Covering mappings. Theory and applications

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1. Covering mappings

Let (X, ρ_X), (Y, ρ_Y) be metric spaces, $\alpha > 0$.

Def. 1. [*] *$F : X \to Y$ is an α-covering mapping if*

$$B_Y(F(x_0), \alpha r) \subset F(B_X(x_0, r)) \ \forall\, x_0 \in X, \ \ r \geq 0.$$

Here $B_X(x_0, r) = \{x \in X : \rho_X(x, x_0) \leq r\}$.

$F : X \times X \to Y$ is α-covering $\Leftrightarrow$ $\forall\ x_0 \in X$, $y \in Y$ $\exists\ x \in X$: $F(x) = y$ and $\rho_X(x_0, x) \leq \dfrac{1}{\alpha} \rho_Y(F(x_0), y)$.

[*] A.V. Arutyunov, Covering mappings in metric spaces and fixed points, Dokl. Math. 76(2)(2007), pp. 665-668.

2. Examples

1. The identity map $F : X \to X$ is 1-covering.

2. Let X, Y be Banach spaces, $F : X \to Y$ be a surjective linear mapping. By Banach Open Mapping Theorem $\exists\ \alpha > 0$ such that F is α-covering.

3. Let $F : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Function F is α-covering if and only if $|F'(x)| \geq \alpha\ \forall\ x \in \mathbb{R}$.

4. A mapping $F : \mathbb{R} \to Y$, $F(x) = |x|$ is not covering if $Y = \mathbb{R}$, F is 1-covering if $Y = \mathbb{R}_+$.

3. Local covering property

Let (X, ρ_X), (Y, ρ_Y) be metric spaces, $\alpha > 0$, $x_0 \in X$.

Def. 2. *$F : X \to Y$ is locally α-covering around x_0 if exists $R > 0$ such that*

$$B_X(x, r) \subset B_X(x_0, R) \Rightarrow B_Y(F(x_0), \alpha r) \subset F(B_X(x_0, r)).$$

Note that if F is α-covering then F is locally α-covering around any $x_0 \in X$.

4. Perturbation theorem

Let (X, ρ_X) be a metric space, $(Y, \|\cdot\|_Y)$ be a normed linear space. Numbers $\alpha > 0$, $\beta \geq 0$, point $x_0 \in X$, mappings $F, G : X \to Y$ are given.

Th.1. [*] [†] *If X is complete, F is continuous and locally α-covering around x_0, G satisfy Lipschitz inequality in a neighborhood of x_0 with constant $\beta < \alpha$, then*

$$F + G : X \to Y, \qquad F + G : x \mapsto F(x) + G(x) \ \forall x \in X$$

is $(\alpha - \beta)$-covering.

[*] L. M. Graves, Some mapping theorems, Duke Math. J., 17(1950), pp. 111-114

[†] A.V. Dmitruk, A.A. Milyutin, N.P. Osmolovskii, Lyusternik's theorem and the theory of extrema, Uspekhi Mat. Nauk, 35:6(216)(1980), pp. 11-46.

5. Corollaries

Corollary 1. * *If X, Y are Banach spaces, F is strictly differentiable at x_0 and $\frac{\partial F}{\partial x}(x_0)X = Y$ then F is locally α-covering around x_0 with some $\alpha > 0$.*

Consider a minimization problem

$$f(x) \to \min, \quad F(x) = 0. \tag{1}$$

$f : \mathbb{R}^n \to \mathbb{R}$, $F : \mathbb{R}^n \to \mathbb{R}^k$ are continuously differentiable in a neighborhood of $x_0 \in \mathbb{R}^n$,

x_0 is a solution of (1).

*B.S. Mordukhovich, Variational Analysis and Generalized Differentiation, V. 1. Springer. 2005.

Define $\overline{F} : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^k$, $\overline{F} : x \mapsto (f(x), F(x))$.

x_0 **is a solution of (1)** $\Rightarrow$ $\overline{F}$ **is not locally covering around** x_0.

Applying Cor.1 we obtain $\dfrac{\partial \overline{F}}{\partial x}(x_0)\mathbb{R}^n \neq \mathbb{R}^k$.

Therefore, $\exists\ \lambda_0 \in \mathbb{R}$, $\lambda \in \mathbb{R}^n$: $(\lambda_0, \lambda) = 0$ and

$$\lambda_0 \frac{\partial f}{\partial x}(x_0) + \lambda \frac{\partial F}{\partial x}(x_0) = 0.$$

Corollary 2 is the Lagrange Multiplier Rule. *

*A.V. Dmitruk, A.A. Milyutin, N.P. Osmolovskii, Lyusternik's theorem and the theory of extrema, Uspekhi Mat. Nauk, 35:6(216)(1980), pp. 11-46.

6. Coincidence points

Let (X,ρ_X), (Y,ρ_Y) be metric spaces, $F,G : X \to Y$.
A solution of the equation

$$F(x) = G(x)$$

is called a coincidence point of F and G.

Th. 2 [*] [†] *Let X be complete, F be continuous and α-covering, G satisfy Lipschitz condition with a constant $\beta < \alpha$. Then* $\forall\, x_0 \in X \quad \exists\, x \in X$:

$$F(x) = G(x) \ \ and \ \ \rho_X(x,x_0) \leq \frac{\rho_Y(F(x_0),G(x_0))}{\alpha-\beta}.$$

[*] A.V. Arutyunov, Covering mappings in metric spaces and fixed points, Dokl. Math. 76(2)(2007), pp. 665-668.

[†] A. Arutyunov, E. Avakov, B. Gel'man B, A. Dmitruk, V. Obukhovskii, Locally covering maps in metric spaces and coincidence points, J. Fixed Points Theory and Applications, 5:1(2009), pp. 105-127.

7. Ordinary differential equations unsolved for the derivative of unknown function

$f : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$, t_0, $t_1 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$

$$f(t, x, \dot{x}) = 0, \;\; x(t_0) = x_0. \tag{1}$$

- $f(\cdot, x, \dot{x})$ is measurable $\forall\ x,\ \dot{x}$;
- $f(t, \cdot)$ is continuous $\dot{\forall}\ t$;
- $\forall\ \rho > 0\ \exists\ \Lambda > 0 : |x| + |v| < \delta \Rightarrow |f(t, x, v)| \leq \Lambda\ \dot{\forall}\ t$.

Def. 3. *Equation (1) is locally solvable if*
$\exists\ \tau > 0,\ x \in AC_\infty[t_0, t_0 + \tau] : x(t_0) = x_0$ *and*
$f(t, x(t), \dot{x}(t)) = 0\ \dot{\forall}\, t \in [t_0, t_0 + \tau]$.

8. Examples

1. $t^2 + x^2 + \dot{x}^2 = 0, \;\; x(0) = 0$ is not locally solvable.

2.

$$f(x, \dot{x}) = 0, \;\; x(t_0) = x_0 \qquad (1')$$

Assume that

- $\dfrac{\partial f}{\partial \dot{x}}(\cdot)$ is continuous;
- $\dfrac{\partial f}{\partial \dot{x}}(x_0, v_0)\,\mathbb{R}^n = \mathbb{R}^k$.

Then there exists a continuous function $F(\cdot)$ such that $\dot{x} = F(x)$. Thus, (1') is locally solvable.

8. One more perturbation theorem

Let (X, ρ_X), (Y, ρ_Y) be metric spaces, $\Gamma : X \times X \to Y$, $x_0 \in X$. For arbitrary $y \in Y$ consider equation

$$\Gamma(x, x) = y.$$

Th. 3. [*] [†] *Let X be complete, Γ be continuous. If*

- $\Gamma(\cdot, x_2)$ *is locally α-covering around x_0 $\forall$ x_2;*
- $\Gamma(x_1, \cdot)$ *satisfy Lipschitz condition with constant β in a neighborhood of x_0 $\forall$ x_1;*
- $\beta < \alpha$

then $\exists\ \delta > 0 : \forall\ y \in B_Y(\Gamma(x_0, x_0), \delta)\ \exists\ x \in X :$
$\Gamma(x, x) = y$ *and* $\rho_X(x_0, x) \leq \frac{\rho_Y(\Gamma(x_0,x_0),y)}{\alpha-\beta}$.

[*] A.V. Arutyunov, E.S. Zhukovskiy, S.E. Zhukovskiy, On the well-posedness of differential equations unsolved for the derivative, Diff. Eq., 47:11(2011), pp. 1–15.

[†] A.V. Arutyunov, E.R. Avakov, E.S. Zhukovskii, Covering mappings and their applications to differential equations unsolved for the derivative, Diff. Equations 45(5)(2009), pp. 627–649.

10. Solvability condition for differential equation

$$f(t,x,\dot{x}) = 0, \;\; x(t_0) = x_0 \qquad (1)$$

- $f(\cdot,x,\dot{x})$ is measurable $\forall\ x,\ \dot{x}$;
- $f(t,\cdot)$ is continuous $\dot{\forall}\ t$;
- $\forall\ \rho > 0\ \exists\ \Lambda > 0 : |x| + |v| < \delta \Rightarrow |f(t,x,v)| \le \Lambda\ \dot{\forall}\ t$.

Th. 4. [*] [†] Assume that

A) $f(t,x,\cdot)$ is locally α-covering around v_0;

B) $f(t,\cdot,v)$ is satisfy Lipschitz condition in a neighborhood of x_0. Then (1) is locally solvable.

[*] A.V. Arutyunov, E.S. Zhukovskiy, S.E. Zhukovskiy, On the well-posedness of differential equations unsolved for the derivative, Diff. Eq., 47:11(2011), pp. 1–15.

[†] A.V. Arutyunov, E.R. Avakov, E.S. Zhukovskii, Covering mappings and their applications to differential equations unsolved for the derivative, Diff. Eq. 45(5)(2009), pp. 627–649.

11. Related problems

1. Solvability of control systems with mixed constraints. Stability problem for the systems. *

2. Solvability of Volterra equations unsolved for the unknown function. †

3. Solvability of differential inclusions unsolved for the derivative of the unknown function. Solutions solvability.

4. Solvability of difference equations in implicit form. Asymptotic behavior of the solutions.

5. Solvability of discrete control systems with mixed constraints.

*A.V. Arutyunov, S.E. Zhukovskiy, Existence of local solutions in constrained dynamic systems, Applicable Analysis, 90:6(2011), pp. 889-898.

†A.V. Arutyunov, E.S. Zhukovskiy, S.E. Zhukovskiy, Covering mappings and well-posedness of nonlinear Volterra equations, Nonlinear Analysis: Theory, Methods and Applications. 75:3(2011), pp. 1026-1044.

Thank you for your attention!