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Apparently Deriving Fictitious Forces

Centrifugal, Coriolis and Euler forces. Their meaning and their mathematical derivation

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ABSTRACT

Since neither Galileo's Law of Inertia nor Newton's Second Law hold true in an accelerating frame of reference (which here we call the "accelerated world"), several challenges arise when trying to describe the movement of bodies in such a type of referential. In simple cases, mainly when the acceleration of the referential is a constant vector in the "inertial world", that is, when there is no acceleration on the acceleration, things become simple because such a vector can be seen by the accelerating observer as coming from a "fictitious" external force in the opposite direction to the force he feels. Why fictitious? Simply because he does not know where it is coming from or what causes it. But, in cases where this referential is subject to an acceleration that accelerates, when seen from the inertial space, things get much more complex to interpret, and all sorts of "fictitious" forces are usually evoked to explain the physics of the accelerating world. Perhaps the case with most academic debate since olden times is the spinning world, from which the concepts of centrifugal, Coriolis and Euler forces result. These are usually considered "fictitious" forces, in order that the laws of physics can be minimally understood from the point of view of the two worlds. This article is not only a discussion on these concepts but also an effort to explain them better, and reclassify them as real and not fictitious. We also argue that the centripetal force which spins the accelerated world is, in fact, a fictitious force.

1. Introduction: the accelerating train

Although our final objective is to study the spinning world, we find it somehow useful to start with the simple case of a linearly accelerating train, where we imagine an observer looking at a mass suspended from the ceiling by a string. The train is running in complete darkness outside, therefore the observer has no means of knowing what is happening in the exterior world (which for simplicity we consider as being an inertial world).

It is usually argued that the observer sees the string making an angle with the normal of the compartment during a period of acceleration. We believe, however, that the observer will not notice any difference between the normal of the compartment (which is given by the direction of his own body standing) but instead a kind of rotation of the whole compartment through an angle θ, as figure 1 suggests, making him feel as if he is standing under gravity on an inclined plane (with a very slight increase in gravity).

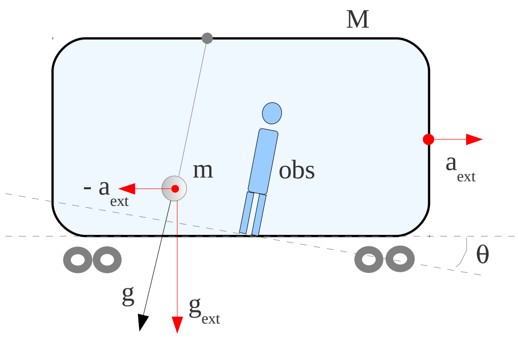

Fig. 1 A wagon of a train accelerating (a_{ext}) with a mass *m* suspended from the ceiling by a string, and an internal observer, as seen from an inertial world (outside world). *M* is the total mass, including the wagon, the observer and the mass *m*.

Thus, for this observer any external forces acting upon the objects surrounding him, and himself also, may all be classified as "fictitious", since they cannot even be distinguished from the situation of someone running up a mountain and with a small increase in the force of gravity. This means that in a certain sense, a rotation and a linear acceleration can be thought of as almost equivalent. Notice that when the net gravity (g) of the accelerated world will be null, the accelerated observer will obviously float, whilst accelerometers would measure zero gravity (0G). When the observer starts levitating, the internal g must be negative, the new gravity is now coming from the ceiling, it is inverted; however, that is not a good sign: it means the wagon is falling even faster than in a free fall situation. That would cause a local *anti-gravity* effect, but not a very interesting one. The exceptional one would be to produce *anti-gravity* in the inertial world.

Focusing on this case, we must notice, however, two interesting things: 1) the string with the mass is inclined to its left, while the observer is inclined to its right. Who is pointing in the direction of the "true" force? While some experiments with rotating objects use a candle (flame) to highlight the "centripetal" force acting on the object, others use a suspended mass to indicate the "centrifugal" force acting on the object. In fact, they are both inclined to the same side. The differences are apparent and only due to the fact that in the first case the fixed point is up, and in the second case it is down, relative to the "free" object[1]. 2) The force acting on the train ($M.a_{ext}$) is, in the perspective of the inertial world, the true force, because it is obvious that it makes the train gain space along its direction of application. But, would it still be considered a true force if the train would not gain space in that direction? We don't believe so. In that case we would call it a fictitious force.

So, in the perspective of the observer inside the train all these "fictitious" external forces that he does not feel are reduced to an inclination of the plane of the train and a small increase in gravity. Of course, if after the journey this observer meets with an inertial observer and tells him: "at some time, you must have seen that the train was inclined upwards...", the inertial observer would think that maybe he was a bit tired. But only if he was not aware of the subtleties of Physics, of course. The relevant question is: was there anything fictitious in the experience of the first observer? Of course not. What he felt different from the inertial observer was the accelerated world as it is, while the other had been standing in an inertial world as it is. But, could they ever manage to understand each other's narratives after some conversation? Yes, absolutely.

2. Going with the carousel, under gravity

We suppose now that our accelerated world is a carousel spinning horizontally, in an inertial world with g_{ext} gravity, and made from a series of masses m interconnected along its radius as suggested in the next figure (Fig. 2). There is an observer inside the rotating world and there are two observers in the inertial world. The rest of the inertial world is again in complete darkness, therefore only these elements can be seen.

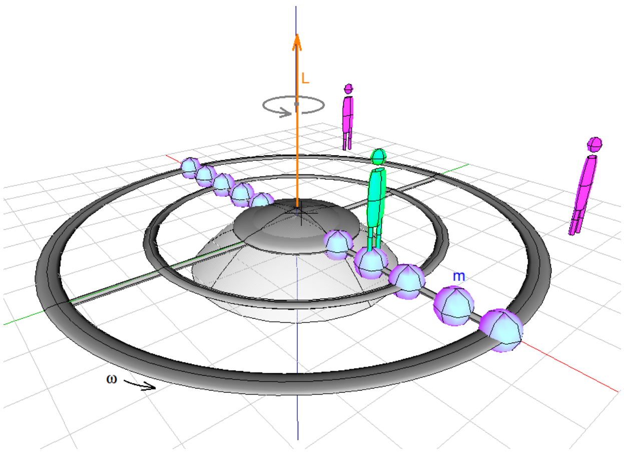

Fig. 2 The spinning world moving around with an angular velocity $\boldsymbol{\omega}$, under gravity g_{ext}, observed by two observers in an inertial world. Each sphere of the rotating world has mass m, while the rest of the structure has no mass. The moving observer is considered to be allowed to move only from sphere to sphere.

Let us now try to imagine how the observers see the two worlds. For each of the inertial observers the spinning observer is simply rotating around an axis if distant from the centre of the carousel, or rotating about himself in the case of being located at the centre. Both the inertial observers agree on what they see, although they cannot really understand which world is rotating. The same impression would be felt if the spinning world would be stopped and

[1] A simple video we have made to demonstrate this effect can be watched at: http://youtu.be/KWvJHdNaPV4

their world rotating with the angular speed $-\omega$. If the spinning observer is moving back and forth between spheres, the inertial observers will see him moving in a kind of an oscillatory motion inscribed on a circumference. But they still do not know which world is responsible for such an oscillation. The situation is ambiguous. And they feel nothing special in their bodies that may help resolve such an ambiguity.

The spinning observer, standing on a sphere, in principle feels precisely the same ambiguity, even if what he sees is quite different: the two "inertial" observers are rotating to the right with an angular speed $-\omega$, but if he fixes his eyes in their direction they will appear oscillating radially, or even running in circles with radius dependent on their speeds, precisely as if each of them would be on his own carousel. But, could the spinning observer feel something in his body that leads him to resolve such an ambiguity? Usually it is argued that the answer to this question is "no". We believe the answer is "yes", as long as none of the observers are located at the centre of rotation: firstly, as in the case of the train, the spinning observer will feel inclined as if the path to the centre of rotation would be upwards, as if he were on a cone:

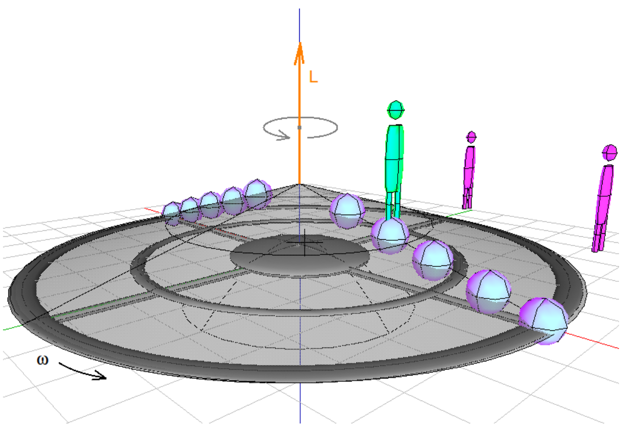

Fig. 3 The spinning world as a creator of "potential energy".

But, since he will also notice an incline in the bodies of the other two observers, the ambiguity is maintained. The new question he asks himself is: are they inclined or is it me that is inclined? And he finds a way to resolve this: if such an inclination makes them feel pushed into changing their distance in respect to the centre of rotation, but not me, then they are rotating and I am not. In the opposite case, I am rotating and they are stationary. If such a push is felt by all of us, then all of us are rotating[2]. But then he finds even another way to resolve it: if while moving to an adjacent sphere he feels a lateral force acting on his body, he at least knows he is rotating. More aware of the subtleties of physics, but still confused anyhow, he asks himself a final question, while remembering the accelerating train situation: is the centripetal force, acting on the sphere where I am standing, a real force? And he finds yet another answer: since such a "force" does not make the sphere gain space in its direction, it will obviously be considered an apparent force; which may even be replaced by a circular restriction to motion. The other force, that is, the one pushing him out of the carousel, as could even be confirmed by the inertial observers, is a real centrifugal force. So he concludes that in this case the centrifugal force is a real force, and is a reaction to a radially constrained motion. On returning to the inertial world, he may also conclude that spinning under gravity creates a kind of deformation in the net gravitational field with the opposite tendency of a gravitational attraction: a gravitational repulsion. Satellites probably move in between these two types of fields. But, in reality, is the rotational field of conical shape? In fact it is a parabolic surface, precisely as in the case of the gravitational field[3]. Of course under no-gravity only the horizontal components of these feelings would exist.

Since the two observers in the inertial world did not undergo any special gravitational effect, it seems reasonable their insistence that such a "centrifugal" force is nevertheless a fictitious force. The only force applied to each sphere, even if the sphere was not moving in direction of it, was a force pushing into the direction of the center of rotation. This force was what was able to move mass away from its tendency to move rectilinearly, as each sphere would do in the absence of such a force. The sphere, animated by the velocity $\underline{v}$, would tend to continue in that direction, should this centripetal force not exist.

The other observer, however, not so convinced by

[2] Here we consider only static observers in relation to their world. This would obviously turn much more complex if we consider them able to move around freely.

[3] There is an important difference between a spinning world and an orbiting world, of course: in the first case the angular velocity ω is constant along r, while in the second case this is not true, it depends on the centripetal acceleration.

these arguments, has decided to propose one last experiment, this time under no gravity, and the apparatus of the carousel was modified as shown in the next figure:

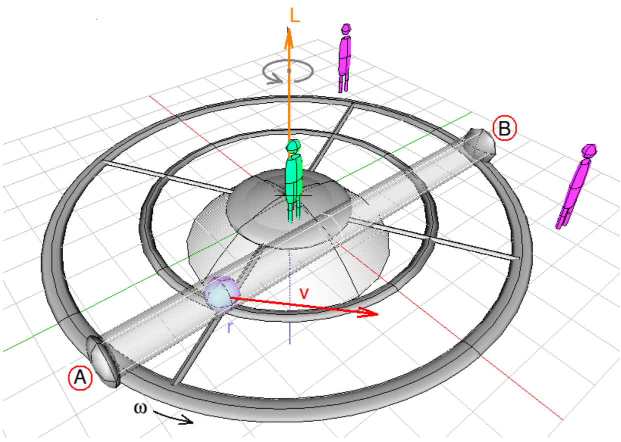

Fig. 4 A single sphere restricted to move (with no friction) inside a transparent tube fixed to the spinning world. Instead of positioned along the radial direction, this tube is inclined, in order to allow the observers to understand better the forces acting on the sphere, which is free to move either to the centre of rotation (to the B side of the tube) or away from it. The sphere is initially at a distance r from the centre, and the carousel starts spinning with very small increments of speed. At the beginning, the velocity $\underline{\boldsymbol{v}}$ of the sphere is perpendicular to its position vector $\underline{\boldsymbol{r}}$, but it has already a component along the tube pointing to B, and the centre. However, the sphere will never move in such a direction, as it would happen in the case of being acted upon by a *centripetal* force. In truth, as the speed of rotation increases, all observers will understand that the sphere will always move into the A side of the tube, away from the centre, as it is acted upon by a real *centrifugal* force.

3. **Feeling the centrifugal, Coriolis, Euler forces**

Before mathematically deriving these forces, let us try to understand physically what in fact happens in a spinning world. Thus, let us forget mathematics, as a way to avoid the tendency of adapting the reality to the model, instead of trying first to understand the reality[4]. For that, we consider again our observer rotating in the spinning world, standing on one of its spheres (Fig. 5), under external zero gravity (0G). The first question now arising is: should the mass of the observer be considered, or not? If we want to talk of a constant angular speed ω, in order not to make things too complex, the movement of the observer around the spinning space should not interfere with ω, as it does if he has mass, due to the need to conserve angular momentum. On the other hand, if we consider him as being a ghost, such a being would feel no force at all acting on it, since its mass is null. So, we decide to let the mass of the observer enter in our thoughts. We must therefore understand that we are dealing with two systems: the *observer* and the *carousel;* and we are trying to infer what is happening in the *carousel* by means of what happens to the *observer*. We must be careful enough not to forget this important fact. Besides, we must avoid studying the mechanics of such a spinning world by any sort of movement of the observer that is not at all times connected to the spinning structure. Any moment that the observer leaves the platform base he will move in a straight-line path. At a certain speed of rotation, it would be enough that the observer jumps vertically such that he would be automatically projected in a straight-line out of the carousel. Why in a straight-line? Because at that precise moment, the circular constraint disappears (and its "centripetal" force) and also the centrifugal force disappears, and Galileo (or should we say Aristotle?) rules again.

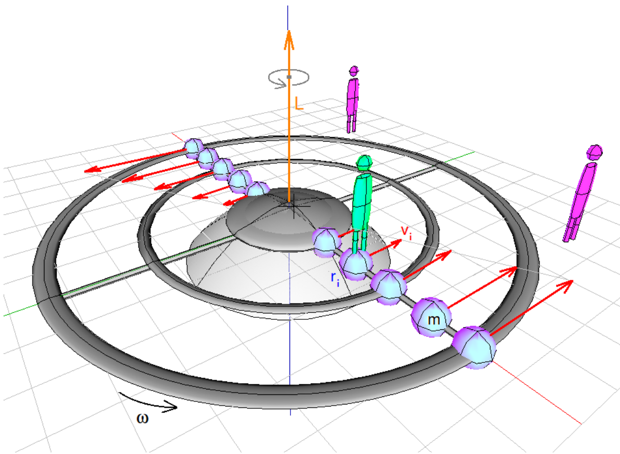

Fig. 5 The spinning world moving around with an angular velocity $\underline{\boldsymbol{\omega}}$, with an observer located at the sphere i at a distance r_i from the centre of rotation, with velocity $\underline{\boldsymbol{v}}_i$. The overall system has an angular momentum $\underline{\boldsymbol{L}}$.

Under these conditions, our observer will be subject to three types of forces: 1) a force due to the position where he stands and its current linear velocity. 2) a force due to moving between spheres along $\underline{\boldsymbol{r}}$. 3) a force due to any acceleration of the

[4] Equations should not be the motto, instead, the intuition, experience and sensibility should; otherwise, fictitious elements may be introduced into our world and an astonishing amount of resources, time and human intellect may be wasted in searching Nature for the existence of such fictitious elements. Science is, of course, a war of models and schools of thought, but it should always be open to questions.

angular speed $d\omega/dt$. Of course, in a gravitational system we would also add to these forces that of the centripetal gravitational force.

1) Centrifugal force:

By the simple fact that it is not positioned at the centre of rotation, the sphere where our observer stands, which moves with a velocity v_i at a distance r_i form the centre, will be acted upon by a force given by $m.v_i^2/r_i$ usually called the *centrifugal force*. This is the main force that was debated in the previous section.

2) Coriolis force:

As shown in figure 5, and by applying the basic relation $v = \omega .r$, we know that the outer spheres will have a superior velocity than the ones nearer the axis of rotation. This is naturally due to the fact that the platform is considered solid and all the points from it will have to rotate with the same angular velocity ω. When the observer at the sphere i moves to the sphere $i+1$, he will of course notice a change in its velocity from v_i to v_{i+1}, which acts as a lateral force while he moves. This a fact and not fiction. It is obvious that the observer will have to feel such a difference of real velocities, also dependent on how fast he moves radially. The reaction of his body to this force is in the opposite direction, thus contrary to the direction of rotation. Since this force results from a real increase in the velocity of the observer it is not an apparent force. It is usually called the Coriolis force.

3) Euler force:

This is the force naturally resulting from any change in the velocity of rotation of the platform, therefore dependent on $d\omega/dt$. In many academic dissertations it is considered null, since it is easier to study situations where ω is constant in time. In the present case, however, we may notice that when the observer moves from the i sphere to the $i+1$ sphere, not only his velocity increases but also the angular momentum of the whole system tends to increase due to the mass of the observer. But, since angular momentum must be conserved, this would tend to slow down the spinning, decreasing ω, and the Coriolis effect would not be as intense as expected. To maintain the system rotating at a constant ω, an external torque would be needed, supplied by an electrical motor, for example. This decrease in the spinning when mass moves outward from the centre and the corresponding increase in speed when it moves towards the centre can be thought of as an Euler force. In reality, however, all these effects are interconnected as a single entity.

In a previous article we presented what we called the *Geometric Law of Motion,* where all these components of force have been condensed in a single expression of geometric algebra, where $\underline{F}$ is the net force and $\underline{\mathcal{M}}$ is the "*modifier*" of the state of the system[5]:

$$\underline{\mathcal{M}} = \underline{r}\,\underline{F} = d\{\underline{r}\,m.\underline{v}\,\}/dt \qquad (1)$$

which can also be written as:

$$\underline{\mathcal{M}} = \underline{r}\,\underline{F} = d\{\underline{r}\,.\,m.\underline{v} \;+\; \underline{r} \wedge m.\underline{v}\}/dt \qquad (2)$$

or, more explicitly:

$$\underline{\mathcal{M}} = \begin{cases} \underline{r}\,.\,\underline{F} = m\,.\,[d\underline{r}/dt]\,.\,\underline{v} \;+ m\,.\,\underline{r}\,.\,\underline{a} & (3) \\ \underline{r} \times \underline{F} = m\,.[d\underline{r}/dt] \times \underline{v} \;+ m\,.\,\underline{r} \times \underline{a} & (4) \end{cases}$$

Notice that it is easy to identify in these equations what may be interpreted as the energies associated with all these forces. In fact, we have:

Centrifugal energy = $m\,.\,[d\underline{r}/dt]\,.\,\underline{v}$
Coriolis energy = $m\,.[d\underline{r}/dt] \times \underline{v}$
Euler energy = $m\,.\,\underline{r} \times \underline{a}$
Any other radial energy = $m\,.\,\underline{r}\,.\,\underline{a}$

4. The common derivation of the fictitious forces

There are several methods for deriving the forces associated with rotational motion, and most of them use the concept of angular velocity vector $\underline{\omega}$, which in itself is already a "fictitious concept", since $\underline{\omega}$ is a *pseudo-vector*, not a real vector. Thus, here we present a more general derivation, similar to that presented in Wikipedia[6], which we consider simpler and clearer than any other. Of course we will

[5] J. Manuel Feliz-Teixeira, "*In Defence of the Centrifugal Force and the Geometric Law of Motion*", first published at http://www.fe.up.pt/~feliz, and YouTube, June 2011

[6] Detail: http://en.wikipedia.org/wiki/Fictitious_force#General_derivation

suitably adapt it to the concepts previously referred. So, let us begin by defining a coordinate system in the rotating world where a vector is represented by its three components along an orthogonal base of *versors* $\underline{\hat{\mathbf{u}}}_1$, $\underline{\hat{\mathbf{u}}}_2$ and $\underline{\hat{\mathbf{u}}}_3$. Thus, the position of a generic particle (P) will be given by the position vector $\underline{\mathbf{r}} = (r_1, r_2, r_3) = r_1.\underline{\hat{\mathbf{u}}}_1 + r_2.\underline{\hat{\mathbf{u}}}_2 + r_3.\underline{\hat{\mathbf{u}}}_3$, in the rotating world. Now, we consider another coordinate system which is fixed to the inertial world, with base *versors* $\underline{\hat{\mathbf{o}}}_1$, $\underline{\hat{\mathbf{o}}}_2$ and $\underline{\hat{\mathbf{o}}}_3$, so that the position of the same generic particle will now be given by the vector $\underline{\mathbf{s}} = (s_1, s_2, s_3) = s_1.\underline{\hat{\mathbf{o}}}_1 + s_2.\underline{\hat{\mathbf{o}}}_2 + s_3.\underline{\hat{\mathbf{o}}}_3$. Finally, the *origin* of the rotating coordinate system, when seen from the inertial system, as shown in the next figure, is $\underline{\mathbf{s}}_0 = s_{01}.\underline{\hat{\mathbf{o}}}_1 + s_{02}.\underline{\hat{\mathbf{o}}}_2 + s_{03}.\underline{\hat{\mathbf{o}}}_3$.

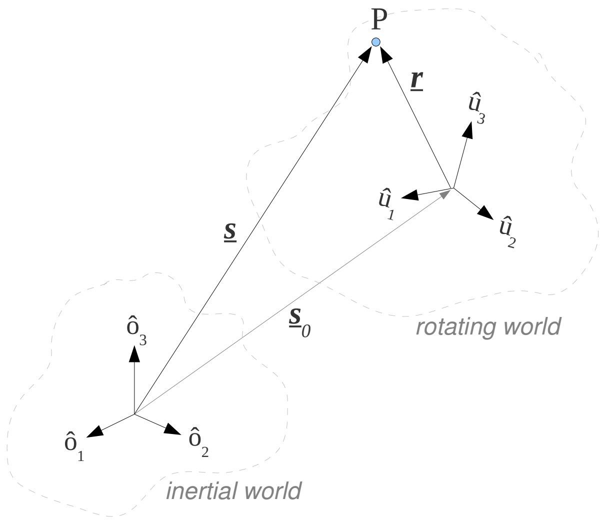

Fig. 6 Two coordinates systems and their interconnection.

Based upon this, we may now verify that the position of the particle in the inertial world may simply be found from the vectorial sum:

$$\underline{\mathbf{s}} = \underline{\mathbf{s}}_0 + \underline{\mathbf{r}} \qquad (5)$$

From now on, all we have to do is a cinematic manipulation. The velocity of the particle as seen from the inertial world will be:

$$\begin{aligned} d\underline{\mathbf{s}}/dt &= d\underline{\mathbf{s}}_0/dt + d\underline{\mathbf{r}}/dt \\ &= d\underline{\mathbf{s}}_0/dt + [d/dt]\{r_1.\underline{\hat{\mathbf{u}}}_1 + r_2.\underline{\hat{\mathbf{u}}}_2 + r_3.\underline{\hat{\mathbf{u}}}_3\} \\ &= d\underline{\mathbf{s}}_0/dt + [d/dt]\{r_i.\underline{\hat{\mathbf{u}}}_i\} \qquad (6) \end{aligned}$$

Using the Einstein convention for summation. Now we must evaluate the derivative of the product related to the components of $\underline{\mathbf{r}}$ in the rotating world, and write:

$$d\underline{\mathbf{s}}/dt = d\underline{\mathbf{s}}_0/dt + (dr_i/dt).\underline{\hat{\mathbf{u}}}_i + r_i.d\underline{\hat{\mathbf{u}}}_i/dt \qquad (7)$$

The first term of the second member ($d\underline{\mathbf{s}}_0/dt$) is simply the velocity of the centre of the coordinate system of the rotating world when seen from the inertial world. In the case of our carousel this is obvious null, since the carousel itself does not move along the inertial world, it only spins. The second term ($(dr_i/dt).\underline{\hat{\mathbf{u}}}_i$), is the velocity of the particle as seen from the spinning world. Finally, the last term ($r_i.d\underline{\hat{\mathbf{u}}}_i/dt$) represents a velocity dependent on how the spinning world rotates, which is also dependent on the distance of the particle from the centre of rotation. One may easily link this term to the $v = \omega.r$ relation, of course. Some say, however, that this is an *apparent velocity*, but honestly we do not understand why, or to whom it is apparent. On the one hand, all this computation is done from the perspective of the inertial world, so it is not apparent for the inertial observer. On the other hand, it is obvious from our previous examples that the inertial observer also feels it, when moving radially in the spinning world, for example. So, in our opinion, to classify it as "fictitious" is ignoring that the coordinate system inside the *spinning world* is not isotropic in what concerns velocity, at least in the plane of rotation. The spinning space introduces in itself a *source of speed*, which is a property dependent on the distance to the axis of spinning. This does not happen in the homogeneous[7] *inertial world;* no speed is gained by simply moving from one place to another. Therefore, the inertial observer should not look at himself as the proper *describer* of a world he cannot even feel, but only observe from afar. So, when he states that these forces that may be felt by the spinning observer are fictitious, he is merely fantasizing. Of course it is mainly a strategy to maintain the use of Newtonian mechanics in the study of accelerated systems; it works, but it would be good not to confuse a strategy with what the reality is. The two worlds are different spaces. In one all the properties are considered homogeneous, whilst in the other, they are not. The term $r_i.d\underline{\hat{\mathbf{u}}}_i/dt$ seen by the inertial observer is in fact embedded in the properties of the rotating frame, and it will be

[7] Here we use *homogeneous* and *isotropic* almost in the same sense. In fact, *isotropic* means the same property along all directions; while *homogeneous* means the same property in all regions of the space.

the source of most of those “fictitious” forces really acting on the spinning observer.

So, let us continue with the mathematics, in this apparent derivation of fictitious forces. To compute the acceleration we have to apply once again the time derivative to equation (7), resulting in:

$$d^2\underline{s}/dt^2 = d^2\underline{s}_0/dt^2 + [d/dt]\{(dr_i/dt).\underline{\hat{u}}_i\} + [d/dt]\{r_i.d\underline{\hat{u}}_i/dt\} \quad (8)$$

Notice that $d^2\underline{s}_0/dt^2$ is simply the acceleration of the centre of the spinning world in the inertial world, so we will call it $\underline{a}_0$. On the other hand, dr_i/dt is the velocity of the particle measured from the rotating world, so we will call it $\underline{v}_i$. And this equation can be written in a more compact form:

$$\underline{a} = \underline{a}_0 + [d/dt]\{v_i.\underline{\hat{u}}_i\} + [d/dt]\{r_i.d\underline{\hat{u}}_i/dt\} \quad (9)$$

By expanding again the derivatives, the second term on the right can be written as:

$$[d/dt]\{v_i.\underline{\hat{u}}_i\} = (dv_i/dt).\underline{\hat{u}}_i + v_i.d\underline{\hat{u}}_i/dt = a_i.\underline{\hat{u}}_i + v_i.d\underline{\hat{u}}_i/dt \quad (10)$$

While the third leads to:

$$[d/dt]\{r_i.d\underline{\hat{u}}_i/dt\} = v_i.d\underline{\hat{u}}_i/dt + r_i.d^2\underline{\hat{u}}_i/dt^2 \quad (11)$$

So, adding everything together, we get:

$$\underline{a} = \underline{a}_0 + a_i.\underline{\hat{u}}_i + 2.v_i.d\underline{\hat{u}}_i/dt + r_i.d^2\underline{\hat{u}}_i/dt^2 \quad (12)$$

In this expression, the terms inside the dotted area represent the accelerations existing in the rotating world, in the perspective of the inertial observer. The term $a_i.\underline{\hat{u}}_i$ is the “usual” linear acceleration, of the type of those acting the inertial world. The second term $2.v_i.d\underline{\hat{u}}_i/dt$ is dependent on the velocity of the rotating world and the *velocity* of the particle in it, and it is named *Coriolis* acceleration. The last term, obviously dependent on the *position* of the particle and the acceleration of the rotating frame, is a single term that includes both the *centrifugal* and *Euler's* accelerations, which in fact can be considered a single mechanism for exchanging energy between the angular and radial dimensions, in order to adjust the motion to the conservation of angular momentum. These two terms, however, will explicitly come out from this equation when we enter with the concept of angular velocity vector $\underline{\omega}$. In affect, by definition we may say that:

$$d\underline{\hat{u}}_i/dt = \underline{\omega} \times \underline{\hat{u}}_i \quad (13)$$

So, when we substitute this into the previous equation (12), we get:

$$\underline{a} = \underline{a}_0 + a_i.\underline{\hat{u}}_i + 2.v_i.\underline{\omega} \times \underline{\hat{u}}_i + r_i.[d/dt](\underline{\omega} \times \underline{\hat{u}}_i) \quad (14)$$

Notice that the last term can now be separated in two, since we can compute another derivative of a product, so, the centrifugal and Euler's accelerations will be separated from each other; taken in account that it holds:

$$[d/dt](\underline{\omega} \times \underline{\hat{u}}_i) = (d\underline{\omega}/dt) \times \underline{\hat{u}}_i + \underline{\omega} \times (d\underline{\hat{u}}_i/dt) = (d\underline{\omega}/dt) \times \underline{\hat{u}}_i + \underline{\omega} \times (\underline{\omega} \times \underline{\hat{u}}_i) \quad (15)$$

We will have:

$$\underline{a} = \underline{a}_0 + a_i.\underline{\hat{u}}_i + 2.v_i.\underline{\omega} \times \underline{\hat{u}}_i + r_i.(d\underline{\omega}/dt) \times \underline{\hat{u}}_i + r_i.\,\underline{\omega} \times (\underline{\omega} \times \underline{\hat{u}}_i) \quad (16)$$

And finally, after some tricks, we can write:

$$\underline{a} = \underline{a}_0 + a_i.\underline{\hat{u}}_i + 2.\underline{\omega} \times \underline{v} + (d\underline{\omega}/dt) \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r}) \quad (17)$$

And, resuming:

$\underline{a}_0$ = acceleration of the centre of the rotating world
$a_i.\underline{\hat{u}}_i$ = linear acceleration in the rotating world
$2.\underline{\omega} \times \underline{v}$ = Coriolis acceleration (intrinsic)
$(d\underline{\omega}/dt) \times \underline{r}$ = Euler acceleration (intrinsic)
$\underline{\omega} \times (\underline{\omega} \times \underline{r})$ = centrifugal acceleration (intrinsic)

What is usually argued is that only the term $a_i.\underline{\hat{u}}_i$ is perceived by the rotating observer, since the others are fictitious accelerations that do not exist in his spinning world. This is not true, in our perspective. He feels the effects of these accelerations as being *intrinsic* to the world he is living in. One of the most interesting examples of this is the going around in circles due to the concurrence of the *Coriolis* and the *centrifugal* forces when a ball (or disc) is pushed from the exterior into the interior of a spinning world[8]. Unlike what happens in the case of the train, a rotating observer knows he is rotating; thus these forces should never be called fictitious (also because they do *work* on the bodies). These forces should instead be called *intrinsic*, in order to better express the kind of deformation they introduce in the space metrics, similarly to what has been proposed by Einstein. So, we would say that the only fiction is that of an inertial observer expecting the rotating world to behave homogeneously, as the inertial world does. This, however, does not reduce the importance of the admirable mathematical work that has been done, in order that such a complex world can still be described by Newton's laws.

5. **Conclusion**

We think we have clearly demonstrated that the *centrifugal*, *Coriolis* and *Euler* forces are real forces, therefore they should not be called "fictitious", since this term induces obvious confusion even to the early student of Physics. Such a seed of confusion tends to spread with the time even to those with some common sense, and in that way contributes to the maintenance of a myth. We wonder how many novel systems and ideas have probably not been allowed to develop, emerge and materialise, through frustration, due to the simple fact that these forces were considered "fictitious", and people who thought on them were simply ridiculed. It is urged, in our opinion, that these kinds of myths are not maintained in science, in order that science will always be an open field for revisionism and evolution.

We believe it will be sufficient that people start to consider these forces as real, so that new studies and proposals in the scientific and technological domains will naturally emerge, and, by the nature of these forces, probably in the field of gravitation. Could it be that the centrifugal force may even be used to produce some kind of centrifugal propulsion or "anti-gravitational" effect? The truth is that probably we will never know the answer if the scientific community continues to intellectually obstruct the study of these effects because of such choices of nomenclature.

[8] A nice example in the video: http://youtu.be/G_lmuI95Kyw

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The airplane test with coffee and ice tea: http://youtu.be/tOZEgKXJMCE

Cars and motorbikes driving around a curved wall: http://youtu.be/hZOekFFSoWI

Rotating reference frame: "*Students rolling a bowling ball on a rotating platform*" : http://youtu.be/PLe2AmmoJjs

Disks and rotating table - round and round: "*The Toledo Imagination Station (used to be Toledo COSI) has a fascinating table with a large rotating circular section. Various disks are provided to get them 'running' around the table.*" : http://youtu.be/G_lmuI95Kyw