Hybrid systems
PDEEC 2009-10

HW #5

Introduction

The project consists of a set of exercises and design problems covering most of the material presented in class. The use of the tools presented in class is strongly recommended, especially in the design problems.

Reach sets

1. Consider the system *Linear model* with A=[1, 0; 0, 1] and B=[1, 0]. Build an approximation to the reach set at time t = 10 when the system departs from the origin at time t = 0.

2. Consider the following optimal control problem

   Min c.x(1) (c is a vector in the unit ball)

   $dx/dt = Ax + Bu$; $x(0)= x_0$; $u\in\Omega$

   Consider that you are given $R(x_0; 0, 1)$, the reach set of the system at time t = 1 when departing from x_0 at time t = 0.

   a) Use $R(x_0, 0, 1)$ to find x*(1), the final state of the solution to the optimal control problem

      Min c.z over all $z \in R(x_0; 0, 1)$

   b) Use formula $x(1) = e^{A}x_0 + \int_0^1 e^{A(1-t)}Bu(t)dt$ in order to show that the optimal control u*(t) maximizes the map $v \rightarrow c^T e^{A(1-t)} B v$ over Ω.

   c) Check that x*(t) minimizes p(t).z over all $z\in R(x_0; 0,t)$, onde $p(t)^T = c^T e^{A(1-t)}$

   Hint: Use the following facts:
   i) The minimization in a) is also satisfied when the set $R(x_0; 0, 1)$ is replaced by $e^{A(1-t)} [R(x_0; 0, t) - x^*(t)] + x^*(1)$.
   Observe that $e^{A(1-t)} [R(x_0; 0, t) - x^*(t)] + x^*(1) \subseteq R(x_0; 0, 1)$.
   ii) $x^*(1)= e^{A(1-t)} x^*(t) + \int_t^1 e^{A(1-s)}Bu^*(s)ds$.
   iii) $z \in A + v$ implies $z = z_1 + v$ for some $z_1 \in A$.

3. Consider the linear system: $dx/dt = Ax + Bu$; $x(0)=x_0$, $u\in\Omega$, $t\in[0,1]$, where $x\in\mathbf{R}^3$ and $u\in\mathbf{R}^1$. Assume that the system is controllable.

   a) Let $\Omega=[\alpha,\beta]$, with $\alpha<\beta$. Show that $R(x_0; 0, 1)$ is convex.

   b) Show that if you replace $\Omega=[\alpha,\beta]$ by $\Omega=\{\alpha,\beta\}$, the reach set $R(x_0;0,1)$ does not change.

      Observation:
      (1) The points on the boundary of $R(x_0;0,1)$ can only be reached by values of the control on the boundary of Ω.
      (2) Points in the interior of $R(x_0;0,1)$ can be reached by values of the control on the boundary of Ω (convexifying effect of the integral).

   Hint: Use the following facts (you may/should verify them):
   i) C is a convex set if $ac_1+(1-a)c_2\in C$, for all $a\in[0,1]$, $c_1\in C$ and $c_2\in C$.
   ii) Check the convexification role of the integral.
   iii) Start with one dimensional state and, then, extend to dimension 2, 3,... n.

4. Inner polyhedral approximation to the reachable set of linear systems.

Use the exercise in 9 to devise an algorithm to compute an inner polyhedral approximation of the reachable $R(x_0;0,1)$.for the linear system $dx/dt = Ax + Bu$; $x(0)=x_0$, $u\in\Omega$.

Justify the following observations:

i) The solution to the minimization in 8.a) is always on the boundary of the reachable set.

ii) Given a set of $N \geq n+1$ points $Z=\{z_1, z_2, z_3,..., z_N\}$, $z_i\in R^n$, $co[Z]\subset R(x_0,0,1)$.

Here co[Z] denotes the convex hull of the set Z. Consider the system in the plane, i.e., n=2, for which $A = [0,1;\ 2,1]$, $B=[0,1]^T$, and $U=[-1,1]$ in order to devise a clever way of generating interesting parameters c_k, k=1…P, $|c_k|=1$ that, by replacing c in the optimization problem in 8.a) will yield convenient vertices $x^*_k(1)$ of the approximating polyhedral.

Hints:

a) The solutions to each one of the optimal control problems can easily be computed by hand as you can see from the optimal control notes.

b) Consider the following scheme:

   a. Initial face: choose an arbitrary vector c_1, $|c_1|=1$, and compute $x^*_1(1)$. Then, let $c_2=-c_1$, and compute $x^*_2(1)$.

   b. Let c_3, $|c_3|=1$ be perpendicular to the vector $x^*_2(1)-x^*_1(1)$ and compute $x^*_3(1)$. Then, let $c_4=-c_3$, and compute $x^*_4(1)$.

   c. Proceed as previously but now minding that the interior of the approximating polyhedron is nonempty and that the recursively generated vectors c have to point outwards.

c) Construct an approximation with 8 points and discuss the "error" of the approximation, i.e., the "distance" (how should it be defined? Is Hausdorff metric ok?) between the sets

$$co\{ x^*_1(1), x^*_2(1),\ldots, x^*_P(1)\} \text{ and } R(x_0;0,1).$$

Observation: The Hausdorff distance between sets A and B is denoted by $d_H(A,B)$ and is defined by

$$d_H(A,B) =: \max\{ \sup\{d\_A(b): b\in B\}, \sup\{d\_B(a): a\in A\} \}$$

where d_C(d) is the usual concept of distance between the point d and the set C, that is

$$d\_C(d) =: \inf\{ ||c-d|| : c\in C\}.$$

Annex - Vehicle models

We consider several models of increasing complexity for the motions of our vehicles in R^2.

The state is $x=(x_1, x_2) \in R^2$

The general model is

dx/dt = f(x, u)

Simple vehicle model

$dx/dt \in B$ (unit closed ball in R^2)

Simple vehicle model under adversarial behaviour

$dx/dt = u + v$ where $u \in B$ (unit closed ball in R^2) and $v \in 0.5B$ (closed ball with radius 0.5 in R^2)

"Arrow model"

$dx_1/dt \in \{-1, 0, 1\}$

$dx_2/dt \in \{-1, 0, 1\}$

Unicycle model

This model has a third state Θ to model the orientation of the vehicle. The inputs are the speed $u_s \in [0, 1]$ and the angular velocity $r \in [-.1, 1]$.

$dx_1/dt = u_s \cos \Theta$

$dx_2/dt = u_s \sin \Theta$

$d\Theta/dt = r$

Planar motions of autonomous vehicles are often described by this simplified model.

Mode controlled unicycle model

The potential behavior of a vehicle modelled as a unicycle is the set of all smooth curves. This is too unstructured. We restrict its behavior to 4 discrete modes of motion:

stop ($u=0$, $\Theta=0$),

straight ($u=5$, $\Theta=0$),

left ($u=0$, $\Theta=1$),

right ($u=0$, $\Theta=-1$).

The vehicle is controlled by selecting a mode of motion.

Linear model

$dx/dt = Ax + Bu$, $u \in U=[-1,1]$