Necessary Conditions for Optimal Control Problems with State Constraints: Theory and Applications

Thesis Research Plan

Md. Haider Ali Biswas [1]

March 30, 2012

[1]FEUP-PDEEC, Department of Electrical and Computer Engineering, Faculty of Engineering, University of Porto, Portugal.

Outline

- Motivation and Contribution
- Main Results
  - State Constrained OCPs: Convex and Nonconvex Cases.
  - Mixed Constrained OCPs: Convex Case.
- Application of OC to Real Problems
  - HIV Model.
- Conclusion and Future Works.

Motivation I

- The 'smallest' set of candidates of local minimizers ensuring the optimal solution.
- Pontryagin Maximum Principle (PMP) is a NCO. For *normal linear-convex* problems, PMP is also sufficient.
- In the *normal linear-convex* case, the NMP is not a sufficient condition.
- In [8], de Pinho and Vinter came up with the necessary conditions of optimality in the "Euler form" in terms of "Joint" adjoint inclusion.
  - These necessary conditions are not a maximum principle because of the absence of *Weierstrass Condition*

Motivation II

- In [4], de Pinho *et al.* extended the work of de Pinho and Vinter to state constrained problems
  - Such generalization remains a sufficient condition for the normal linear-convex problems.
- Quite recently in [3], Clarke and de Pinho derived a new nonsmooth maximum principle in the vein of [8].
  - Lipschitz continuity of dynamics with respect to both state and control is assumed.
  - Sufficiency of the nonsmooth maximum principle when applied to normal linear-convex problems.

Our Contribution

- We obtain a NMP for state constrained problems in the vein of [5].
- We apply a ”strong version” of the results of Clarke and de Pinho [3] to our problems.
- Our result covers state constrained problems in two steps; first the convex case is treated in the vein of [4] using techniques based on [9] and then convexity is removed in vein of [5].
- We add the Weierstrass conditions to adjoint inclusions using the joint subdifferentials with respect to the state and the control.
- Our Nonsmooth Maximum Principle (NMP) is a sufficient condition for normal *linear-convex* problems.

Problem Formulation

We consider the optimal control problem of our interest as

$$(P)\quad \begin{cases} \text{Minimize } l(x(a),x(b)) + \int_a^b L(t,x(t),u(t))\, dt \\ \text{subject to} \\ \quad \dot{x}(t) = f(t,x(t),u(t)) \quad \text{a.e. } t \in [a,b] \\ \quad h(t,x(t)) \le 0 \quad \text{for all } \ t \in [a,b] \\ \quad u(t) \in U(t) \quad \text{a.e. } \ t \in [a,b] \\ \quad (x(a),x(b)) \in E. \end{cases}$$

Here $[a,b]$ is a fixed interval, the functions $f\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ describes the system dynamics, $h\colon [a,b]\times\mathbb{R}^n \to \mathbb{R}$ defines the pathwise state constraint, $L : [a,b]\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}$ is a scalar function and $U\colon [a,b] \to \mathbb{R}^m$ is a multifunction.

Basic Assumptions

We consider the following hypotheses throughout:

- the functions L and f are $\mathcal{L} \times \mathcal{B}$-measurable,
- the multifunction $t \to U(t)$ has $\mathcal{L} \times \mathcal{B}$-measurable graph,
- the set E is closed,
- f is locally Lipschitz with respect to x and
- l is locally Lipschitz.

Notations

- $\mathbb{B}$ represents the closed unit ball centered at the origin.
- $(\bar{x}, \bar{u})$ denotes the *optimal solution* of the problem under consideration.
- A function $h\colon [a,b] \to \mathbb{R}^p$ lies in $W^{1,1}([a,b];\mathbb{R}^p)$ if and only if it is absolutely continuous; in $L^1([a,b];\mathbb{R}^p)$ iff it is integrable; and in $L^\infty([a,b];\mathbb{R}^p)$ iff it is essentially bounded.
- We say that the process $(\bar{x}, \bar{u})$ is a *strong local minimum* if, for some $\varepsilon > 0$, it minimizes the cost over admissible processes (x,u) such that $|x(t) - \bar{x}(t)| \leq \varepsilon \quad$ for all $t \in [a,b]$.

Reformulated Problem

We consider the following problem in absence of state constraints:

$$(S)\quad \begin{cases} \text{Minimize } l(x(a),x(b)) + \int_a^b L(t,x(t),u(t))dt \\ \text{subject to} \\ \quad \dot{x}(t) = f(t,x(t),u(t)) \text{ a.e. } t \in [a,b] \\ \quad u(t) \in U(t) \text{ a.e. } t \in [a,b] \\ \quad (x(a),x(b)) \in E. \end{cases}$$

where all symbols and notations are same as in problem (P).

Assumptions I

A1 There exist constants k_x^ϕ and k_u^ϕ such that for almost every $t \in [a,b]$ and every (x_i, u_i) $(i = 1, 2)$ such that

$$x_i \in \{x : |x - \bar{x}(t)| \leq \varepsilon\}, \quad u_i \in U(t)$$

we have

$$|\phi(t, x_1, u_1) - \phi(t, x_2, u_2)| \leq k_x^\phi |x_1 - x_2| + k_u^\phi |u_1 - u_2|.$$

When **A1** is imposed on f and/or L, then the Lipschitz constants are denoted by k_x^f, k_u^f, k_x^L and k_u^L.

Assumptions II

A2 The set valued function $t \to U(t)$ is closed valued and there exists a constant $c > 0$ such that for almost every $t \in [a,b]$ we have

$$|u(t)| \leq c \quad \forall u \in U(t).$$

A3 For all x such that $|x(t) - \bar{x}(t)| \leq \varepsilon$ the function $t \to h(t,x)$ is continuous. Furthermore, there exists a constant $k_h > 0$ such that the function $x \to h(t,x)$ is Lipschitz of rank k_h for all $t \in [a,b]$.

Our assumptions also assert that following conditions are satisfied:

Assumptions III

- $$|\phi(t,\bar{x}(t),u)-\phi(t,\bar{x}(t),\bar{u}(t))| \leq k_u^{\phi}|u-\bar{u}(t)| \text{ for all } u \in U(t) \text{ a.e. } t$$

  and there exists an integrable function k such that

  $$|\phi(t,\bar{x}(t),u)| \leq k(t) \text{ for all } u \in U(t) \text{ a.e. } t.$$

  In the above ϕ is to be replaced by f and L.
- The sets

  $$f(t,x,U(t)) \text{ and } L(t,x,U(t)) \text{ are compact for all } x \in \bar{x}(t)+\varepsilon B.$$

Auxiliary Results

Theorem

(adaption of Theorem 3.1 in [3]) *Let $(\bar{x},\bar{u})$ be a strong local minimum for problem (S). If the basic assumptions are satisfied, f and L satisfy* **A1** *and U is closed valued, then there exist $p \in W^{1,1}([a,b];\mathbb{R}^n)$ and a scalar $\lambda_0 \geq 0$ satisfying*

$$(p(t),\lambda_0) \neq 0 \quad \forall\, t \in [a,b],$$

$$(-\dot{p}(t),0) \in \partial^C_{x,u}\left[\langle p(t),\bar{f}(t)\rangle - \lambda_0 \bar{L}(t)\right] - \{0\} \times N^C_{U(t)}(\bar{u}(t)) \; a.e.$$

$$\forall u \in U(t),\ \langle p(t),f(t,\bar{x}(t),u)\rangle - \lambda_0 L(t,\bar{x}(t),u) \leq \langle p(t),\bar{f}(t)\rangle - \lambda_0 \bar{L}(t) \; a.e.$$

$$(p(a),-p(b)) \in N^L_E(\bar{x}(a),\bar{x}(b)) + \lambda_0 \partial l(\bar{x}(a),\bar{x}(b)).$$

where $\bar{f}(t)$ and $\bar{L}(t)$ represent the function evaluated at $(t,\bar{x}(t),\bar{u}(t))$.

Main Result: Convex Case

- We now impose the following convexity assumption on the "velocity set" for problem (P):

[C] The velocity set $\{(v,l)=(f(t,x,u),L(t,x,u)),\ u\in U(t)\}$ is convex for all $(t,x)\in[a,b]\times\mathbb{R}^n$.

- Introduce the following subdifferential

$$\bar{\partial}_x h(t,x) := \text{co}\,\{\lim \xi_i \ :\ \xi_i \in \nabla_x h(t_i,x_i), (t_i,x_i)\to(t,x)\}. \quad (2)$$

Main Result: Convex Case

Theorem

Let $(\bar{x},\bar{u})$ be a strong local minimum for problem (P). Suppose that f and L satisfy **A1**, *and that assumptions* **A2** *and* **C** *hold and h satisfies* **A3**, *then there exist $p \in W^{1,1}([a,b];\mathbb{R}^n)$, $\gamma \in L^1([a,b];\mathbb{R})$, a measure $\mu \in C^{\oplus}([a,b];\mathbb{R})$, and a scalar $\lambda_0 \geq 0$ satisfying*

$$\mu\{[a,b]\} + ||p||_\infty + \lambda_0 > 0,$$

$$(-\dot{p}(t),0) \in \partial^C_{x,u}\left[\langle q(t),\bar{f}(t)\rangle - \lambda_0 \bar{L}(t)\right] - \{0\}\times N^C_{U(t)}(\bar{u}(t)) \ a.e.$$

$$\forall\ u \in U(t),\ \ \langle q(t),f(t,\bar{x}(t),u)\rangle - \lambda_0 L(t,\bar{x}(t),u) \leq \langle q(t),\bar{f}(t)\rangle - \lambda_0 \bar{L}(t) \ a.e.$$

$$(p(a),-q(b)) \in N^L_E(\bar{x}(a),\bar{x}(b)) + \lambda_0 \partial l(\bar{x}(a),\bar{x}(b)),$$

$$\gamma(t) \in \bar{\partial} h(t,\bar{x}(t)) \quad \mu\text{-}a.e.,$$

$$\text{supp}\{\mu\} \subset \{t \in [a,b] : h(t,\bar{x}(t)) = 0\},$$

Main Result: Convex Case

where

$$q(t) = \begin{cases} p(t) + \int_{[a,t)} \gamma(s)\mu(ds) & t \in [a,b) \\ p(t) + \int_{[a,b]} \gamma(s)\mu(ds) & t = b. \end{cases} \tag{3}$$

Main Result: Non Convex Case

We derive a NMP for nonconvex case of the problem (P).

- Convexity assumption [C] is removed.
- Replace the subdifferential $\bar{\partial}_x h$ by a more refined subdifferential $\partial_x^> h$ defined by

$$\partial_x^> h(t,x) := \text{co}\,\{\xi : \exists (t_i, x_i) \xrightarrow{h} (t,x) : h(t_i,x_i) > 0\ \forall i,\ \nabla_x h(t_i,x_i) \to \xi\}. \tag{4}$$

Main Result: Non Convex Case

Theorem

Let $(\bar{x}, \bar{u})$ be a strong local minimum for problem (P). Assume that f and L satisfy **A1**, *h satisfies* **A3** *and that* **A2** *as well as the basic assumptions are satisfied. Then there exist an absolutely continuous function p, integrable functions ξ and γ, a non-negative measure $\mu \in C^{\oplus}([a,b];\mathbb{R})$, and a scalar $\lambda_0 \geq 0$ such that conditions (i)–(vi) of Theorem 2 hold with $\partial_x^{>} h$ as in (4) replacing $\bar{\partial}_x h$ and where q is as defined in* (3).

A Refinement of Theorem 3

- $(\bar{x}, \bar{u})$ to be a weak local minimum instead of a strong local minimum.
- Our results in Theorem 3 have been extended to cover with a $W^{1,1}$ local minimum for problem (P) following the approach in [10].

Theorem

Let $(\bar{x}, \bar{u})$ be merely a $W^{1,1}$ local minimum for problem (P). Then the conclusions of Theorem 3 hold.

The Convex Case

We validate the Theorem 2 to the following problem

$$(Q)\quad \begin{cases} \text{Minimize } l(x(b)) \\ \text{subject to} \\ \qquad \dot{x}(t) = f(t,x(t),u(t)) & \text{a.e.} \quad t \in [a,b] \\ \qquad u(t) \in U(t) & \text{a.e.} \quad t \in [a,b] \\ \qquad h(t,x(t)) \leq 0 & \text{for all} \quad t \in [a,b] \\ \qquad (x(a),x(b)) \in \{x_a\} \times E_b. \end{cases}$$

Problem (Q) is a special case of (P) in which $E = \{x_a\} \times E_b$ and $l(x_a, x_b) = l(x_b)$.

Sequence of Problems

- Penalize state-constraint violation.
  Define the following problem for each $i \in \mathbb{N}$:

$$(Q_i) \quad \begin{cases} \text{Minimize } l(x(b)) + i \int_a^b h^+(t,x(t))\, dt \\ \text{subject to} \\ \quad \dot{x}(t) = f(t,x(t),u(t)) \quad \text{a.e. } t \in [a,b] \\ \quad (x(a),x(b)) \in \{x_a\} \times E_b, \end{cases}$$

  where $h^+(t,x) := \max\{0, h(t,x)\}$. This differs from (Q) by shifting the state constraint into the objective function. Following the approach in [9], we assume that penalization is effective, i.e., we suppose the interim hypothesis

[IH] $\lim_{i\to\infty} \inf\{Q_i\} = \inf\{Q\}$.

Application of Ekeland's theorem

Let W denote the set of measurable functions $u\colon [a,b] \to \mathbb{R}^m$, $u(t) \in U(t)$ a.e. for which there exists an absolutely continuous function x such that $\dot{x}(t) = f(t,x(t),u(t))$, for almost every $t \in [a,b]$, $x(t) \in \bar{x}(t) + \varepsilon\mathbb{B}$ for all $t \in [a,b]$, $x(a) = x_a$ and $x(b) \in E_b$. We provide W with the metric $\Delta(u,v) := \| u - v \|_{L_1}$ and define $J_i\colon W \to \mathbb{R}$ using the arc x mentioned above:

$$J_i(u) := l(x(b)) + i \int_a^b h^+(t,x(t))\,dt.$$

Then (W,Δ) is a complete metric space in which the functional $J_i\colon W \to \mathbb{R}$ is continuous. Moreover, problem (Q_i) above is closely related to the abstract problem

$$(R_i) \quad \begin{cases} \text{Minimize} & J_i(u) \\ \text{subject to} & u \in W. \end{cases}$$

Study of perturbed problem

- Clearly $(\bar{u}, \bar{x}(b))$ is admissible for (R_i), with $J_i(\bar{u}) = l(\bar{x}(b)) = \inf Q$ since for all $t \in [a,b]$, $h^+(t, \bar{x}(t)) = 0$.
- Let $\varepsilon_i = J_i(\bar{u}) - \inf Q_i$.
  We have $\varepsilon_i \geq 0$ and, taking into account [IH], $\varepsilon_i \to 0$. Ekeland's variational principle (see [10]) applies. It asserts the existence of $u_i \in W$ such that

$$\| u_i - \bar{u} \|_{L_1} \leq \sqrt{\varepsilon_i} \tag{5}$$

and u_i minimizes over W the perturbed cost functional

$$u \mapsto J_i(u) + \sqrt{\varepsilon_i} \| u_i - \bar{u} \|_{L_1} . \tag{6}$$

Let x_i be the trajectory corresponding to u_i.

Optimality conditions for the perturbed problem

Ekeland's Theorem shows that the process (x_i, u_i) solves the following optimal control problem:

$$
(D_i) \quad \left\{ \begin{array}{l}
\text{Minimize } l(x(b)) + i\int_a^b h^+(t,x(t))\,dt + \sqrt{\varepsilon_i}\int_a^b |u(t)-u_i(t)|\ dt \\
\text{subject to} \\
\begin{array}{rclrl}
\dot{x}(t) & = & f(t,x(t),u(t)) & \text{a.e.} & t\in[a,b] \\
u(t) & \in & U(t) & \text{a.e.} & t\in[a,b] \\
x(t) & \in & \bar{x}(t)+\varepsilon\mathbb{B} & \text{for all} & t\in[a,b] \\
x(a) & = & x_a & & \\
x(b) & \in & E_b. & &
\end{array}
\end{array} \right.
$$

Since $\varepsilon_i \to 0$ (by [IH]) it follows from (5) that $u_i \to \bar{u}$ strongly. Apply Theorem 1 to each (D_i), rewriting the conditions and taking the limits as in [4] we get the required conclusions.

The 'minimax' problem

- We consider the following 'minimax' optimal control problem

$$(\tilde{R}) \begin{cases} \text{Minimize } \tilde{l}(x(a), x(b), \max_{t\in[a,b]} h(t, x(t))) \\ \text{over } x \in W^{1,1} \text{ and measurable functions } u \text{ satisfying} \\ \quad \dot{x}(t) = f(t, x(t), u(t)) \text{ a.e. } t \in [a,b] \\ \quad u(t) \in U(t) \text{ a.e. } t \in [a,b] \\ \quad (x(a), x(b)) \in E_a \times \mathbb{R}^n. \end{cases}$$

A4 The integrable function $\tilde{l}$ is Lipschitz continuous on a neighbourhood of

$$(\bar{x}(a), \bar{x}(b), \max_{t\in[a,b]} h(t, \bar{x}(t)))$$

and $\tilde{l}$ is monotone in the z variable, in the sense that $z' \geq z$ implies $\tilde{l}(y, x, z') \geq \tilde{l}(y, x, z)$, for all $(y, x) \in \mathbb{R}^n \times \mathbb{R}^n$.

Apply Ekeland's theorem for sequence of problems

We consider the set

$$V := \{(x,u,e) : (x,u) \text{ satisfies } \dot{x}(t) = f(t,x(t),u(t)),$$
$$u(t) \in U(t) \text{ a.e., } e \in \mathbb{R}^n, (x(a),e) \in E \text{ and } \|x - \bar{x}\|_{L^\infty} \leq \varepsilon\}$$

and let $d_V : V \times V \to \mathbb{R}$ be a function defined by

$$d_V((x,u,e),(x',u',e')) = |x(a) - x'(a)| + |e - e'| + \int_a^b |u(t) - u'(t)|dt$$

The problem

The problem of interest is

$$(P') \quad \begin{cases} \text{Minimize } l(x(a),x(b)) \\ \text{subject to} \\ \quad \dot{x}(t) = f(t,x(t),u(t)) \quad \text{a.e. } t \in [a,b] \\ \quad h(t,x(t)) \leq 0 \quad \text{for all } \ t \in [a,b] \\ \quad (x(t),u(t)) \in S(t) \quad \text{a.e. } t \in [a,b] \\ \quad (x(a),x(b)) \in E. \end{cases}$$

where we define

$$S(t) := \{(x,u) : (t,x,u) \in S\} \text{ for all } t \in [a,b].$$

and for some $\varepsilon > 0$

$$S_*^{\varepsilon}(t) = \{(x,u) \in S(t) : \ |x - \bar{x}(t)| \leq \varepsilon\}.$$

Assumptions I

- We assume that a function $\phi(t,x,u)$ satisfies $[L_*^\varepsilon]$ if:
  $[\mathbf{L}_*^\varepsilon]$ There exist constants k_x^ϕ and k_u^ϕ such that for almost every $t \in [a,b]$ and every $(x_i,u_i) \in S_*^\varepsilon(t)$ $(i = 1,2)$ we have

$$|\phi(t,x_1,u_1) - \phi(t,x_2,u_2)| \leq k_x^\phi |x_1 - x_2| + k_u^\phi |u_1 - u_2|.$$

- For $S(t)$ we consider the following **bounded slope condition**:
  $[\mathbf{BS}_*^\varepsilon]$ There exists a constant k_S such that for almost every $t \in [a,b]$ the following condition holds

$$(x,u) \in S_*^\varepsilon(t),\ (\alpha,\beta) \in N_{S(t)}^P(x,u) \Longrightarrow |\alpha| \leq k_S|\beta|.$$

Assumptions II

- [$\mathbf{CS}_*^\varepsilon$] The set $S_*^\varepsilon(t)$ is closed and there exists an integrable function c such that for almost every $t \in [a,b]$ the following holds

$$S_*^\varepsilon(t) \text{ is closed and } (x,u) \in S_*^\varepsilon(t) \implies |(x,u)| \le c(t).$$

[C'] The velocity set $\{v \in \mathbb{R}^n : \; v = f(t,x,u),\; u \in S(t,x)\}$ is convex for all $t \in [a,b]$.

[H1] For all $x \in \bar{x}(t) + \varepsilon\mathbb{B}$ the function $t \to h(t,x)$ is continuous and there exists a scalar $k_h > 0$ such that the function $x \to h(t,x)$ is Lipschitz of rank k_h for all $t \in [a,b]$.

[H2] For almost every $t \in [a,b]$ the following condition holds: for all $u \in S(t,\bar{x}(t))$ and all sequence $x_n \to \bar{x}(t)$ there exists a sequence $u_n \in S(t,x_n)$ such that $u_n \to u$.

Main Results

Theorem

Let $(\bar{x},\bar{u})$ be a strong local minimum for problem (P'). Assume that the basic hypotheses, [C'], [H1], [H2], $[BS^{\varepsilon}_{*}]$ and $[CS^{\varepsilon}_{*}]$ hold and that f satisfies $[L^{\varepsilon}_{*}]$. Then there exist $p \in W^{1,1}([a,b];\mathbb{R}^n)$, $\gamma \in L^1([a,b];\mathbb{R})$, a measure $\mu \in C^{\oplus}([a,b];\mathbb{R})$, and a scalar $\lambda_0 \geq 0$ such that

$$\mu\{[a,b]\} + ||p||_{\infty} + \lambda_0 > 0,$$

$$(-\dot{p}(t),0) \in \partial^{C}_{x,u}\langle q(t), f(t,\bar{x}(t),\bar{u}(t))\rangle - N^{C}_{S(t)}(\bar{x}(t),\bar{u}(t)) \;\; a.e.,$$

$$(\bar{x}(t),u) \in S(t) \Longrightarrow \langle q(t), f(t,\bar{x}(t),u)\rangle \leq \langle q(t), f(t,\bar{x}(t),\bar{u}(t))\rangle \;\; a.e.,$$

$$(p(a),-q(b)) \in N^{L}_{E}(\bar{x}(a),\bar{x}(b)) + \lambda_0 \partial l(\bar{x}(a),\bar{x}(b)),$$

$$\gamma(t) \in \bar{\partial} h(t,\bar{x}(t)) \quad \mu\text{-}a.e.,$$

$$\text{supp}\{\mu\} \subset \{t \in [a,b] : h(t,\bar{x}(t)) = 0\},$$

Existing Model of HIV

We consider the HIV model as in [7]

$$\frac{dT_A}{dt} = \frac{s}{1+V(t)} - \mu_{T_A} T_A(t) + rT_A(t)\left(1 - \frac{T_A(t)+T_L(t)+T_I(t)}{T_{\max}}\right) - \mu_i V(t) T_A$$

$$\frac{dT_L}{dt} = \mu_i V(t) T_A(t) - \mu_{T_L} T_L(t) - \mu_c T_L(t),$$

$$\frac{dT_I}{dt} = \mu_c T_L(t) - \mu_{T_I} T_I(t),$$

$$\frac{dV}{dt} = (1-u(t)) N \mu_{T_I} T_I(t) - \mu_i V(t) T_A(t) - \mu_V V(t),$$

Objective function

The objective is

$$\text{Minimize } \ J(u) = \int_{t_s}^{t_f} \Big(- T_A(t) + \frac{1}{2} B u^2(t) \Big) dt$$

subject to

the dynamics (i)– (iv) defined in the model.

where $B > 0$ denotes the weight parameter.

- with the initial conditions

$$T_A(0) = T_{A0}, \ \ T_L(0) = 0, \ \ T_I(0) = 0, \ \ \text{and} \ \ V(0) = V_0,$$

for the case of infection by free virus, or

$$T_A(0) = T_{A0}, \ \ T_L(0) = T_{L0}, \ \ T_I(0) = T_{I0}, \ \ \text{and} \ \ V(0) = V_0,$$

for the case of infections by both free virus and infected cells.

Proposed Model of HIV

Our proposed HIV model

$$\frac{dT_A}{dt} = \frac{s}{1+V(t)} - \mu_{T_A} T_A(t) + rT_A(t)\left(1 - \frac{T_A(t)+T_L(t)+T_I(t)}{T_{\max}}\right) - \mu_i V(t) T_A$$

$$\frac{dT_L}{dt} = \mu_i V(t) T_A(t) - \mu_{T_L} T_L(t) - \mu_c T_L(t),$$

$$\frac{dT_I}{dt} = \mu_c T_L(t) - \mu_{T_I} T_I(t),$$

$$\frac{dV}{dt} = (1-u(t)) N \mu_{T_I} T_I(t) - \mu_i V(t) T_A(t) - \mu_V V(t),$$

$$T_A(t) \geq \tilde{T},$$

Here $\tilde{T}$ is a lower bound belonging to $\mathbb{R}$.

Objective functional

Our objective functional

$$\text{Minimize } \; J(u) = -T_A(t_f) + \int_{t_s}^{t_f} Bu(t)dt$$

- with the initial conditions

$$T_A(0) = T_{A0}, \;\; T_L(0) = T_{L0}, \;\; T_I(0) = T_{I0}, \;\; \text{and} \;\; V(0) = V_0,$$

The Challenge

- Proposed model may be difficult due to nonlinearity of the dynamics and the presence of state constraints.
- New chemotherapeutic strategy for the HIV infections.
- Analytical solution for optimal chemotherapy and then compare the result by numerical illustration.

Conclusion and Future Works

1. Final refinements of our developed results,
2. Try to derive new MPs for OCPs with mixed constraints for nonconvex case,
3. Study the feasibility of imposing state constraints on the HIV model,
4. Analytical optimality conditions of our proposed HIV model and compare the results with numerical simulations.

Bibliography I

[1] F. Clarke, “ Optimization and Nonsmooth Analysis,” *John Wiley and Sons*, New York, 1983.

[2] F. Clarke and MdR de Pinho, *The nonsmooth maximum principle*, *Control Cybernet.*, **38** (2009) 1151–1167.

[3] F. Clarke and MdR de Pinho. *Optimal control problems with mixed constraints*, *SIAM J. Control Optim.*, **48** (2010), 4500–4524.

[4] MdR de Pinho, M. M. A. Ferreira, and F. A. C. C. Fontes, *An Euler-Lagrange inclusion for optimal control problems with state constraints*, *Dynam. Control Systems*, **8** (2002), 23–45.

Bibliography II

[5] MdR de Pinho, M. M. A. Ferreira, and F. A. C. C. Fontes, *Unmaximized inclusion necessary conditions for nonconvex constrained optimal control problems*, *ESAIM Control Optim. Calc. Var.*, **11** (2005) 614–632.

[6] MdR de Pinho, P. Loewen and G. N. Silva, *A weak maximum principle for optimal control problems with nonsmooth mixed constraints*, *Set-Valued and Variational Analysis*, **17** 2009, 203–2219.

[7] D. Kirschner, S. Lenhart and S. Serbin, *Optimal Control of the Chemotherapy of HIV*, *J. Math. Biol*, **35**, (1997) 775–792.

Bibliography III

[8] MdR de Pinho and R. B. Vinter, *An Euler-Lagrange inclusion for optimal control problems*, *IEEE Trans. Automat. Control*, **40** (1995), 1191-1198.

[9] R. B. Vinter and G. Pappas, *A maximum principle for nonsmooth optimal-control problems with state constraints*, *J. Math. Anal. Appl.*, **89** (1982), 212–232.

[10] R. B. Vinter, "Optimal Control," *Birkhäuser, Boston*, 2000.

Thank You for Attentions.

Questions?