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ON AN APPLICATION OF COVERING MAPPINGS TO AN INITIAL VALUE PROBLEM FOR DIFFERENTIAL INCLUSION

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Keywords: Covering mappings, set-valued mappings, differential inclusions, local solvability.
Abstract: The solvability of inclusions unsolved with respect to the derivative of the unknown function is studied. The reduction of this problem to an inclusion in a functional metric space is presented. The sufficient conditions for solvability of the inclusion are derived in the terms of covering mappings.

A large amount of ordinary differential equations, functional-differential equations and some other problems can be introduced as an equation of the form

$$f(x, x) = y, \tag{1}$$

where $f : X \times X \to Y$, X and Y are functional metric spaces, $x \in X$ is an unknown function, y is a given function. In [1–3], there the solvability conditions for equations of the type (1) solvability were established and used to obtain theorems on the solvability and well-posedness of control systems, ordinary differential equations, Volterra equations, etc. The proposed in the mentioned above papers ideas can be applied to differential inclusions unsolved with respect to the derivative of the unknown function.

Given a vector $x_0 \in \mathbb{R}^n$ and a set-valued mapping $F : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^k$ such that $F(\cdot, x, u)$ is measurable for any (x, u), $F(t, \cdot, \cdot)$ is continuous for a.a. t, consider an initial value problem for differential inclusion

$$0 \in F(t, x, \dot{x}), \ \ x(0) = x_0. \tag{2}$$

An absolutely continuous function $x(\cdot)$ is a solution of (2) if $0 \in F\big(t, x(t), \dot{x}(t)\big)$ for a.a. $t \in [0,1]$. Under certain assumptions on F this initial value problem is equivalent to the inclusion

$$0 \in \Gamma(u, u), \tag{3}$$

where $\Gamma : L^n_\infty \times L^n_\infty \rightrightarrows L^k_\infty$,

$$\Gamma(u, v)(t) = F\left(t, x_0 + \int_0^t v(s)ds, u(t)\right) \ \dot{\forall} t \in [0,1],$$

L^n_∞ states for the set of all measurable essentially bounded functions $u : [0,1] \to \mathbb{R}^n$, the solutions x of (2) and u of (3) are related by formula $\dot{x}(t) = u(t)$ for a.a. t.

The investigation of the inclusion (3) can be performed using the covering mappings theory. The known results related to solvability of abstract equations and inclusions can be found at [4–6], where the coincidence points theorems were established in the terms of covering and Lipschitz mappings. Another results that guaranties the existence of a solution for equations in metric spaces was proved in [3] in the form of a theorem on Lipscitz perturbations of covering mappings. Below we present a similar statement on Lipscitz perturbations of covering mappings which can be applied to obtain solvability conditions for inclusion (3).

Let (X, ρ_X), (Y, ρ_Y) be metric spaces, a set-valued mapping $G : X \rightrightarrows Y$ correspond a closed set of $G(x) \subset Y$ to each $x \in X$, a set $\mathfrak{A} \subset X \times [0, +\infty)$ be nonempty. For an arbitrary set U, a point $x \in X$ and a number $r \in [0, \infty)$ we denote by $B_X(x,r)$ the closed ball in X centered in x with radius r,

$$B_X(U,r) = \bigcup_{x \in U} B_X(x,r), \quad G(U) = \bigcup_{x \in U} G(x).$$

We will say that the set-valued mapping G is **covering on the system** $\mathfrak{A}$ if

$$B\big(G(x), \alpha r\big) \subset G\big(B(x,r)\big) \quad \forall (x,r) \in \mathfrak{A}.$$

Let $\Gamma : X \times X \rightrightarrows Y$ be a set-valued mapping that corresponds a closed set of $\Gamma(x_1, x_2) \subset Y$ to each point $(x_1, x_2) \in X \times X$. Let $x_0 \in X$, $y_0 \in \Gamma(x_0, x_0)$, $y \in Y$, $R \geqslant 0$, $\alpha > 0$, $\beta \geqslant 0$ be given. Denote

$$\mathfrak{B}(x) = \big\{(x,r) : 0 \leqslant r \leqslant R - \rho_X(x_0, x)\big\} \quad \forall\, x \in B_X(x_0, R).$$

T h e o r e m 1. *Assume that the set-valued mapping* Γ *is closed and*

A) $\Gamma(x_1, \cdot)$ *is* β*-Lipschitz for any* $x_1 \in B_X(x_0, R)$ *;*

B) $\Gamma(\cdot, x_2)$ *is covering on the system* $\mathfrak{B}(x_2)$ *for any* $x_2 \in B_X(x_0, R)$ *;*

C) $\alpha > \beta$;

D) $\rho_Y(y_0, y) < (\alpha - \beta)R$.

Then there exist a point $\xi \in X$ *such that*

$$y \in \Gamma(\xi, \xi) \ \text{ and } \ \rho_X(x_0, \xi) \leqslant \frac{1}{\alpha - \beta} \rho_Y(y_0, y).$$

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О ПРИЛОЖЕНИЯХ НАКРЫВАЮЩИХ ОТОБРАЖЕНИЙ К ЗАДАЧЕ КОШИ ДЛЯ ДИФФЕРЕНЦИАЛЬНЫХ ВКЛЮЧЕНИЙ

Аннотация: Изучается разрешимость дифференциальных включений, не разрешенных относительно производной неизвестной функции. Дифференциальное включение сводится к включению в функциональных пространствах. Для полученной задачи приводятся достаточные условия разрешимости в терминах накрывающих отображений.

Ключевые слова: Накрывающие отображения, многозначные отображения, дифференциальные включения, локальная разрешимость.

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