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Deducing Kepler and Newton from Avicenna (ابن سينا), Huygens and Descartes

Impetus (momentum), Centrifugal force, Analytic geometry.

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ABSTRACT

With the advent of Einstein's Theory of Relativity and Quantum Mechanics, the interest in Classical Mechanics, as a field of research and philosophy, has naturally declined and almost vanished, being now practically reduced to its didactic relevance and a powerful tool for engineering. However, a fascination for the primordial questions of Physics has compelled me as a justification for once again looking at those principles regulating the movement of bodies, and to deduce from them the laws of gravitation and celestial dynamics. This time, however, this was done without the help from either Kepler or Newton. In effect, one may even say that Newton and Kepler are derived from the conceptual and mathematical manipulations presented herein, based on very simple and universal truths, some of them which have been perceived since at least the time of Avicenna[1]. Then, both the deduced gravitational force and the laws governing the planets will be compared with Newton's gravity and Kepler's laws. Finally, some considerations are drawn about Newton's gravitational constant G and proposed an orbital constant G_0 and a gravitation force dependent on $1/r^3$.

1. Introduction

What is *Gravity*?, what is *Time*?, what in reality is a *Force;* does it exist? Has anyone answered these questions? In fact not. In truth, however, we still are able to make surprising and accurate calculations based on what we perceive of all this, even without answering those questions. It is normally the *logos* that we understand, and not properly the origin of the movements. *I know who I am but I do not know where I am coming from, neither where I go to*. It is perhaps this strange conscience of a reality, a sort of a dream suspended between two unknown abysses, that sustains our curiosity and perhaps even our pleasures and fears. Life is a plan fluctuating in the unknown, which at least we know is fluctuating.

These first thoughts serve to justify the admirable contributions of our ancestors to the knowledge we proudly exhibit today, and to argue, to a certain extent, that all of them have been right in what they believed. Knowledge is a collective heritage, and those who do not expose it under such a perspective are not contributing to doing justice to their past. Avicenna, for example, nearly 700 years before Galileo Galilei, not only had already recognized the extreme importance of the product of *mass* times *velocity* (*momentum*) and called it *impetus*, but also had exposed the notion that masses would move ad infinitum due to such an *inclination* if there would be no force opposing it[2]. This ancestral perception of the causes of motion will also be used in this text as a fundamental idea, as we will see. Around 650

[1] Of Persian origin, and perhaps the most notable universal thinker of the so-called "Islamic Renascence". He was born near 1000 AD.

[2] A. Sayili (1987), "Ibn Sīnā and Buridan on the Motion of the Projectile", *Annals of the New York Academy of Sciences* **500** (1), p. 477 – 482.

years later, Galileo and Kepler, as we know, finally launched the basis of modern science, the first with its definitive genius splitting from the remnants of an Aristotelian view of the universe, by means of its *Law of Inertia* (later called the first law of Newton), and the second with its extraordinary spirit of measuring and geometrical reasoning, which detected that planets were describing elliptical orbits around the Sun, while sweeping out equal areas during equal intervals of time, with each orbital time being a constant related to the semi-major axis of such orbits. And when Descartes develops *analytic geometry,* which made it possible to represent these geometrical forms by means of algebraic equations, and Huygens unveils his famous *centrifugal force* equation ($F_c = m.v^2/r$), all the bases have in fact been launched so that Isaac Newton could brilliantly integrate this knowledge into his three laws of mechanics and a theory of gravitation. Nevertheless, and in a certain sense, since Newton, mechanics have started to be more focused on how systems evolve in time and less on their geometrical secrets. Differential calculus becomes powerful, and through it the genius of Newton has definitely transformed the perspective of his times, turning it into a more reductionist view, contrary to the previous holistic tendency for geometrical laws and properties, which frequently were independent of time. But, is *time* really important to the dynamics of the universe, or only to the human being? Does it matter *when* something will happen, if we already know it *will* happen?

2. **Center of mass and universal time**

Thinking of Avicenna, what is the source of the "force" which moves the masses? Is there really a force, or simply an *inclination* for the movement, that can even be a spatial *inclination*, for example. Do masses feel each other? Or do they not, and what they feel is an *individual* tendency forcing them to move into a certain and *same point* in the space? Indeed, suppose there is no mass at rest, instead, all masses are being accelerated into a single point somewhere in the universe, the *center of mass of the universe* (CM_u). Could such a tendency be the *inclination* already perceived by Avicenna? Could it be that masses are not attracting each other but instead *falling together* into the same point, their center of mass? That way, the idea that *momentum* is exchanged between masses in order to produce a force of attraction, as certain modern theories proclaim, is not necessary. If the only and single force comes from the *total mass* concentrated in a single point (*CM*), the rest are artefacts, impressions, illusions of attraction.

Let us start to imagine that this is true. In an imaginary system of only three masses, as depicted in figure 1, we assume that none of the masses *feels* the other two. Instead, each mass feels the "call" of a *single* and same point in the space: the center of mass (*CM*). This brings something interesting related to what we have been talking about: masses already "know" that their destiny is to be pushed into that point in the space. Whether they will reach it, or simply orbit around it, will depend only on their initial conditions of movement, or their *impetus*. Certain is, however, that each of the masses will only feel completely at rest (free of tension) when it will share such a special point, either physically or geometrically.

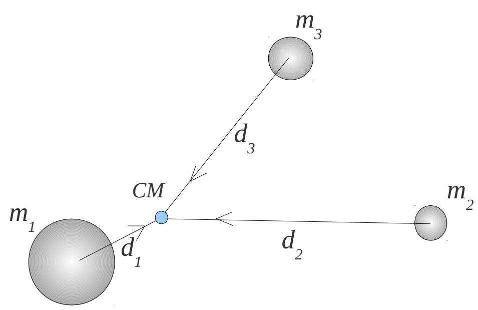

Fig. 1 Three masses falling into the center of mass (*CM*).

Another important aspect is related to *time*: any solution must imply that all the masses will spend the same amount of time[3] on reaching the CM, as they naturally will meet there in a single *event*. Since we know this *event* is "marked" to happen in the future, each mass must spend precisely the *same time* travelling into it. *Time* may, therefore, be seen as an almost irrelevant parameter, something that simply tells us how *fast* or *slow* the process will occur, but nothing special beyond that. *Time* can be seen, in this perspective, as a simple parameter with which humans express geometric laws of nature in terms of parametric equations. Here we may call this the *universal time* (t_u).

[3] notice that we are considering *free fall*, therefor the initial velocity is null.

Thus, since t_u is a constant from the perspective of each mass, we may say that $t_1 = t_2 = t_3 = t_u$. Or, which is the same, that t_u will be a constant of the system. It can also be written, however as:

$$t_u = d_1/v_1 = d_2/v_2 = d_3/v_3 = constant \qquad (1)$$

It is this *constant* that simply tells us whether what is to happen will be fast or slow, but nothing about what is to happen. Notice that the previous expression will be valid for any instant of time, therefore we may write, for any of the masses:

$$d(t)/v(t) = constant \qquad (2)$$

That is, as we can see, motion will be ruled by the geometric parameters and tendencies expressed by $d(t)$ and $v(t)$. And velocity may be seen as the tendency of the changing of a geometry in relation to the *universal time*. The *universal time*, being a constant, will be almost irrelevant. As an example: the flower is contained in the seed of the plant. What does velocity of the development of the plant mean? What if systems always move as a whole?

3. Generalizing the conservation of momentum

Thinking still on the *impetus* of Avicenna, as well as on the *Law of Inertia* of Galileo Galilei, we feel impelled to look at the conservation of *momentum* ($m.v$) as being a particular case of a more general principle: the principle of *conservation of angular momentum* ($r \times m.v$). In fact, if we consider any observer to be represented as a single point (the origin of a coordinate system), then, in truth, there are only angular motions in relation to that point, except in the very particular cases of when the movement is made along a straight line containing such a point. Only in this case is the motion rectilinear relative to the observer, like in the case of the *free fall*, for example. As the next figure suggests, any other "rectilinear" motion can be seen as angular. From this we conclude that we should mostly use the principle of *conservation of angular momentum*.

By figure 1, which represents a general case of rectilinear motion in relation to the observer *obs*, it is easy to see that the distance from *obs* to the body of mass *m* (moving with constant velocity vector $\underline{v}$) is dependent on the position $\underline{r}$ of the object. Such a distance can be represented as something like a conic figure[4], a hyperbole in this case. So, with the same basic mathematical expression of the ellipse and the parabola, which are figures typically related to planetary motion. Thanks, René Descartes.

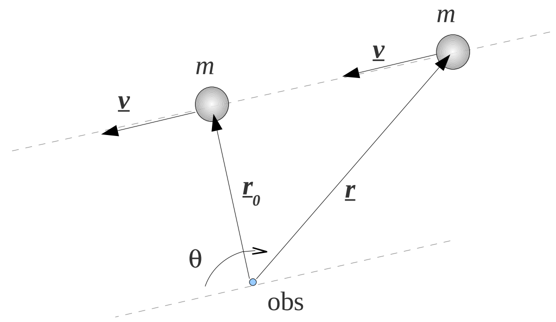

Fig. 2 Rectilinear motion as a form of an angular motion.

Notice that the observer really *sees* the body approaching him/her till the minimum distance of $\underline{r}_0$, and then going away again, as the angle θ goes form 180º to 0º. This angle is, of course, the angle between $\underline{r}$ and $\underline{v}$ at the instant of time. So, it is as if the body would be moving along a curve, as it does in the space-time diagram (Fig. 3), and not along a straight line.

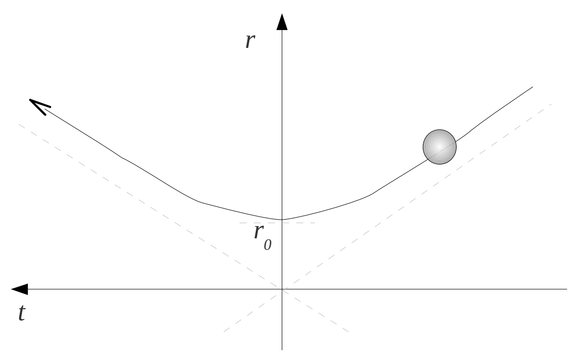

Fig. 3 Space-time diagram of a rectilinear "angular" motion. Notice that the parabola inclination depends on the *velocity*.

Curiously, the body is still maintaining constant its angular momentum $\underline{L}$, which is given by:

$$\underline{L} = \underline{r} \times m.\underline{v} \qquad (3)$$
$$\underline{L} = r \,.\, m.v \,.\, \sin\theta \,.\, \underline{u}_L$$
$$\underline{L} = m \,.\, r.v.\sin\theta \,.\, \underline{u}_L \qquad (4)$$

Where $\underline{u}_L$ represents a unitary vector (*versor)*

[4] Ellipse, parabola, hyperbole: $ax^2 + bxy + cy^2 + dx + ey = f$

perpendicular to the plan of motion. We thus may conclude that $r.v.sin\theta$ is a constant too.

Since we know $r.v.sin\theta$ is the *area swept per unit of time*, we may simply understand that *angular momentum* in fact means *mass* times the *rate of change of the swept area* (or *area speed*), which is a constant of the movement. *Angular momentum* is therefore a double quantity: a cinematic quantity times the "inertia" imposed by a mass.

Noticing that $v_\perp$, the component of $\underline{v}$ along the perpendicular to $\underline{r}$, is precisely given by $v.sin\theta$, as shown in figure 4, and we can then write the area swept per unit of time as:

$$dA/dt = r\ .\ v_\perp = constant \qquad (5)$$

We are, of course, deducing here that the two areas represented in figure 4 are in fact equal, and their value is constant for any instant of time.

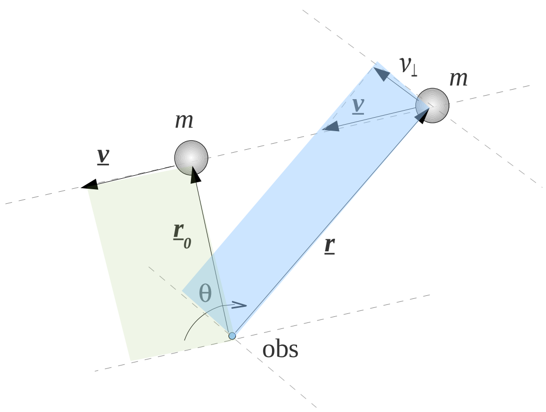

Fig. 4 Conservation of the *area swept per unit of time*.

When Johannes Kepler discovered "his" second law[5] he was, as we see from these thoughts, simply discovering the *angular momentum* properties, or, saying better, the properties of the *area speed*. The very interesting thing here is: such a law does not only apply to planets in orbit but also to bodies moving freely and in "rectilinear" motion, under no applied forces at all. So, this law of Kepler is neither a law of the planets nor related to the gravitational field. It is a more universal law of motion which is called *conservation of angular momentum*.

[5] A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

But there is another interesting property of angular momentum: since it defines a *pseudo-vector* which naturally tends to avoid change, it also tends to reduce the *degrees of freedom* of the motion, by "compacting" it into a planar surface. This is intimately related to the motion of the planets, of course. And the power of such a "compaction" is obviously contained in the mass, for a given *area speed*. The power of a thing is contained in its mass, the rest are probably cinematic properties of the geometry. It is much easier to change the orbit of our Earth, for example, than if Jupiter would fly in our orbit, which has around 300 times more mass than the Earth.

4. Free fall in a two-body system

Based on the previous ideas, we will now start to derive an expression for the *central force* (F_{CM})[6] in a two-body system. Thus, let us imagine two masses, M and m, separated by a distance $r_M + r_m$, and let us also assume the observer is positioned at the *center of mass* (*CM*), as depicted in the figure:

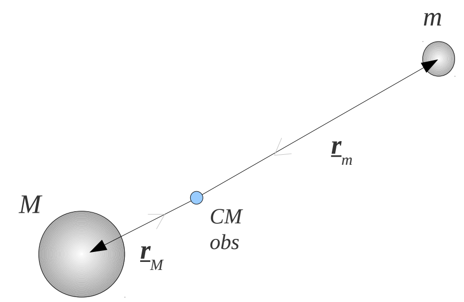

Fig. 5 Two bodies falling into the center of mass (*CM*).

Then, we may express the position vector of the center of mass as:

$$\underline{CM} = (-\underline{r}_M .\ M + \underline{r}_m .\ m)/(M+m) = \underline{0} \qquad (6)$$

Since the vector of the center of mass in relation to an observer located at the center of mass is a *null* vector, we can write:

$$(-\underline{r}_M .\ M)/(M+m) + (\underline{r}_m .\ m)/(M+m) = \underline{0}$$

or, after obvious manipulation, deduce that:

[6] We use this term instead of centripetal force in order to better link it to the idea of a *central acceleration*.

$$M \,.\, \underline{r}_M = m \,.\, \underline{r}_m \qquad (7)$$

This is an important result, since it tells us that, for any instant of time, the system must obey this “law” of “balance” between the two masses. The *CM* is the point where these masses are in equilibrium. So, both masses naturally will tend to either “fall” into or rotate around that point. We are at the moment considering *free fall*, however; but we may conclude something more from this simple equation. If we differentiate it with respect to time, we get:

$$M \,.\, d\underline{r}_M/dt = m \,.\, d\underline{r}_m/dt$$

$$M \,.\, \underline{v}_M = m \,.\, \underline{v}_m \qquad (8)$$

Which represents the “law” of *conservation of momentum* (*m.v*), curiously. Notice that, by this expression, the velocities of the two bodies *must* at any time be related by the *ratio of their masses*, no matter how fast the masses move; and this must hold not only for constant velocities but also in the presence of an acceleration. In this last case we may even get, by differentiating again:

$$M \,.\, \underline{a}_M = m \,.\, \underline{a}_m \qquad (9)$$

From where we conclude that, if there is a force acting, then each mass has to feel the same force.

Finally, dividing equation 7 by equation 8, we also find the result:

$$\underline{r}_M/\underline{v}_M = \underline{r}_m/\underline{v}_m \qquad (10)$$

Which is precisely the *constant* of the system we previously have named the “*universal time*” (t_u); the same as saying, *time is the same for every body*.

We must not forget, anyhow, that the velocities and the accelerations used herein are only those components parallel to the irrespective position vectors, so, only a part of the reality if compared with general motion and orbital systems. From now on we will distinguish them by the symbols $_{//}$ for parallel and $_{\perp}$ for perpendicular.

5. Angular momentum

Let us now go a bit further and notice that for any kind of motion the two bodies will be involved in, the *total angular momentum* of the system must be a constant too. So, in relation to our observer located at the center of mass we must write that the total angular momentum is, of course, the sum of the angular momenta of the two bodies:

$$\underline{L}_{\,total} = \underline{r}_M \times M \,.\, \underline{v}_M + \underline{r}_m \times m \,.\, \underline{v}_m \qquad (11)$$

$$\underline{L}_{\,total} = M \,.\, \underline{r}_M \times \underline{v}_M + m \,.\, \underline{r}_m \times \underline{v}_m$$

$$\underline{L}_{\,total} = (M \,.\, r_M \,.\, v_{M\perp} + m \,.\, r_m \,.\, v_{m\perp}) \,.\, \underline{u}_L \qquad (12)$$

Which is an expression already using the concept of *area speed*. Since we know that $\underline{u}_L$ will also be constant, it defines the plan of the motion, we may now represent this expression in the scalar form:

$$L_{\,total} = M \,.\, r_M \,.\, v_{M\perp} + m \,.\, r_m \,.\, v_{m\perp} = constant \qquad (13)$$

Figure 6 represents a general case of motion where both bodies have parallel and perpendicular velocities. Notice, though, that no external forces are considered to be applied to them, so, the parallel components of the velocities are due to their natural *attraction* into the *CM*.

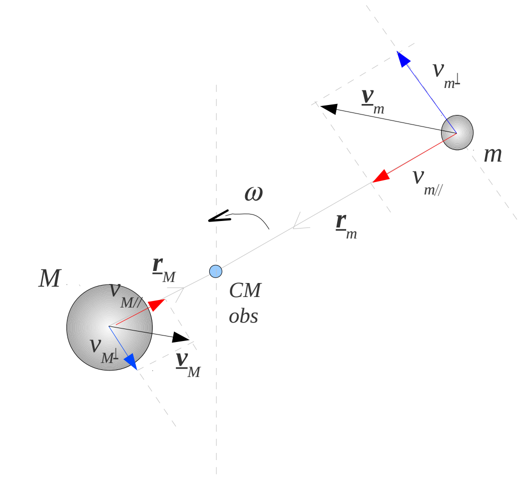

Fig. 6 Two bodies moving around the center of mass (*CM*).

It is not difficult to see that in a system like this, in every instant both masses have to rotate with the same angular speed *ω*, even if this speed will not be constant in the time; otherwise, the center of mass could get out from the straight line connecting the two bodies, which is mathematical nonsense. We may, therefore, by the definition of *angular speed*, write the equalities:

$$\omega_M = v_{M\perp} / r_M = \omega_m = v_{m\perp} / r_m = \omega(t) \qquad (14)$$

So, equation 13 can be written as:

$$L_{total} = M \,.\, \omega \,.\, r_M^{\,2} + m \,.\, \omega \,.\, r_m^{\,2} = const \qquad (15)$$

Now, using the expression 7 in order to let us eliminate r_M from this equation, so that we may treat the problem focusing the attention more on the motion of the smaller body, we get:

$$L_{total} = M \,.\, \omega \,.\, (m/M)^2 \,.\, r_m^{\,2} + m \,.\, \omega \,.\, r_m^{\,2} \qquad (16)$$

$$L_{total} = \omega \,.\, r_m^{\,2} \,.\, m \,.\, (M+m)/M \qquad (17)$$

Compressing the constant term which depends on the masses into:

$$K_M = m \,.\, (M+m)/M \qquad (18)$$

We get:

$$L_{total} = \omega \,.\, r_m^{\,2} \,.\, K_M \qquad (19)$$

Notice that, for the general case, everything is a constant in this equation except possibly ω and r_m, which may change with time. But, in the particular case of a *circular orbit*, these two quantities are also constant. Thus, when Galileo Galilei used to argue that the perfect orbits for the planets should be circles he was in fact right. Those are the most stable orbits, although other orbits can also occur as long as the product $\omega \,.\, r_m^{\,2}$ remains constant. Notice that this product is precisely, and once again, the *area speed*. For reasons of convenience, let us for now rewrite the previous equation as:

$$\omega = \{ L_{total} / K_M \} / r_m^{\,2} \qquad (20)$$

6. **The central force**

We already know that, if there is a *central force* acting on the bodies, the force on the mass *m* is precisely the same (although in the opposite direction) as the force acting on mass *M*. We may now use Huygens testimony about the existence of a force acting on bodies when these move in a circle with radius *r* and linear speed *v*, which pull them apart from the center of the rotation. As we know, Huygens gave to it the name "*centrifugal force*", and expressed it by:

$$F_c = m \,.\, v^2 / r \qquad (21)$$

We may now look into this in different ways. We may imagine that the *centrifugal force* appears naturally every time a body is made to change its direction of motion, as a sort of *reaction* to what *forced* such a change: a *central force*; or we may think of it as an *apparent force* due to inertia, in fact non-existent, that seems to appear under the action of a *central force*. That is, however, the force needed to act on the body in order to restrain its motion into a circle, usually called *centripetal force*. For long time a discussion about this issue remains in the academic circles, as we know. Perhaps in a future article we will address such a debate, but for now it is not our intent. Let us simply pick up the second position, and think that a body needs to be acted upon by a *centripetal force* given by equation 21 in order to exhibit an angular motion. So, if the body gets angular motion, it obviously also gets *angular speed* ($\omega = v/r$), and the expression of our *centripetal force* can now be written as:

$$F_c = m \,.\, r \,.\, \omega^2 \qquad (22)$$

Using equation 20 for the *angular speed*, in the case of two bodies that we are analysing, we can deduce that the *central force* originating from the *center of mass* and acting on the smaller body is given by:

$$F_{CM} = m \,.\, \{ L_{total} / K_M \}^2 / r_m^{\,3} \qquad (23)$$

This expression seems to tell us that the *central force* responsible for planetary motion, for example, is dependent on $1/r^3$, and not on $1/r^2$ as proposed by Newton. And, in a beautiful and simple way, it tells us also that the *gravitational constant*[7] of Newton probably results from two other constants of the system: the *total angular momentum* and a *relation between masses*. But this expression may even raise another question: could it be that the *universal gravitational constant* (G) is not really universal, but and approximation dependent on the system?

[7] $G = 6.67428 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$

7. Comparing with Newton's gravitation

We will now try to compare the last results, mainly equation 23, with what is commonly known as the *Theory of Gravitation* of Newton, which equation for the gravitational force is:

$$F_{newton} = m \,.\, M \,.\, G \,/\, r^2 \qquad (24)$$

Notice, however, that this expression uses the distance between the two bodies, which is given by $r = r_M + r_m$, while our expression uses the distance of the mass m to the *CM*. We must, therefore, rewrite it properly. Considering equation 7, we can write:

$$r = r_M + r_m = (m/M)r_m + r_m = r_m \,.\, (m/M + 1)$$

$$r = r_m \,.\, (M+m)/M$$

But, since from equation 18 we have:

$$K_M = m \,.\, (M+m)/M$$

we can write:

$$r = r_m \,.\, K_M / m \qquad (25)$$

Newton's force may now be arranged to become:

$$F_{newton} = m \,.\, M \,.\, G \,.\, \{ m / K_M \}^2 / r_m^{\,2} \qquad (26)$$

Which can finally be compared to the expression previously derived for the *central force* (equation 23), written below in a more appropriate form and called F_{feliz}.

$$F_{feliz} = m \,.\, \{ L_{total} / K_M \}^2 \,.\, (1/r_m) / r_m^{\,2} \qquad (27)$$

For the two forces to be equal, it must hold:

$$M \,.\, G \,.\, \{ m / K_M \}^2 = \{ L_{total} / K_M \}^2 \,.\, (1/r_m)$$

$$G = L_{total}^{\,2} / \{ M \,.\, m^2 \,.\, r_m \} \qquad (28)$$

This result is somehow surprising because we were expecting to find G as a constant. On the contrary, G appears here dependent on r_m. This way, it would only be a constant in those particular cases of circular orbits, or near, where r_m can be seen as a constant too. Anyway, testing this relation by means of *dimensional analysis* results in a truth, meaning that, somehow, Newton has included a distance into his universal constant. It is time, therefore, for us to inspect the way Newton had deduced his *Law of Gravitation*.

8. Newton's deduction of gravitation

The idea of Newton was, somehow, very simple, although probably genial for his times, when a structured knowledge of Physics and mathematics were at its beginning. Newton's ability for cleaning away the fog around the subjects and transforming them into coherent systems of thought, or models, was in fact extraordinary for that time. Even so, perhaps there were some small uncertainties related to his calculations that may explain the slight divergence between F_{newton} and F_{feliz}.

It seems[8] Newton has started by considering that the force acting on a body describing a circle or ellipse had to be of the same order asthe centrifugal force discovered by Huygens. Only that way the body would be able to maintain its closed loop trajectory. Thus, Newton considers a *gravitational central force* (F_g) coming from a single point, and starts with the equation:

$$F_g = F_c = m \,.\, v^2 / r \qquad (29)$$

Then, deciding to use the period of the orbit (T) as well as the orbital path ($2\pi\, r$), he computes the orbital speed as $v = 2\pi\, r/T$ and writes:

$$F_g = m \,.\, \{ 2\pi \,.\, r / T \}^2 / r \qquad (30)$$

$$F_g = m \,.\, 4\pi^2 \,.\, \{ r^2 / T^2 \} / r$$

As we can see, there are already some questions here. Why compute an *average speed* by means of an *average time* (T) and an *average radius* (r) and use them in what should be an *instantaneous* equation? This seems to result from a strategy of adapting his formula to what he already knew empirically from Kepler. Newton, in fact, obviously does it when he

[8] I didn't read his "*Philosophiæ Naturalis Principia Mathematic*", but only an analysis of his method, in Steffen Ducheyne, "*Mathematical Models in Newton's Principia: A New View of the "Newtonian Style"*", Centre for Logic and Philosophy of Science, Ghent University, Belgium.

decides to multiply this equation by *r/r*, that way achieving the form:

$$F_g = m \cdot 4\pi^2 \cdot \{ r^3 / T^2 \} / r^2 \qquad (31)$$

Since he knows from Kepler that $r^3 / T^2 = const$, he can finally deduce that the *central forces* responsible for planetary motion are of the $1/r^2$ type. From here to the more general equation known today it is only a small step, if one imposes the presence of the mass *M*, and, by dimensional manipulation, discovers the constant *G*. That is, if we say that:

$$4\pi^2 \cdot \{ r^3 / T^2 \} = M \cdot G \qquad (32)$$

We finally arrive to the well-known equation of Newton's gravitation:

$$F_g = m \cdot M \cdot G / r^2 \qquad (33)$$

9. **Gnewton versus Gfeliz**

Kepler's more exact expression for the third law must take into account the two masses involved, that is (M+m) in place of M, in equation 32. Then, it can be written in a simpler form if we use the concept of angular speed $\omega = 2\pi / T$. That is:

$$\omega^2 = G \cdot (M+m) / r^3 \qquad (34)$$

We must, however, remember that this equation was derived based on *average quantities*, therefore it should, in a more realistic way, be written as:

$$\omega_{avr}^2 = G \cdot (M+m) / r_{avr}^3 \qquad (35)$$

Since we have, from the ellipse geometry:

$$r_{avr} = (r_{min} + r_{max}) / 2 = a$$

where *a* represents the ellipse *major semi-axis*, we finally can write an equation representing what was really observed by Kepler in his experiments:

$$\omega_{avr}^2 = G \cdot (M+m) / a^3 \qquad (36)$$

It is now easier to have a better idea of what Newton's *universal constant G* means. By properly arranging this equation we get:

$$G_{newton} = \omega_{avr}^2 \cdot a^3 / (M+m) \qquad (37)$$

Which obviously shows that G_{newton} represents an average quantity. This must now be compared with the expression for *G* derived by our method, that is, with equation 28, which we reproduce here as:

$$G_{feliz} = L_{total}^2 / \{ M \cdot m^2 \cdot r_m \}$$

Knowing, that $L_{total} = \omega \cdot r_m^2 \cdot K_M$, as deduced in equation 19, this expression can be written:

$$G_{feliz} = \omega^2 \cdot r_m^4 \cdot K_M^2 / \{ M \cdot m^2 \cdot r_m \} \qquad (38)$$

We now need, first of all, to transform r_m into *r*, in order that the two approaches can be compared, which can easily be done using equation 25 again, that is $r = r_m \cdot K_M / m$, and transforming the previous relation, with a bit of manipulation, into:

$$G_{feliz} = \{ \omega^2 \cdot r^3 / M \} \cdot (m / K_M) \qquad (39)$$

Since we know that $K_M = m \cdot (M+m)/M$, we can deduce that:

$$G_{feliz} = \omega^2 \cdot r^3 / (M+m) \qquad (40)$$

From all this, of course, results:

$$\{ G_{feliz} / G_{newton} \} = \{\omega/\omega_{avr}\}^2 \cdot \{r/a\}^3$$

which shows that the two quantities are equal only in the case of circular orbits, not in general. And, if we look again into equation 36 arranged in the form:

$$\omega_{avr}^2 \cdot a^3 = G \cdot (M+m) = constant \qquad (41)$$

these thoughts also lead us to try to better understand what Kepler's observations mean, since we know that <u>*area speed*</u> (let us called it *A*), which is given by the product $\omega \cdot r^2$, is also a constant of motion for each orbit. On one side, we know from Kepler that dimensionally $[\omega^2 \cdot r^3]$ is a constant of motion; on the other, *area speed* squared is also a constant of motion:

$$\omega^2 . r^4 = A^2 = constant \quad (42)$$

If we call them C_K and C_0, for example, we have:

$$C_0 = r . C_K$$

which makes us think that the two constants may perhaps be a single one, which is dependent on r:

$$C_K = C_K(r) = C_0 / r \quad (43)$$

This means that the Kepler constant is in effect a universal constant, but can be seen as an *area speed squared* (energy per unit mass?) losing strength as it gets away from the *center of force*, precisely at the same pace of the energy density: $1/r$.

10. An orbital constant G_0 and $1/r^3$ forces?

Keeping in mind that the total *area speed* (A) is in fact a *constant of motion* for each orbit, we can again use the expression previously deduced for the force "pushing" masses into the *CM* (equation 27):

$$F_{feliz} = m . \{ L_{total} / K_M \}^2 . (1/r_m) / r_m^2$$

$$F_{feliz} = m . \{ L_{total} / K_M \}^2 / r_m^3 \quad (44)$$

Since $L_{total} = \omega . r_m^2 . K_M$, we may write it again in the form:

$$F_{feliz} = m . \{ \omega . r_m^2 . K_M / K_M \}^2 / r_m^3$$

$$F_{feliz} = m . \{ \omega . r_m^2 \}^2 / r_m^3 \quad (45)$$

Expressing r_m in terms of r, since we know it holds $r = r_m . K_M/m$, and considering $A = \omega . r^2$, we will find:

$$F_{feliz} = m . A^2 . (m/K_M) / r^3 \quad (46)$$

Once again using $K_M = m . (M+m)/M$, we get:

$$F_{feliz} = m . A^2 . \{ M/(M+m)\} / r^3 \quad (47)$$

And so, rearranging:

$$F_{feliz} = m . M . \{ A^2 /(M+m)\} / r^3 \quad (48)$$

If now we start to use as an <u>*orbital constant*</u> the quantity defined by:

$$G_0 = A^2 /(M+m) \quad (49)$$

We finally deduce the new equation for the gravitational field, which we expect to be valid under any circumstances, as:

$$F_{feliz} = m . M . G_0 / r^3 \quad (50)$$

Two aspects fast emerging from this expression appear of interest: first, the fact that G_0 will be a true constant[9] of the *orbital motion* in question, obviously related to the conservation of angular momentum; second, the $1/r^3$ dependency makes us think on what could perhaps be a curious similarity between gravitation and magnetism.

11. Some aspects and conclusions

With a bit of manipulation of equations 41 and 49 we can deduce that <u>$G_0 \simeq G . a$</u>. So, the previous equation can be written as:

$$F_{feliz} = m . M . G . a / r^3$$

$$F_{feliz} = \{a/r\} . m . M . G / r^2 \quad (51)$$

$$F_{feliz} = \{a/r\} . F_{newton} \quad (52)$$

from where we can deduce that F_{feliz} contains in itself the term $\{a/r\}$ which is usually treated as a perturbation to the Newton's equation. We expect, therefore, that F_{feliz} automatically corrects F_{newton}. Notice that Newton was linking this term to the following relation of masses:

$$\{a/r\}^3 = M / (M+m) \quad (53)$$

<u>**Author's Biography:**</u>

J. Manuel Feliz-Teixeira graduated in Physics in the Faculty of Sciences of the University of Porto, and holds an MSc and a PhD from the Faculty of Engineering of the same university. His PhD thesis is on "*Flexible Supply Chain Simulation*". Lately he is also dedicated to researching new approaches for renewable energy, as well as interested in Classical Mechanics specially related to *gravity* and *anti-gravity* phenomena.

[9] Notice that G_0 can be easily computed by squaring the product of *distance* times the *speed* at either the *Perihelium or* the *Aphelium*, and dividing it by the *total mass*.