to Theorem 4, there exists an index $\bar{N} > N_3$ and a sequence $\{p^N\} \subset B_X(p, R)$ such that $S^N(p^N) = D^N(p^N)$ and $p^N \longrightarrow p$ as $N \longrightarrow \infty$. Since $p^N \in B_X(p, R) \subset M^\varepsilon \subset P^N$ for any $N > \bar{N}$, we conclude that p^N is an equilibrium price vector in the model σ^N.

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