NONLINEAR CONTROL - AN OVERVIEW

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Introduction

Problem description - system components, objectives and control issues.

Example 1: telescope mirrors control

PICTURE

Most of real systems involve nonlinearities in one way or another.

However, many concepts for linear systems play an important role since some of the techniques to deal with nonlinear systems are:

- Change of variable in phase space so that the resulting system is linear.
- Nonlinearities may be cancelled by picking up appropriate feedback control laws.
- Pervasive analogy ...

Introduction

Example 2: Pendulum

Modeling

PICTURE

Dynamics (Newton's law for rotating objects):

$$m\ddot{\theta}(t) + mgsin(\theta(t)) = u(t)$$

State variables: $\theta, \dot{\theta}$
Control/Input variable: u (external applied torque)

Let us assume $m = 1, l = 1, g = 1$.

Introduction

Control Design

$$\text{Stationary positions:} \begin{cases} (0,0) & \text{stable equilibrium} \\ (\pi,0) & \text{unstable equilibrium.} \end{cases} \quad (1)$$

Consider the local control around the later.

Objective:
Apply u so that, for small $(\theta(0), \dot{\theta}(0))$, $(\theta(t), \dot{\theta}(t)) \to (\pi, 0)$ as $t \to \infty$.

Step 1 - Analysis - Linearization

For θ close to π: $sin(\theta) = -(\theta - \pi) + o(\theta - \pi)$.
Let $\varphi = \theta - \pi$, then:

$$\ddot{\varphi}(t) - \varphi(t) = u(t)$$

Introduction

Step 2 - Control Synthesis

Take $\varphi > 0$ and pick $u(t) = -\alpha\varphi(t)$ with $\alpha > 0$. Then

$$\ddot{\varphi}(t) + (\alpha - 1)\varphi(t) = 0.$$

$$\begin{cases} \alpha > 1 & \text{oscillatory behavior.} \\ \alpha = 1 & \text{only stable point: } \dot{\varphi}(0) = 0 \\ \alpha < 1 & \text{stable points: } \dot{\varphi}(0) = -\varphi(0)\sqrt{1-\alpha} \end{cases} \tag{2}$$

Conclusion: "proportional control" does not suffice.
Some anticipative control action is required.
This achieved by "proportional and derivative control", i.e.,

$$u(t) = -\alpha\varphi(t) - \beta\dot{\varphi}(t)$$

with $\alpha > 1$ and $\beta > 0$.
Then

$$\ddot{\varphi}(t) + \beta\dot{\varphi}(t) + (\alpha - 1)\varphi(t) = 0.$$

The roots of the characteristic polynomial are $\frac{-\beta \pm \sqrt{\beta^2 - 4(\alpha-1)}}{2}$.

Introduction

Linearization Principle

Linear designs apply to linearized nonlinear systems operating locally.

Extension: controller scheduling
Organize the phase space into regions and "patch" together local linear designs.

Introduction

Issues in control system design:

1 - Study the system of interest and decide on sensors and actuators

2 - Model the resulting system to be controlled

3 - Simplify the model so that it becomes tractable

4 - Analyze the system to determine its properties

5 - Define performance specifications

6 - Design the controller to meet the specifications

7 - Evaluate the design system via, say, simulation

8 - If not happy go to the beginning; otherwise

9 - Proceed to implementation

10 - Being the case, tune the controller on-line

Introduction

Strategy of the course:

- Easy introduction to the main concepts of nonlinear control by making as much use as possible of linear systems theory and with as little as possible Mathematics.
- Example-oriented introduction to concepts.
- Matlab hands-on exercises

Introduction

Structure of the course:

- Selected topics of linear systems
- Issues and specific background topics for nonlinear systems
- Main issues Lyapunov Stability
- Examples of feedback linearization design
- Basic issues in optimal control
- Introduction to adaptive control

Selected topics of linear systems

State-space representation of Linear Dynamic Systems

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \quad [t_0, t_f] - a.e., \qquad x(t_0) = x_0 \\ y(t) &= C(t)x(t) + D(t)u(t) \quad [t_0, t_f] - a.e. \end{cases}$$

where

- $x \in \Re^n$, $y \in \Re^q$, and $u \in \Re^m$
- $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{q\times n}$, and $D \in \Re^{q\times m}$

System's state trajectory given by a closed form solution

$$x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)d\tau$$

where $\phi(t, s)$ is the State Transition Matrix from s to t.

Selected topics of linear systems

Definition of State Transition Matrix Fundamental matrix $X(\cdot)$ of the pair $(A(\cdot), X_0)$ is the solution to $\dot{X}(t) = A(t)X(t),\ X(t_0) = X_0$.
The State Transition Matrix $\phi(t, t_0)$ is given by $X(t)$ when $X_0 = I$.
Thus, it satisfies the $n \times n$ matrix differential equation

$$\frac{\partial}{\partial t}\phi(t, t_0) = A(t)\phi(t, t_0).$$

Main Properties

1. $\phi(t, t_0)$ is uniquely defined
2. The solution to $\dot{x}(t) = A(t)x(t),\ x(t_0) = x_0$, is given by $\phi(t, t_0)x_0$
3. For all t, t_0, t_1, $\phi(t, t_0) = \phi(t, t_1)\phi(t_1, t_0)$
4. $\phi(t_1, t_0)$ is nonsingular and $[\phi(t, t_0)]^{-1} = \phi(t_0, t)$
5. $\phi(t, t_0) = X(t, t_1)[X(t_0)]^{-1}$
6. $\phi(t, t_0) = I + \int_{t_0}^{t} A(\sigma_1)d\sigma_1 + \int_{t_0}^{t} A(\sigma_1)\int_{t_0}^{\sigma_1} A(\sigma_2)d\sigma_2 d\sigma_1 + \int_{t_0}^{t} A(\sigma_1)\int_{t_0}^{\sigma_1} A(\sigma_2)\int_{t_0}^{\sigma_2} A(\sigma_3)d\sigma_3 d\sigma_2 d\sigma_1 + ...$

Selected topics of linear systems

"Variations-by-parts" formula

Exercise: Check "variations-by-parts" formula by noting that

$$\dot{x}(t) = \dot{\phi}(t,t_0)x_0 + \phi(t,t_0)B(t)u(t) + \int_{t_0}^{t} \dot{\phi}(t,s)B(s)u(s)ds$$

Heuristic derivation

Let $u(\sigma) = u(s)$ when $\sigma \in [s, s+ds]$ where $t_0 < s < t$. Then

- $x(s) = \phi(s,t_0)x_0$
- $x(s+ds) \simeq x(s) + [A(s)x(s) + B(s)u(s)]\, ds$
  $= [\phi(s,t_0) + A(s)\phi(s,t_0)ds]\, x_0 + B(s)u(s)ds$
  $\simeq \phi(s+ds,t_0)x_0 + B(s)u(s)ds$
- $x(t) = \phi(t,s+ds)x(s+ds) \simeq \phi(t,t_0)x_0 + \phi(t,s+ds)B(s)u(s)ds$
- Summing the input contributions for all s:

$$x(t) = \phi(t,t_0)x_0 + \int_{t_0}^{t} \phi(t,s)B(s)u(s)ds$$

Selected topics of linear systems

Computation of the State Transition Matrix

$$\phi(t,t_0) = exp(\int_{t_0}^{t} A(s)ds) = \sum_{i=0}^{\infty} \frac{1}{i!}(\int_{t_0}^{t} A(s)ds)^i$$
$$= \sum_{i=0}^{N} \alpha_i(t,t_0)(\int_{t_0}^{t} A(s)ds)^i \quad (3)$$

for some $N \leq n-1$. The last equality holds due to the Cayley Hamilton Theorem:
$p(A) = 0$.
Here p is the characteristic polynomial of A.

Algorithm

- Compute eigenvalues and eigenvectors of $\int_{t_0}^{t} A(s)ds$
- Compute coefficients $\alpha_i(t,t_0)$ by solving a system of linear functional equations obtained by noting that A and its eigenvalues satisfy (3)
- Plug the $\alpha_i(t,t_0)$'s in (3) in order to obtain $\phi(t,t_0)$

Selected topics of linear systems

Observations:

- The above algorithm is more generic: it enables the computation of any sufficiently regular function of the matrix A.
- When all the eigenvalues are distinct, $N = n - 1$ and one equation is written for each eigenvalue. Otherwise, N is the degree of the Minimal polynomial.
- When an eigenvalue has multiplicity $d > 1$ and its greatest Jordan block of its eigenspace has dimension m, then q equations associated with this eigenvalue have to be added by taking the successive derivatives (from 0 to $q - 1$) w.r.t. the eigenvalue of both sides.
- Geometric interpretation plays a key role.

Def. The minimal polynomial ψ is the one of least degree s. t. $\psi(A) = 0$.
Let A have σ distinct eigenvalues, λ_i whose multiplicity is d_i and let m_i be the dimension of the associated greatest invariant subspace. Then:

$$\psi(s) = \Pi_{i=1}^{\sigma}(\lambda - \lambda_i)^{m_i} \text{ and } p(s) = \Pi_{i=1}^{\sigma}(\lambda - \lambda_i)^{d_i}.$$

Selected topics of linear systems

Jordan Representation

Let $\aleph_k = \aleph[(A - \lambda_k I)^{m_k}]$ and B_k a basis for $\aleph_k$. Then:

- $\dim(\aleph_k) = d_k$
- $\Re^n = \aleph_1 \bigoplus \aleph_2 \bigoplus ... \bigoplus \aleph_\sigma$
- A is represented by $diag(A_i)$ in $B = \bigcup_1^\sigma B_i$ where B_i is a basis for $\aleph_i$
- $det(A - \lambda I) = \Pi_{i=1}^\sigma det(A_i - \lambda I_i) = \Pi_{k=1}^\sigma \alpha_k (\lambda - \lambda_k)^{d_k}$

Function of a Matrix

$$f(A) = \sum_{k=1}^\sigma \sum_{l=0}^{m_k-1} f^{(l)}(\lambda_k) p_{kl}(A) \qquad \text{where}$$

- $p_{kl}(\lambda) = \frac{(\lambda-\lambda_k)^l \phi_k(\lambda)}{l!}$, $l = 0, ..., m_k - 1$
- $\phi_k(\lambda) = n_k(\lambda) \frac{\psi(\lambda)}{(\lambda-\lambda_k)^{m_k}}$, and
- $n_k(\lambda)$ are the coefficients of a partial fraction expansion of $\frac{1}{\psi(\lambda)}$, i.e., $\frac{1}{\psi(\lambda)} = \sum_{k=1}^\sigma \frac{n_k(\lambda)}{(\lambda-\lambda_k)^{m_k}}$

Selected topics of linear systems

Geometric Interpretation of Eigenvalues and Eigenvectors

Assume that the eigenvalues of A, λ_i, $i = 1, ..., n$ are distinct and denote by v_i the associated eigenvectors.
Let $x(0) = \sum_{i=1}^{n} \alpha_i v_i$. Then $x(t) = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} v_i$.

Exercise 1 Let A be a real valued square matrix. Show that, under the above assumptions, the set of eigenvectors can be considered a basis for the state space.

Exercise 2 Let M be a matrix whose columns are the eigenvectors of A. Show that if ξ is the representation of x in the basis formed by the eigenvalues of A, then

$$\dot{\xi}(t) = \Lambda \xi(t) + M^{-1} B u(t), \qquad y(t) = CM\xi(t) + Du(t)$$

where $\Lambda = M^{-1}AM$ is a diagonal matrix.

Exercise 3 Let λ_i be a complex eigenvalue of A. Show that its conjugate is also an eigenvalue of A and that if $x(0) = Re(\lambda_i)$, then

$$x(t) = e^{Re(\lambda_i)t}[cos(Im(\lambda_i)t)Re(v_i) + sin(Im(\lambda_i)t)Im(v_i)].$$

Selected topics of linear systems

Trajectory graphical analysis

Representation in $\Re^2$ suffices to cover all the situations.

a) Stable node $\lambda_1 < \lambda_2 < 0$

b) Unstable node $\lambda_1 > \lambda_2 > 0$

c) Saddle point $\lambda_1 > 0 > \lambda_2$

d) Unstable focus $Re(\lambda_1) > 0$

e) Stable focus $Re(\lambda_1) < 0$

f) Center $Re(\lambda_1) = 0$

Selected topics of linear systems

Realization theory: controllability

Definition - A system representation is completely controllable on a given time interval if every initial state is controllable in that time interval, i.e., there is a control that transfers the initial state to the origin at the final time.

<u>Characterization</u>
Complete Controllability on $[t_0, t_1] \Longleftrightarrow M(t_0, t_1) := \int_{t_0}^{t_1} \phi(t_1, s)B(s)B^T(s)\phi^T(t_1, s)ds > 0$.

<u>Exercise 1</u> Check that $u(t) = -B^T(t)\phi^T(t_1, t)M^{-1}(t_0, t_1)\phi(t_1, t_0)x_0$ drives x_0 at t_0 to 0 at t_1.

Observation: A contradiction argument shows that, for invariant systems, the controllability condition is equivalent to rank of Q is n, where $Q := [B|AB|...|A^{n-1}B]$.

<u>Exercise 2</u> Show that the Range of Q, $\Re(Q)$, is
a) the set of states that can be reached from the origin.
b) invariant under A.

Selected topics of linear systems

Realization theory: observability

$$y(t) = C(t)\phi(t,t_0)x_0 + C(t)\int_{t_0}^{t} \phi(t,s)B(s)u(s)ds + D(t)u(t)$$

Definition - A state is unobservable if its input response is identically zero. A system representation is completely observable on $[t_0, t_1]$ if no state is unobservable on that time interval.

<u>Characterization</u>
Complete Observability on $[t_0, t_1] \Longleftrightarrow N(t_0,t_1) := \int_{t_0}^{t_1} \phi^T(s,t_0)C^T(s)C(s)\phi(s,t_0)ds > 0.$

<u>Exercise 1</u> Check that $x_0 = N^{-1}(t_0,t_1)\int_{t_0}^{t_1} \phi^T(s,t_0)C^T(s)y(s)ds$.

An invariant system is completely observable if and only if rank of R is n, where $R := col[C|CA|...|CA^{n-1}]$.

<u>Exercise 2</u> Let $\aleph(R)$ denote the null space of R.
a) x_0 is unobservable if and only if $x_0 \in \aleph(R)$.
b) $\aleph(R)$ is invariant under A.

Selected topics of linear systems

Minimal Realization

Definition - A representation is a minimal representation if it is completely observable and completely controllable

Fact - Minimal representation $\Longleftrightarrow \text{rank}(RQ) = n$

Issues and specific background topics for nonlinear systems

Examples of nonlinear control systems

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(0) = x_0$$

Issues: Existence, Uniqueness

- $\dot{x}(t) = -sign(x(t)), \quad x(0) = 0$
- $\dot{x}(t) = \frac{1}{2x(t)}, \quad x(0) = 0$

Definitions:

Autonomy - No t-dependence; even through u

Equilibrium Point - $f(t, x_0) = 0, \ \forall t > t_0$
Relevance: $x(t) = x_0 \ \forall t > t_0$

Isolated Equilibrium point - There is a neighborhood of x_0 where no additional equilibrium points can be found.

Exercise - List the Pendulum equilibrium points

Issues and specific background topics for nonlinear systems

Second order Systems

$$\begin{aligned} \dot{x}_1(t) &= f_1(t, x_1, x_2) \\ \dot{x}_2(t) &= f_2(t, x_1, x_2) \end{aligned}$$

The direction of the vector field $f : \Re^2 \rightarrow \Re^2$, also referred to as velocity vector field is given by $\theta_f(x) = \arctan(\frac{f_2(t,x_1,x_2)}{f_1(t,x_1,x_2)})$.

Methods to "compute" state plane trajectories:

- Linearization - Consider the first order approximation to the dynamics and use results of planar linear systems. The correspondence ok for all but center equilibrium points. Why?
  Example: $\ddot{y}(t) - \mu[1 - y^2(t)]\dot{y}(t) + y(t) = 0$
- Graphical Euler - forward Euler numerical integration
  $x_2 + \triangle x_2 = x_2 + s(x_1, x_2) \triangle x_1$ where $s(x_1, x_2) := \frac{f_2(t,x_1,x_2)}{f_1(t,x_1,x_2)}$
- Isocline - Efficient when sketching trajectories from a set of initial points. For various values of α, find (x_1, x_2) s.t. $s(x_1, x_2) := \alpha$.
- Vector Field - As before but the resulting plot is the vector field itself.

Issues and specific background topics for nonlinear systems

Periodic Solutions

Consider the following pair of planar dynamic systems:

$$\begin{cases} \dot{x}_1(t) & = x_2(t) \\ \dot{x}_2(t) & = -x_1(t) \end{cases} \qquad \begin{cases} \dot{x}_1(t) & = x_2(t) + \alpha x_1(t)(\beta^2 - x_1^2(t) - x_2^2(t)) \\ \dot{x}_2(t) & = -x_1(t) + \alpha x_2(t)(\beta^2 - x_1^2(t) - x_2^2(t)) \end{cases}$$

Observation: Although both exhibit periodic solutions, they are of quite different nature. Contrary to the linear system, the nonlinear one has only one isolated periodic solution which is approached as $t \to \infty$ regardless of the initial condition.

Bendixson's Theorem - condition for the nonexistence of limit cycles
If D be a simply connected domain in $\Re^2$ in which

$$\nabla f(x) := \frac{\partial f_1}{\partial x_1}(x_1, x_2) + \frac{\partial f_2}{\partial x_1}(x_1, x_2)$$

neither is identically null nor changes sign, then D does not contains closed trajectories.

Exercise: Test this result with the linear and nonlinear systems given above for different domains.

Issues and specific background topics for nonlinear systems

Periodic Solutions

Poincare Bendixson Theorem - existence of periodic trajectories.
Let L be the set of limit points of a trajectory S such that L is contained in a closed, bounded region M which does not contain equilibrium points.
Then: either L or S is a periodic trajectory.

By *Limit set* it is meant the set of all those points in the state space which are visited infinitely often as time goes to ∞.

Exercise 1: Take $M := \{(x, y) : 1/2 \leq x^2 + y^2 \leq 3/2\}$ and the nonlinear system in the previous slide.

Exercise 2: Analyze the system $f(x) := col(-x_1 + x_2, -x_1 - x_2)$ in the closed unit disk.
Suggestion: consider a change to polar coordinates.

Issues and specific background topics for nonlinear systems

Index Theory

Let J denote a simple, closed, positively (counterclockwise) oriented Jordan curve in an open, simply connected subset D of $\Re^n$ not passing through any equilibrium point of f.

Definitions - $I_f(J) := \frac{1}{2\pi}\int_J d\theta_f(x)$ - The index of a curve J w.r.t. f is the net change in the direction of f as x traverses around J divided by 2π.
If p is the only equilibrium point lying in J, then the index of a p, $I_f(p)$, is $I_f(J)$.
Here, $\theta_f(x) := \arctan(f_2/f_1)(x)$.

Facts:

- If J has no equilibrium points in its interior, then $I_f(J) = 0$.
- The index of a center, focus and node is 1, and the index of a saddle is -1.
- If $p_i,\ i = 1, ..., N$, are equilibrium points "in J", then $I_f(J) = \sum_{i=1}^{N} I_f(p_i)$.
- If J is a trajectory of f, then $I_f(J) = 1$.
- If f and g are such that $|\theta_f - \theta_g| < \pi$ along a J that does not pass through any equilibrium point of either f or g, then $I_f(J) = I_g(J)$

Issues and specific background topics for nonlinear systems

Index Theory (cont.)

Theorem

Every closed trajectory of f has at least one equilibrium point in its interior.
If f has only isolated equilibrium points, then it has only a finite number of such points and the sum of their indices is 1.

Exercise

Consider the Volterra predator-prey equations:

$$\begin{cases} \dot{x}_1 = -x_1 + x_1 x_2 \\ \dot{x}_2 = x_2 - x_1 x_2 \end{cases} \tag{4}$$

a) Which variable represents the number of prey and that of predators?
b) Determine and classify the system equilibrium points.
c) Sketch the vector field.

Issues and specific background topics for nonlinear systems

A simple method to compute closed trajectories

Take $\begin{cases} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{cases}$ and $V : D \to \Re$ s.t. $D \subset \Re^2$ is open and

$$\dot{V}(x_1, x_2) = \frac{\partial V}{\partial x_1} f_1(x_1, x_2) + \frac{\partial V}{\partial x_2} f_2(x_1, x_2) \equiv 0 \text{ on } D.$$

Let $\bar{S} := \{(x_1(t), x_2(t)) :$ trajectory for $f,\ (x_1(0), x_2(0)) = (x_1^0, x_2^0),\ t > 0\}$ and $S := \{(x_1, x_2) : V(x_1, x_2) = V(x_1^0, x_2^0)\}$.

Certainly $\bar{S} \subset S$.
Under some assumptions: If S is a closed curve, then $\bar{S}$ is a closed trajectory of f.
Furthermore: $\{(x_1, x_2) : V(x_1, x_2) = constant\}$ may define a continuum of closed trajectories of f

Exercise
Compute a family of closed trajectories of the prey-predator system by taking $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$.

Lyapunov Stability

Concept of Stability

It pertains the property of how a dynamic system drives or not its state to a given initial equilibrium point from which it was removed by some perturbation.

This is concept is independent of the control activity.
An important application is to support the design of a feedback control so that the closed loop system has the desired stability properties at the equilibrium points of interest.

There are several ways of going back to the equilibrium point.

Lyapunov Stability

Definitions of Stability Assume with no loss of generality that $x_0 = 0$ is an equilibrium of $\dot{x}(t) = f(t, x(t))$ at $t = t_0$.

Types of stability

$$x_0 \text{ is } \left\{ \begin{array}{ccc} \underline{\text{Stable}} \text{ at } t_0 & & \underline{\text{Uniformly Stable}} \text{ over } [t_0, \infty) \\ \text{iff} & & \text{iff} \\ \forall \epsilon \; \exists \delta(t_0, \epsilon) \;\; \text{such that} & & \forall \epsilon \; \exists \delta(\epsilon) \;\; \text{such that} \\ \|x(t_0)\| \leq \delta(t_0, \epsilon) & \text{and} & \|x(t_1)\| \leq \delta(\epsilon) \text{ and } t_1 \geq t_0 \\ \Downarrow & & \Downarrow \\ \|x(t)\| \leq \epsilon \; \forall t \geq t_0 & & \|x(t)\| \leq \epsilon \; \forall t \geq t_1 \end{array} \right\}$$

$$x_0 \text{ is } \left\{ \begin{array}{ccc} \underline{\text{Asymptotically Stable}} & & \underline{\text{Uniformly Asymptotically Stable}} \\ \text{at } t_0 \text{ iff} & & \text{over } [t_0, \infty) \text{ iff} \\ \text{it is } \underline{\text{stable}} \text{ at } t_0 \text{ and} & & \text{it is } \underline{\text{uniformly stable}} \text{ over } [t_0, \infty) \text{ and} \\ \exists \gamma(t_0) > 0 \text{ such that} & \text{and} & \exists \gamma > 0 \text{ such that} \\ \|x(t_0)\| \leq \gamma(t_0) & & \|x(t_1)\| \leq \gamma \text{ and } t_1 \geq t_0 \\ \Downarrow & & \Downarrow \\ \|x(t)\| \rightarrow 0 \text{ as } t \rightarrow \infty & & \|x(t)\| \rightarrow 0 \text{ as } t \rightarrow \infty \end{array} \right\}$$

Lyapunov Stability

Definitions of Stability (cont.)

Exercise 1 - Check if $(0,0)$ is a stable equilibrium point of

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_1(t) + (1 - x_1^2(t))x_2(t) \end{cases}$$

Exercise 2 - Stability and Uniform Stability

a) Check that the solution to $\dot{x}(t) = (6t\sin(t) - 2t)x(t)$ is given by $\ln(x(t)/x(t_0)) = t_0^2 - t^2 + 6t_0\cos(t) - 6t\cos(t) - 6\sin(t_0) + 6\sin(t)$.

b) Apply the definition of stability, i.e., show that you can pick $\delta(\epsilon, t_0) = \frac{\epsilon}{c(t_0)}$ for a suitable $c(t_0)$. How would you choose $c(t_0)$? (get a formula!)

c) Can such a constant $c(t_0)$ be chosen for any t_0? Give a counterexample.

Observation: Stability and Uniform Stability coincide for Time invariant or periodic systems. Why?

Exercise 3 - Check that $(0,0)$ is an asymptotically stable equilibrium point for the system:

$$\begin{cases} \dot{x}_1(t) &= x_1(t)(x_1^2(t) + x_2^2(t) - 1) - x_2(t) \\ \dot{x}_2(t) &= x_1(t) + x_2(t)(x_1^2(t) + x_2^2(t) - 1) \end{cases}$$

Lyapunov Stability

Auxiliary Definitions

α is a Function of class $\mathcal{K}$ if $\alpha(\cdot)$ is nondecreasing, $\alpha(0) = 0$, and $\alpha(p) > 0$, $\forall p > 0$.

$V : \Re^n \times \Re^+ \to \Re$ is a Decrescent Function if $\exists \beta(\cdot)$ of class $\mathcal{K}$ s.t. $V(t, x) \leq \beta(\|x\|)$, $\forall t \geq 0$, $\forall x$ s.t. $\|x\| \leq r$ for some r.

$V : \Re^n \times \Re^+ \to \Re$ is a Locally Positive Definite Function if $V(t, 0) = 0$, $\forall t \geq 0$, $V(t, x) \geq \alpha(\|x\|)$, $\forall t \geq 0$, $\forall x$ s.t. $\|x\| \leq r$ for some r.

V is a Positive Definite Function if it is Locally Positive Definite Function with "$r = \infty$" and $\alpha(r) \uparrow \infty$ as $r \uparrow \infty$.

Observation: The time independent versions of the above definitions can be expressed without the need of the $\mathcal{K}$ class functions, i.e.,

$$\bar{V}(0) = 0, \ \bar{V}(x) > 0 \ \forall x \text{ s.t. } \|x\| \leq r, ...$$

.

Fact A time dependent function is a l.p.d. (p.d.) iff it "dominates" a time independent l.p.d.f. (p.d.f.).

Lyapunov Stability

Lyapunov's Direct Method

This is a criterion to test the stability of equilibria whose Basic Idea is:

- Take 0 to be the identified equilibrium point
- Let V be an "energy function" which is 0 at 0 and positive everywhere else.
- The system is perturbed to a new nonzero initial point
- The system may be stable if the system's dynamics are s.t. its energy level does not increase with time.
- The system may is asymptotically stable if the system's dynamics are s.t. its energy level decreases monotonically with time.

Lyapunov Stability

Lyapunov's Direct Method

Main Results The equilibrium point 0 at time t_0 is (uniformly) stable (over $[t_0, \infty)$) if $\exists$ a C^1 (decrescent) l.p.d.f. V s.t.

$$\dot{V}(t,x) \leq 0, \qquad \forall t \geq 0,\ \forall x \text{ s.t. } \|x\| \leq r \text{ for some } r.$$

Given ϵ, let $\bar{\epsilon} := min\{\epsilon, r, s\}$, where s is such that $V(t,x) \geq \alpha(\|x\|)$, $\forall t > 0$, $\forall \|x\| \leq s$. To check that $\delta > 0$ s.t. $\beta(t_0, \delta) := sup\{V(t_0, x) : \|x\| \leq \delta\} < \alpha(\bar{\epsilon})$ is as required in the definition of stability, note that since $\dot{V}(t,x) \leq 0$ whenever $\|x\| < \delta$, we have $\alpha(\|x(t)\|) \leq V(t, x(t)) \leq V(t_0, x(t_0)) \leq \alpha(\bar{\epsilon})$, and thus $\|x(t)\| \leq \epsilon$.

Example Take $V(x_1, x_2) = \frac{1}{2}x_2^2 + \int_0^{x_1} g(s)ds$ and apply Lyapunov theorem to the system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -f(x_2) - g(x_1) \end{cases}$$

where f (friction) and g (spring restoring force) are continuous and, $\forall s \in [-s_0, s_0]$, satisfy $sf(s) \geq 0$ and $sg(s) > 0$ for $s \neq 0$.

Lyapunov Stability

Lyapunov's Direct Method

Main Results (cont.)

The equilibrium point 0 at time t_0 is uniformly asymptotically stable over $[t_0, \infty)$ if $\exists$ a C^1 (decrescent) l.p.d.f. V s.t. $-\dot{V}$ is an l.p.d.f..
Global asymptotic stability requires the p.d.f.V to satisfy $\dot{V} \leq -\gamma(\|x\|), \forall t \geq t_0, x \in \Re^n$ for some $\mathcal{K}$ function γ.

Examples

Check the asymptotic stability of

a) $V(x_1, x_2) = x_1^2 + x_2^2$ and $\begin{cases} \dot{x}_1 = x_1(x_1^2 + x_2^2 - 1) - x_2 \\ \dot{x}_2 = x_1 + x_2(x_1^2 + x_2^2 - 1) \end{cases}$

b) $V(t, x_1, x_2) = x_1^2 + (1 + e^{-2t})x_2^2$ and $\begin{cases} \dot{x}_1 = -x_1 - e^{-2t}x_2 \\ \dot{x}_2 = x_1 - x_2 \end{cases}$

Lyapunov Stability

Lyapunov's Direct Method

$M \subset \Re^n$ is an invariant set for $\dot{x} = f(t,x)$ if $x(t_0) \in M$ for some $t_0 > 0$ implies $x(t) \in M \forall t \geq t_0$.

A set $S \subset \Re^n$ is the positive limit set for a trajectory $x(\cdot)$ if $\forall x \in S$, $x = lim_{t_n} x(t_n)$ for some sequence $\{t_n\}$ s.t. $t_n \to \infty$.

Facts

a) For periodic or autonomous systems, the positive set of any trajectory is an invariant set.
b) The positive limit set of a bounded trajectory is closed and bounded.
c) Let $x(\cdot)$ be bounded and S be the system's positive limit set
$\lim_{t\uparrow\infty} \sup_{y\in S} \|x(t) - y\| = 0$.

Another Fact

Consider $\dot{x} = f(x)$ and $V : \Re^n \to \Re$ to be C^1 s.t.: $S_c := \{x \in \Re^n : V(x) \leq 0\}$ is bounded; V is bounded from below on S_c; and $\dot{V}(x) \leq 0$ on S_c.
Then, $\forall x_0 \in S_c$, $\lim_{t\uparrow\infty} x(t; x_0, 0) \in M$ where M is the largest invariant subset in $\{x \in S_c : \dot{V}(x) = 0\}$.

Lyapunov Stability

Lyapunov's Direct Method

Exercise - understanding the last fact

a) Show that $x(t;t_0,0) \in S_c \quad \forall t > 0$.
b) Let L be the limit set of $x(t;t_0,0)$. Show $V(y) = \lim_{t\uparrow\infty} V(t;t_0,0)\ \forall y \in L$.
c) Why is L an invariant set? Why does the above fact hold?

Notes

- $x_0 \in \Omega_c$ and $\dot{V}(x) \leq 0 \forall x \in \Omega_c \Longrightarrow V(x(t;x_0,0))$ nonincreasing
- $V(x(t;x_0,0))$ nonincreasing and bounded from below $\Longrightarrow \exists c_0$ s.t. $V(x(t;x_0,0)) \rightarrow c_0$
- L is the limit set of $x(t;x_0,0) \Longrightarrow V(y) = c_0\ \forall y \in L$
- L is invariant $\Longrightarrow [\ \dot{V}(y) = 0\ \forall y \in L \Rightarrow L \subset S\ ]$
- $L \subset M \Longrightarrow$ the conclusion

Lyapunov Stability

Lyapunov's Direct Method

LaSalle's local Theorem

Let $\dot{x} = f(x)$ and $V : \Re^n \rightarrow \Re$ be C^1 l.p.d s.t. $\dot{V}(x) \leq 0 \ \forall \ \|x\| \leq r$. Assume $S := \{x \in \Re^n : V(x) \leq m, \ \dot{V}(x) = 0\}$, $m := \sup_{\|x\| \leq r}\{V(x)\}$ has only the trajectory $x \equiv 0$. Then, 0 is asymptotically stable.

LaSalle's global Theorem

Let $\dot{x} = f(x)$ and V be autonomous or of period T. V be C^1, p.d. and s.t. $\dot{V}(x) \leq 0$ $\forall x \in \Re^n$. Also $S := \{x \in \Re^n : \ \dot{V} = 0 \ \forall t \geq 0\}$ contains no trivial trajectories. Then, 0 is globally asymptotically stable.

Observation:

The advantage of LaSalle's theorems is that Asymptotic Stability is concluded only by requiring $\dot{V}(x) \leq 0$ and not $-\dot{V}(x) \leq \alpha(\|x\|)$.

The price to pay? The system has to be autonomous or periodic in time.

Exercise - Apply LaSalle's theorem to $\ddot{y} + f(\dot{y}) + g(y) = 0$ with $V(y, \dot{y}) = \frac{1}{2}\int_0^y g(s)ds$.
f, g are continuous, $f(0) = g(0) = 0$, $sf(s) > 0$, $sg(s) > 0$, $\forall s \neq 0$.

Lyapunov Stability

Conditions for equilibria instability

"0 is unstable at t_0" if $\exists$ a C^1 $V : \Re \times \Re^n \to \Re$,

(i) s.t.: V is decrescent, $\dot{V}$ is l.p.d., $V(t, 0) = 0$ and $\forall \epsilon > 0$, $\exists x \in \epsilon B$, $V(t_0, x) > 0$.

(ii) s.t.: V is decrescent, $V(t, 0) = 0$, $\forall \epsilon > 0$, $\exists x \in \epsilon B$, $V(t_0, x) > 0$ and $\dot{V}(t, x) = \lambda V(t, x) + W(t, x)$ with $\lambda > 0$ and $W(t, x) \forall t \geq t_0 \; \|x\| \leq r$.

(iii) closed Ω and open $\bar{\Omega} \subset \Omega$ s.t. $0 \in \text{int}\Omega$, $0 \in \partial\bar{\Omega}$, and, $\forall t \geq t_0$, V is bounded above in Ω, uniformly in t, $V(t, x) = 0$, on $\partial\bar{\Omega}$, $\forall x \in \bar{\Omega}$, $V(t, x) > 0$, and $\dot{V}(t, x) \geq \gamma(\|x\|)$ for some γ of class $\mathcal{K}$.

Issue: How to construct a Lyapunov candidate function?

a) Physical considerations: Specify the system's total energy function.

b) Trial method: Guess stability or instability and try functions of increasing order of complexity until the criterion is met.

Lyapunov Stability

Linear Systems

For $\dot{x}(t) = A(t)x(t)$, $t > 0$, 0 is an isolated equilibrium.

Let $\{\lambda_i : i = 1, ..., n\}$ be the eigenvalues of A and Φ denote the State Transition Matrix.

Necessary and Sufficient Conditions for Stability

Stable
(Also Stable $\forall t_1 \geq t_0$)
$\sup_{t \geq t_0} \|\Phi(t, t_0)\| := m(t_0) < \infty$

Asymptotically Stable
(Also Globally Asympt. Stable)
$\lim_{t \to \infty} \|\Phi(t, t_0)\| = 0$

Uniformly Stable
$\sup_{t_0 \geq 0} m(t_0) < \infty$

Uniformly Asympt. Stable over $[0, \infty)$
$\sup_{t_0 \geq 0} m(t_0) < \infty$ and
$\lim_{t \to \infty} \|\Phi(t, t_0)\| = 0$ unif. in t_0

$\exists m,\ \lambda > 0$ s.t., $\forall t \geq t_0\ \forall t_0$,
$\|\Phi(t, t_0)\| \leq m e^{-\lambda(t - t_0)}$

Lyapunov Stability

Linear Systems

Consider the autonomous system $\dot{x}(t) = Ax(t)$.

Classical result

a) Glob. Asympt. Stability iff $\mathrm{Re}\{\lambda_i\} < 0$ and

b) Stability iff $\mathrm{Re}\{\lambda_i\} = 0$ only if λ_i is a minimal polynomial simple zero.

Lyapunov equation : $\quad A'P + PA + Q = 0 \quad (LE)$

$$V(x) = \frac{1}{2}x'Px \implies \dot{V}(x) = x'(A'P + PA)x = -x'Qx$$

$$\begin{array}{llll} (P,Q) \text{ s.t.:} & P > 0,\ Q > 0 \ \Leftrightarrow & V \text{ and } \dot{V} \text{ p.d.} & \Leftrightarrow\ 0 \text{ is G. A. S.} \\ & Q > 0,\ \exists \lambda_i(P) \ \Leftrightarrow & -\dot{V} \text{ p.d. and } \exists \epsilon & \Leftrightarrow\ 0 \text{ is unstable} \\ & Re(\lambda_i(P)) < 0 & \|x\| < \epsilon\, V(x) > 0 & \end{array}$$

Lyapunov Stability

Linear Systems

Given A, there are two approaches:

either pick P and study the resulting Q,
or pick Q and study the resulting P.

While the first requires an apriori guess on the stability (of 0), the second is a more straightforward trial test.
However, there is a problem of nonuniqueness. Hence:

Theorem A

$\forall\ Q,\ \exists^1\ P$ sol. to (LE) iff $\lambda_i + \lambda_j^* \neq 0\ \ \forall\ i,\ j$.

Theorem B

$Re(\lambda_i) < 0\ \ \forall i$ iff $\exists Q > 0$ s.t. $\exists^1\ P$ sol. to (LE), $P > 0$ iff $\forall Q > 0, \exists^1\ P$ sol. to (LE), $P > 0$.

Theorem C

Suppose $\lambda_i + \lambda_j^* \neq 0$ and let (LE) be s.t. $\exists^1$ solution P for each Q. If $Q > 0$, then P has many negative eigenvalues as A has eigenvalues with positive real part.

Lyapunov Stability

Indirect Method

Key idea: derive (local) stability conclusions for nonlinear systems from results for linear systems.

Take $\dot{x}(t) = f(t, x(t))$ where f is C^1 in x, $f(t, 0) = 0 \ \forall t \geq 0$, and $\lim_{\|x\| \to 0} \frac{\|g(t,x)\|}{\|x\|} = 0$ where $g(t, x) = f(t, x) - A(t)x$ with $A(t) = \frac{\partial f}{\partial x}(t, x)|_{x=0}$.

Theorem Assume $A(\cdot)$ is bounded and $\lim_{\|x\| \to 0} \sup_{t \geq 0} \frac{\|g(t,x)\|}{\|x\|} = 0$. Then, if 0 is uniformly asymptotically stable (UAS) over $[0, \infty)$ for the linearized system (LS), $\dot{z}(t) = A(t)z(t)$, so is for the nonlinear system $\dot{x}(t) = f(t, x(t))$.

Exercises (Take the assumptions and definitions of the above theorem).

1- Let $P(t) := \int_t^\infty \Phi'(s, t)\Phi(s, t)ds$. Show that:

  1.1 $\forall t \geq 0 \ P(t) > 0$ and $\exists b > a > 0$ s.t. $ax'x \leq x'P(t)x \leq bx'x$.

  1.2 $\dot{P}(t) + A'(t)P(t) + P(t)A(t) + I = 0$.

2- Let $V(t, x) := x'P(t)x$. Show that $V(t, x)$ is a decrescent p.d.f. with $\dot{V}(t, x) = -x'x + 2x'P(t)g(t, x)$.

3- Take $r > 0$ s.t. $\|x\| \leq r \Rightarrow \|g(t, x)\| \leq \|x\|/(3b) \ \forall t \geq 0$. Show that $\dot{V}(t, x) \leq -x'x/3$.

Lyapunov Stability

Indirect Method

Theorem'

Take the data of the previous theorem and assume that $A(t) = \bar{A}\ \forall t \geq 0$. Then, if $\bar{A}$ has at least an eigenvalue with positive real part, 0 is an unstable equilibrium point for the nonlinear system.

Exercises Determine the stability of the origin for the following systems (including the domain of attraction).

a) $\ddot{y} = (1 - y^2)\dot{y} - y$ with $\mu > 0$

b) $\begin{cases} \dot{x}_1 &= x_1 + x_2 + x_1 x_2 \\ \dot{x}_2 &= -x_1 + x_2^2 \end{cases}$

Lyapunov Stability

The Feedback Stabilization Problem

Take the control system $\dot{x}(t) = f(x(t), u(t))$ and specify a feedback control law $u(t) = g(x(t))$ so that the reference equilibrium point of the closed loop system $\dot{x}(t) = f(x(t), g(x(t)))$ is asymptotically stable.

Assumptions

a) f is C^1 in $\Re^n \times \Re^m$ and $f(0,0) = 0$.

b) $\text{rank}[B|AB|...|A^{n-1}B] = n$, with $A = \frac{\partial f}{\partial x}(x,u)|_{x=0,u=0}$, $B = \frac{\partial f}{\partial u}(x,u)|_{x=0,u=0}$.

Observation The assumptions imply that the linearized system around $(0,0)$, $\dot{z}(t) = Az(t) + Bv(t)$, is controllable.

Fact There is a matrix K s.t. all the eigenvalues of $A - BK$ have negative real parts and, thus 0 is G.A.S. for the closed loop system $\dot{z}(t) = (A - BK)z(t)$.

Another Fact (how to compute such a K? Use LQ results) Given (A, B) as above, $K = Q^{-1}B'M$ where M is the solution to the Riccati equation, $-P - A'M - MA + MBQ^{-1}B'M = 0$ for given $P > 0$ and $Q > 0$.

Lyapunov Stability

Feedback Stabilization

Theorem (Nonlinear stabilization)
Take $\dot{x}(t) = f(x(t), u(t))$ where f s.t. b) and let A and B defined as above.
Take $K \in$ s.t all the eigenvalues of $A - BK$ have negative real parts. Then,
$u(t) = -Kx(t) \Longrightarrow 0$ is an asymptoptically stable equilibrium point of
$\dot{x}(t) = f(x(t), -Kx(t))$.

Approach
a) Linearize the nonlinear system
b) Compute K stabilizing the linear system
c) Feed the nonlinear system input with $-Kx$.

Exercise
Find a feedback stabilizing control for the system

$$\begin{cases} \dot{x}_1 & = 3x_1 + x_2^2 + g(x_2, u) \\ \dot{x}_2 & = sin(x_1) - x_2 + u \end{cases} \qquad \text{where } g(a,b) = \begin{cases} 2a+b & \text{if } 2a+b \leq 1 \\ 1 & \text{if } 2a+b > 1 \\ -1 & \text{if } 2a+b < -1 \end{cases}.$$

Input/Output Stability

The (dynamical) system regarded as an input-output transformation

Formalization requires:

Definition 1 $L_p[0,\infty)$, $(L_{pe}[0,\infty))$, $p = 1, ...\infty$ is the set of all measurable $f(\cdot)(f_T(\cdot)) : [0,\infty) \to \Re)$ s.t. $\int_0^\infty |f(t)|^p dt < \infty$ $(\int_0^\infty |f_T(t)|^p dt < \infty \quad \forall T > 0)$.

Note: f might be vector valued.

Definition 2 $A : L_{pe}^n \to L_{pe}^m$ is L_p-stable if
a) $f \in L_p^n$ implies that $Af \in L_p^m$; and
b) $\exists k, c$ s.t. $\|Af\|_p \leq k\|f\| + b \; \forall f \in L_p$.

Note: Bounded input/Bounded output Stability - $p = \infty$

Example 1 $(Af)(t) = \int_0^t e^{-\alpha(t-\tau)} f(\tau d\tau$
Find k and c.

Example 2 $(Af)(t) = f^2(t)$
Is it a L_p map? Can you find k and c fulfilling the definition of stability.

Input/Output Stability

$$(1)\begin{cases} e_1 = u_1 - y_2 \\ e_2 = y_1 + u_2 \\ y_1 = G_1 e_1 \\ y_2 = G_2 e_2 \end{cases} \quad \begin{matrix} \nearrow \\ \searrow \end{matrix} \quad \begin{matrix} (2)\begin{cases} y_1 = G_1(u_1 - y_2) \\ y_2 = G_2(u_2 + y_1) \end{cases} \\ \begin{cases} e_1 = u_1 + G_2 e_2 \\ e_2 = u_2 + G_1 e_1 \end{cases} \end{matrix}$$

Theorem
Take (1) with $(G_1 x)(t) := \int_0^t G(t,\tau) n_1(\tau, x(\tau)) d\tau$ and $(G_2 x)(t) := n_2(t, x(t))$.
Here, $G(\cdot,\cdot)$ and $n_i : \Re_+ \times \Re^n \longrightarrow \Re^n$, $i = 1, 2$, are continuous with $n_i(\cdot, 0) = 0$ and the K_i-Lipschitz continuity of $n_i(t, \cdot)$.
Then, for $i = 1, 2$, $G_i : L^n_{pe} \to L^m_{pe}$ and

$$\forall u_1, u_2 \in L^n_{pe}, \exists^1 (e_1, e_2, y_1, y_2) \in L^n_{pe} \text{ s.t. (1) holds.}$$

Definition 3 - (2) is L_p-stable if,
$\forall u_1, u_2 \in L^n_{pe}$, y_1, y_2 s.t. (2) holds are in L^n_p, and
$\exists k, b$ s.t., for $i = 1, 2$, $\|y_i\|_p \leq k(\|u_1\|_p + \|u_2\|_p) + b$, whenever u_1, u_2, y_1, y_2 are s.t. (2) holds.

Input/Output Stability and Lyapunov Stability

Theorem

Consider the system $$(1)\begin{cases} \dot{x}(t) = Ax(t) - f(t, x(t)) \\ x(0) = x_0 \end{cases},$$

where $Re\{\lambda_i(A)\} < 0$ and $f(\cdot, \cdot)$ is continuous, and define the corresponding nonlinear feedback system:

$$(2)\begin{cases} e(t) = u(t) - \int_0^t e^{A(t-\tau)y(\tau)d\tau} \\ y(t) = f(t, e(t)) \end{cases}$$

Then, if (2) is L_2-stable, then the equilibrium 0 of (1) is GAS (Globally Asymptotically Stable).

Observation: Input/Output techniques yield either GAS or nothing!
Difficult to estimate regions of attraction.

State Feedback Linearization

Addressed class of systems

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i(t), \qquad x(0) = x_0 \in \Re^n$$
$$y_i(t) = h_i(x(t)) \qquad i = 1, \ldots, p$$

General objectives:

Central issue: synthesis of state feedback control linearizing the dynamic system.

Nonlinear change of phase of coordinates so that only one dynamic component depends nonlinearly on the state variable and affinely on the control.

The geometric insight provided by this framework allows to address various problems: stabilization, tracking, regulation ....

Required background topics:
Differential calculus on vector fields,
Distributions,
Nonlinear change of coordinates,
Assorted dynamic systems results ....

State Feedback Linearization

Notation

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \qquad x(0) = x_0 \in \Re^n$$
$$y_i = h_i(x) \qquad i = 1, \ldots, p$$

Exercise: Linear systems as a special case.

f and $g_i : \Re^n \to \Re^n$ $i = 1, \ldots, p$ are smooth maps on an open set U (in particular $U = \Re^n$) are referred to as Vector fields.
Geometrically, they define tangent directions to trajectories at x.

Covector field, the dual of a vector field, maps each $x \in U$ with a point in the dual of the image space of the vector field.
Covectors can be regarded as linear functionals on spaces of vector fields.

Example
$d\lambda(x) = \frac{\partial \lambda}{\partial x}(x)$ is the differential/exact differential or gradient of $\lambda : U \to \Re$, $U \subset \Re^n$.

State Feedback Linearization

Notation

A) Derivative of λ along f - $L_f\lambda(x) := d\lambda(x)\cdot f(x) = \sum_{i=1}^{n}\frac{\partial\lambda}{\partial x_i}(x)f_i(x)$.

Exercise: Check that $L_gL_f\lambda(x) = \frac{\partial(L_f\lambda(x))}{\partial x}g(x)$ and $L_f^k\lambda(x) = L_f(L_f^{k-1}\lambda(x))$.

B) Lie product or bracket of f and g - $[f,g](x) := L_fg(x) - L_gf(x)$.

Note that $ad_f^kg(x) = [f, ad_f^{k-1}g](x)$, where $ad_f^0g(x) = g(x)$.

Exercise: Show that $[\cdot,\cdot]$ is bilinear, skew commutative, and
$[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$ (Jacobi identity).

C) Derivative of a covector w along f - $L_fw(x) := f'(\frac{\partial w'}{\partial x})' + w\frac{\partial f}{\partial x}$.

Exercise: Check the following Properties

a) $L_{\alpha f}\lambda(x) = (L_f\lambda(x))\alpha(x)$.
b) $[\alpha f, \beta g](x) = \alpha(x)\beta(x)[f,g](x) + (L_f\beta(x))\alpha(x)g(x) - (L_g\alpha(x))\beta(x)f(x)$.
c) $L_{[f,g]}\lambda(x) = L_fL_g\lambda(x) - L_gL_f\lambda(x)$.
d) $L_{\alpha f}\beta w(x) = \alpha(x)\beta(x)L_fw(x) + \beta(x)w(x)\cdot f(x)d\alpha(x) + (L_f\beta(x))\alpha(x)\beta(x)$
e) $L_fd\lambda(x) = dL_f\lambda(x)$.
f) $L_f(w\cdot g)(x) = (L_fw(x))\cdot g(x) + w(x)\cdot[f,g](x)$.

State Feedback Linearization

State Space Change of Coordinates

Why?
highlights properties (observ., reachab.) or problem solution (decoupl., stabilaz.).

Example for linear systems:

$$\bar{A} = TAT^{-1}$$

, $\bar{B} = TB$ and, $\bar{C} = CT^{-1}$
where matrix T represents a linear change of phase coordinates.

A map $z = \Phi(x)$ is a nonlinear change of phase coordinates if it is a (global) (local) diffeomorphism, i.e., invertible and, both Φ and Φ^{-1} are smooth.

Exercise:

a) $det(\frac{\partial \Phi}{\partial x}) \neq 0$ at x_0 implies that Φ is a local diffeomorphism.
b) Check that $\bar{f} = (\frac{\partial \Phi}{\partial x} f) \circ \Phi^{-1}(z)$, $\bar{g} = (\frac{\partial \Phi}{\partial x} g) \circ \Phi^{-1}(z)$ and $\bar{h} = h \circ \Phi^{-1}(z)$.

State Feedback Linearization

Distributions

The map $x \to \Delta(x) := span\{f_i(x) : i = 1, \ldots, d\}$, where $f_i : U \to \Re^n$ is a (smooth) vector field, is a (smooth) distribution.

Let $F = col(f_1, \cdots, f_d)$. Then, $\Delta(x) = Im\{F(x)\}$ and, $dim(\Delta) = rank(F)$.

x_0 is a regular point of Δ if $\exists$ open U, $x_0 \in U$ where Δ is nonsingular, i.e., F is full rank.

Observations:
Distributions inherit pointwisely vector spaces operations and properties.
For any smooth vector field $\tau \in \Delta$, $\exists c_i(x)$ smooth s.t. $\tau(x) = \sum_{i=1}^{d} c_i(x) f_i(x)$.

Δ is involutive if $\forall \tau, \sigma \in \Delta$, $[\tau, \sigma] \in \Delta$.

Criteria: $rank(F) = rank(col(F|[f_i, f_j]))$, $\forall i \neq j$, $\forall x (\in U)$.

Exercise: Show that the intersection of two involutive distributions is also involutive. What about the sum?

Let Δ be noninvolutive. The involutive closure of Δ is the smallest involutive distribution containing it.

State Feedback Linearization

Distributions

Codistribution, Ω, is the dual of distribution, Δ, in the sense that is spanned by the associated covectors.

Example: The annihilator of Δ, $\Delta^{\perp}(x) := \{w^* \in (\Re^n)^* : w^* \cdot v = 0 \; \forall v \in \Delta\}$.

Properties (Let Δ (Ω) is spanned by the rows of F (W)):
a) $dim(\Delta) + dim(\Delta^{\perp}) = n \;\; \forall x$.
b) $[\Delta_1 \bigcap \Delta_2]^{\perp} = \Delta_1^{\perp} + \Delta_2^{\perp}$.
c) $\Delta^{\perp} = \{w^* : w^* F(x) = 0\}$.
d) $W(x)v = 0 \; \forall v \in \Delta$ and $\Omega^{\perp}(x) = Ker(W(x))$.

Frobenius Theorem: A nonsingular distribution is completely integrable iff it is involutive.

Δ, $dim(\Delta) = d$, is completely integrable at $x \in U$ if $\exists \; \lambda_i : U \to \Re$, $i = 1, \ldots, n-d$, s.t. $span\{d\lambda_1, \ldots, d\lambda_{n-d}\} = \Delta^{\perp}$.

In other words, the p.d.e. $\frac{\partial \lambda}{\partial x} F(x) = 0$ has $n - d$ independent solutions.

State Feedback Linearization

Distributions

Application of Frobenius theorem: Easy way to solve the p.d.e. $\frac{\partial \lambda_i}{\partial x} F(x) = 0$, $i = 1, \ldots, n-d$.

Procedure:

(1) Complete Δ with additional $n-d$ independent vector fields.

(2) Solve o.d.e.s $\dot{x} = f_i(x)$ with $x(0) = x_0$, yielding $x^i(t) = \Phi_t^{f_i}(x_0)$, $t \in [0, z_i]$, in $U_\epsilon = \{z \in \Re^n : z_i < \epsilon\}$.

(3) Take $\Psi : U_\epsilon \to \Re^n$, $\Psi(z) := \prod_{i=1}^n \Phi_{z_i}^{f_i}(x_0) = \Phi_{z_1}^{f_1} \circ \cdots \circ \Phi_{z_n}^{f_n}(x_0)$.

(4) The last $n-d$ rows of Ψ^{-1} are a sol. to the p.d.e.

To verify this, note that $(\frac{\partial \Psi^{-1}}{\partial x}|_{x=\Phi(z)})(\frac{\partial \Psi}{\partial x}) = I$ and since the first d columns of $\frac{\partial \Psi}{\partial x}$ form a basis for Δ, the last $n-d$ rows of $\frac{\partial \Psi^{-1}}{\partial x}|_{x=\Phi(z)}$ annihilate Δ.

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Invariant Distributions - Δ is invariant under the vector field f if $[f, \Delta] \subset \Delta$.

Extension of invariant sub-spaces for linear systems: Take $f_A(x) = Ax$, $\Delta = span\{\tau_1(x), \ldots, \tau_d(x)\}$ and $\tau_i(x) = v_i$. Then $[f_A, \tau_i] = -Av_i$.

Advantages: For completely integrable distributions, it allows a simplified representation of the dynamic system.

Proposition Let Δ be a nonsingular, involutive smooth distribution of dimension d invariant under f. Then,
(i) $\forall x_0 \; \exists$ open U_0, $x_0 \in U_0$ and a transformation $z = \Phi(x)$ on U_0 for which f is represented by $\bar{f}(z) = \text{col}(\bar{f}^1(z^1, z^2), \bar{f}^2(z^2))$.
(ii) Furthermore, the codistribution $\Delta^\perp$ is also invariant under f. The converse is also true.

Exercise Consider $\Delta = span\{v_1, v_2\}$, $v_1 = \text{col}(1, 0, 0, x_2)$, $v_2 = \text{col}(0, 1, 0, x_1)$, and $f = \text{col}(x_2, x_3, (x_4 - x_1 x_2)x_3, sin(x_3) + x_2^2 + x_1 x_3)$.

a) Check that Δ is involutive and invariant under f.
b) Apply Frobenius theorem, i.e., find $\lambda_1, \ldots \lambda_k$ s.t. $span\{d\lambda_1, \ldots, d\lambda_k) = \Delta^\perp$.
c) Form the change of coordinates mapping $z = \Phi(x)$ and compute $\bar{f}(z)$.