Covering mappings. Theory and applications

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1. Covering mappings

Let (X, ρ_X), (Y, ρ_Y) be metric spaces, $\alpha > 0$.

Denote $B_X(x_0, r) = \{x \in X : \rho_X(x, x_0) \leq r\}$, $x_0 \in X$, $r \geq 0$.

Def. 1. *$F : X \to Y$ is an α-covering mapping if*

$$B_Y(F(x_0), \alpha r) \subset F(B_X(x_0, r)) \ \forall\, x_0 \in X, \ \ r \geq 0.$$

Here $B_X(x_0, r) = \{x \in X : \rho_X(x, x_0) \leq r\}$.

$F : X \times X \to Y$ is α-covering $\Leftrightarrow$ $\forall\ x_0 \in X$, $y \in Y$ $\exists\ x \in X$: $F(x) = y$ and $\rho_X(x_0, x) \leq \dfrac{1}{\alpha}\rho_Y(F(x_0), y)$.

Examples

1. The identity map $F : X \to X$ is 1-covering.

2. Let $F : \mathbb{R} \to \mathbb{R}$ be absolutely continuous.

F is α-covering if and only if

$\Big(F'(x) \geq \alpha$ for a.a. $x \in \mathbb{R} \Big)$ or $\Big(F'(x) \leq -\alpha$ for a.a. $x \in \mathbb{R} \Big)$.

3. Let X, Y be Banach spaces, $F : X \to Y$ be a surjective continuous linear mapping. By Banach Open Mapping Theorem, $\exists\ \alpha > 0$ such that F is α-covering.

2. Local covering property

Let (X, ρ_X), (Y, ρ_Y) be metric spaces, $\alpha > 0$, $x_0 \in X$.

Def. 2. $F : X \to Y$ *is locally* α*-covering around* x_0 *if exists* $R > 0$ *such that*

$$B_X(x, r) \subset B_X(x_0, R) \Rightarrow B_Y(F(x_0), \alpha r) \subset F(B_X(x_0, r)).$$

Note that if F is α-covering then F is locally α-covering around any $x_0 \in X$.

3. Perturbation theorem

Let (X, ρ_X) be a metric space, $(Y, \|\cdot\|_Y)$ be a normed linear space. Numbers $\alpha > 0$, $\beta \geq 0$, point $x_0 \in X$, mappings $F, G : X \to Y$ are given.

Th.1. * *If X is complete, F is continuous and locally α-covering around x_0, G is β-Lipschitz in a neighborhood of x_0, and $\beta < \alpha$, then $F + G$ is locally $(\alpha - \beta)$-covering around x_0.*

A similar result was obtained by L.M. Graves. †

*A.V. Dmitruk, A.A. Milyutin, N.P. Osmolovskii, Lyusternik's theorem and the theory of extrema, Uspekhi Mat. Nauk, 35:6(216)(1980), pp. 11-46.

†L. M. Graves, Some mapping theorems, Duke Math. J., 17(1950), pp. 111-114

Corollaries

Let X, Y be Banach spaces, $F : X \to Y$.

Th. 2. * *If F is strictly differentiable at x_0 and*

$$\frac{\partial F}{\partial x}(x_0)X = Y, \tag{1}$$

then F is locally α-covering around x_0 with some $\alpha > 0$.

Th. 3. *If F is strictly differentiable at x_0 and (1) holds, then there exists $\varepsilon > 0$, $c > 0$, $x : B_Y(F(x_0), \varepsilon) \to X$:*

1) $F(x(y)) \equiv y$;

2) $x(F(x_0)) = x_0$;

3) $\|x(y) - x_0\| \le c\|y - F(x_0)\| \ \forall \ y$.

*B.S. Mordukhovich, Variational Analysis and Generalized Differentiation, V. 1. Springer. 2005.

4. Coincidence points

Let (X, ρ_X), (Y, ρ_Y) be metric spaces, $F, G : X \to Y$.
A solution to the equation

$$F(x) = G(x)$$

is called a coincidence point of F and G.

Th. 4 [*] [†] *Let X be complete, F be continuous and α-covering, G satisfy Lipschitz condition with a constant $\beta < \alpha$. Then* $\forall\, x_0 \in X \quad \exists\, x \in X$:

$$F(x) = G(x) \ \ and \ \ \rho_X(x, x_0) \le \frac{\rho_Y(F(x_0), G(x_0))}{\alpha - \beta}.$$

[*] A.V. Arutyunov, Covering mappings in metric spaces and fixed points, Dokl. Math. 76(2)(2007), pp. 665-668.

[†] A. Arutyunov, E. Avakov, B. Gel'man B, A. Dmitruk, V. Obukhovskii, Locally covering maps in metric spaces and coincidence points, J. Fixed Points Theory and Applications, 5:1(2009), pp. 105-127.

5. Ordinary differential equations unsolved for the derivative of unknown function

$f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$, $x_0 \in \mathbb{R}^n$

$$f(t, x, \dot{x}) = 0, \;\; x(0) = x_0. \tag{2}$$

- $f(\cdot, x, \dot{x})$ is measurable $\forall\ x,\ \dot{x}$;
- $f(t, \cdot)$ is continuous $\dot{\forall}\ t$;
- $\forall\ \rho > 0\ \exists\ \Lambda > 0 : |x| + |v| < \delta \Rightarrow |f(t, x, v)| \le \Lambda\ \dot{\forall}\ t$.

Equation (2) is locally solvable if

$\exists\ \tau > 0,\ x \in AC_\infty[0, \tau] : x(0) = x_0$ and

$f(t, x(t), \dot{x}(t)) = 0\ \dot{\forall} t \in [0, \tau]$.

6. One more perturbation theorem

Let (X, ρ_X), (Y, ρ_Y) be metric spaces, $\Gamma : X \times X \to Y$, $x_0 \in X$. Given a point $y \in Y$, consider the equation

$$\Gamma(x, x) = y.$$

Th. 5. [*] [†] *Let X be complete, Γ be continuous. If*

- $\Gamma(\cdot, x_2)$ *is α-covering around* $\forall\ x_2$;
- $\Gamma(x_1, \cdot)$ *is β-Lipscitz* $\forall\ x_1$;
- $\beta < \alpha$;

then $x \in X : \Gamma(x, x) = y$ *and* $\rho_X(x_0, x) \le \frac{\rho_Y(\Gamma(x_0, x_0), y)}{\alpha - \beta}$.

[*] A.V. Arutyunov, E.S. Zhukovskiy, S.E. Zhukovskiy, On the well-posedness of differential equations unsolved for the derivative, Diff. Eq., 47:11(2011), pp. 1–15.

[†] A.V. Arutyunov, E.R. Avakov, E.S. Zhukovskii, Covering mappings and their applications to differential equations unsolved for the derivative, Diff. Equations 45(5)(2009), pp. 627–649.

7. Solvability condition for the ODEs

$$f(t,x,\dot{x})=0, \;\; x(0)=x_0 \tag{2}$$

- $f(\cdot,x,\dot{x})$ is measurable $\forall\ x,\ \dot{x}$;
- $f(t,\cdot)$ is continuous $\dot{\forall}\ t$;
- $\forall\ \rho>0\ \exists\ \Lambda>0:\ |x|+|v|<\delta \Rightarrow |f(t,x,v)|\le\Lambda\ \dot{\forall}\ t.$

Th. 6. [*] [†] *Assume that*

A) $f(t,x,\cdot)$ *is* α*-covering* $\dot{\forall}\ t\in[0,1]$, $\forall\ x\in\mathbb{R}^n$;

B) $f(t,\cdot,u)$ *is* k*-Lipschitz* $\dot{\forall}\ t\in[0,1]$, $\forall\ u\in\mathbb{R}^n$;

Then (2) is locally solvable.

[*] A.V. Arutyunov, E.S. Zhukovskiy, S.E. Zhukovskiy, On the well-posedness of differential equations unsolved for the derivative, Diff. Eq., 47:11(2011), pp. 1–15.

[†] A.V. Arutyunov, E.R. Avakov, E.S. Zhukovskii, Covering mappings and their applications to differential equations unsolved for the derivative, Diff. Eq. 45(5)(2009), pp. 627–649.

8. Differential inclusions unsolved for the derivative of unknown function

Consider a differential inclusion

$$0 \in F(t, x, \dot{x}), \quad x(0) = x_0. \tag{3}$$

Here $x_0 \in \mathbb{R}^n$, $F : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to K(\mathbb{R}^k)$,

- $F(\cdot, x, u)$ is measurable for any (x, u);
- $F(t, \cdot, \cdot)$ is continuous for almost all t;
- $\forall\ d \geq 0\ \exists\ m \geq 0 : |x| \leq d$ and $|u| \leq d \Rightarrow$

$|y| \leq m\ \forall y \in F(t, x, u),\ \dot{\forall} t \in [0, 1]$.

Inclusion (3) is locally solvable if $\exists\ \tau > 0$,
$x \in AC_\infty[0, \tau] : x(0) = x_0$ and $0 \in F(t, x(t), \dot{x}(t))\ \dot{\forall}\, t$.

Given metric spaces X, Y, $G : X \rightrightarrows Y$ is α-covering if $B_Y(G(x), \alpha r) \subset G(B_X(x, r))\ \forall\ x \in X,\ r \geq 0$.

$$0 \in F(t, x, \dot{x}), \quad x(0) = x_0. \tag{3}$$

Th. 6. *Assume that*

A) $F(t, x, \cdot)$ *is* α*-covering* $\dot{\forall}\ t \in [0, 1]$, $\forall\ x \in \mathbb{R}^n$;

B) $F(t, \cdot, u)$ *is* k*-Lipschitz* $\dot{\forall}\ t \in [0, 1]$, $\forall\ u \in \mathbb{R}^n$;

Then (3) is locally solvable.

Thank you for your attention!