Reachability and control for continuous-time systems

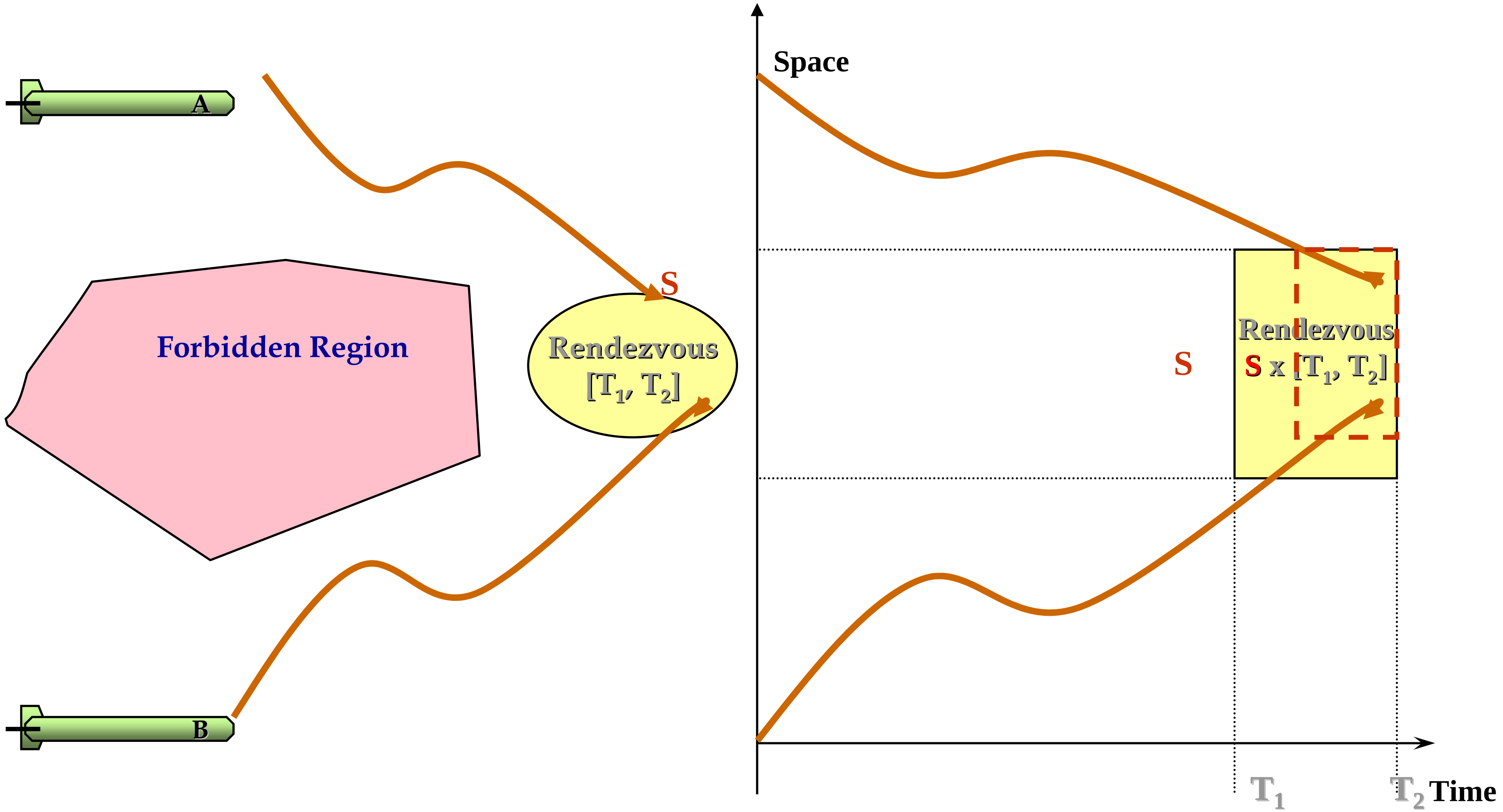
- João Borges de Sousa

Outline

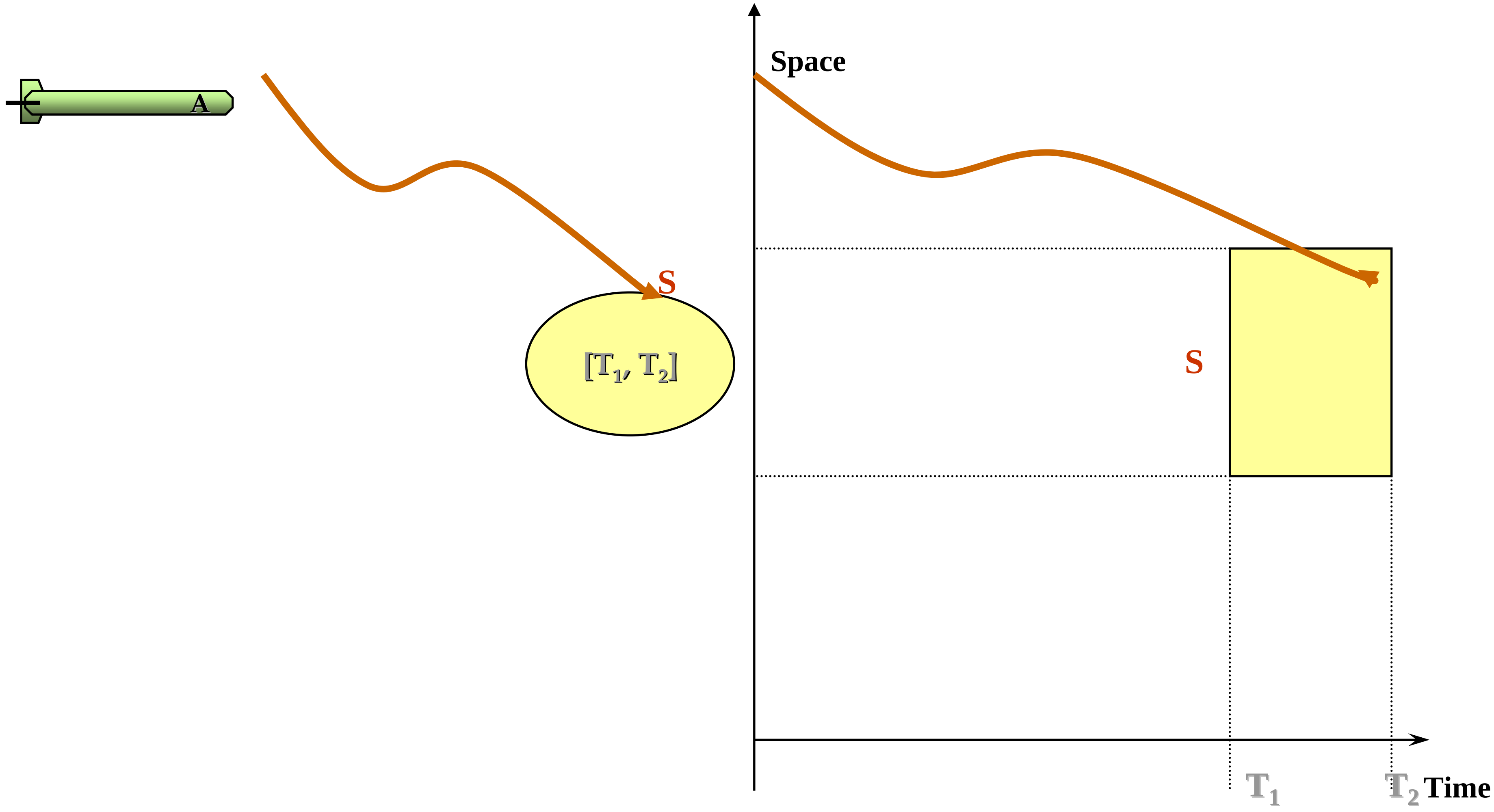
- Motivation
- Definitions
  - Forward reachability
  - Backward reachability
- Reach sets, invariance, and control
- Why are we interested in reach sets?
- How to compute reach sets?
- Dynamic optimization for reach set computation
  - Introduction
  - The maximum principle and reach sets for linear systems
  - Value functions and Hamilton-Jacobi-Bellman equation
  - A direct method

Motivation

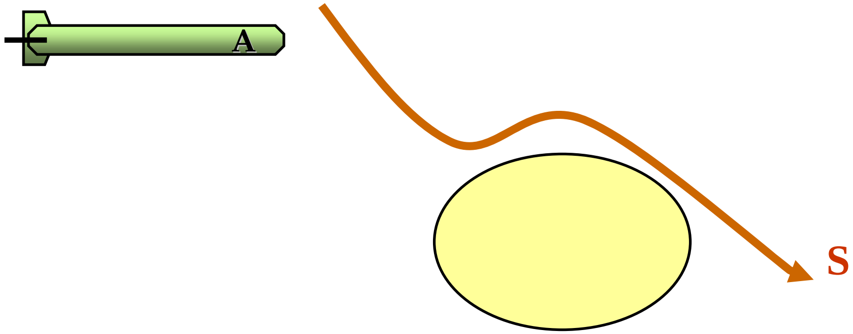
Practical problem: rendezvous

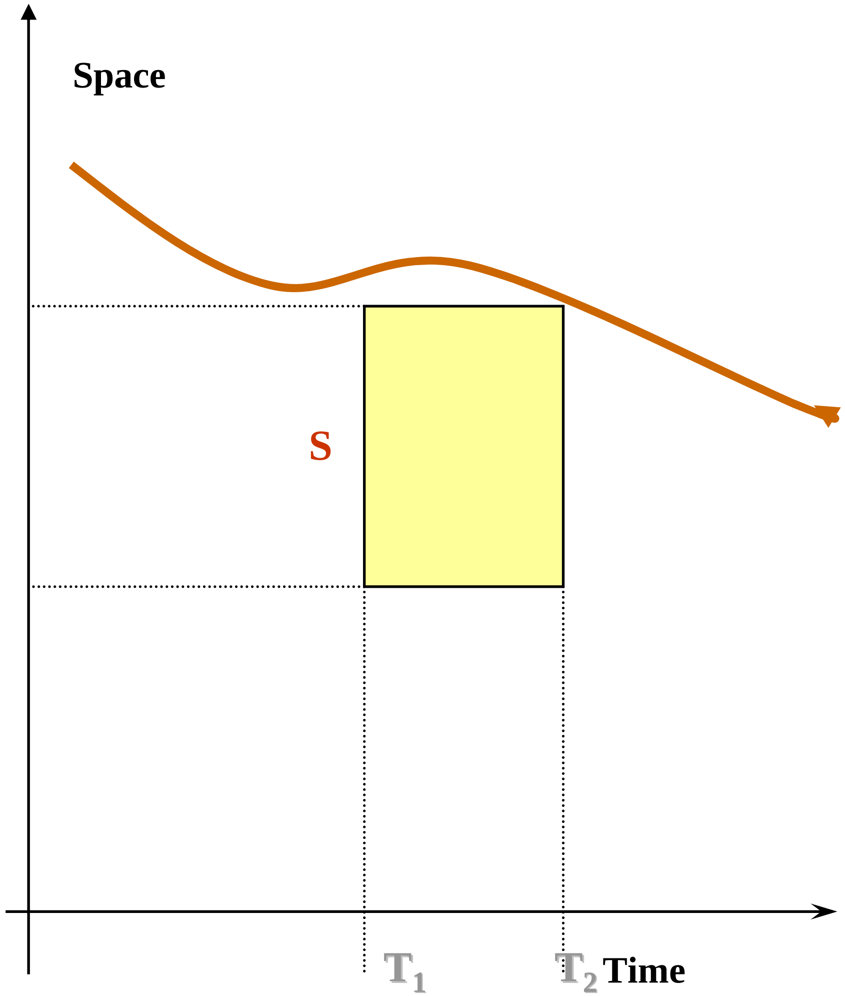


Practical problem: reaching a target

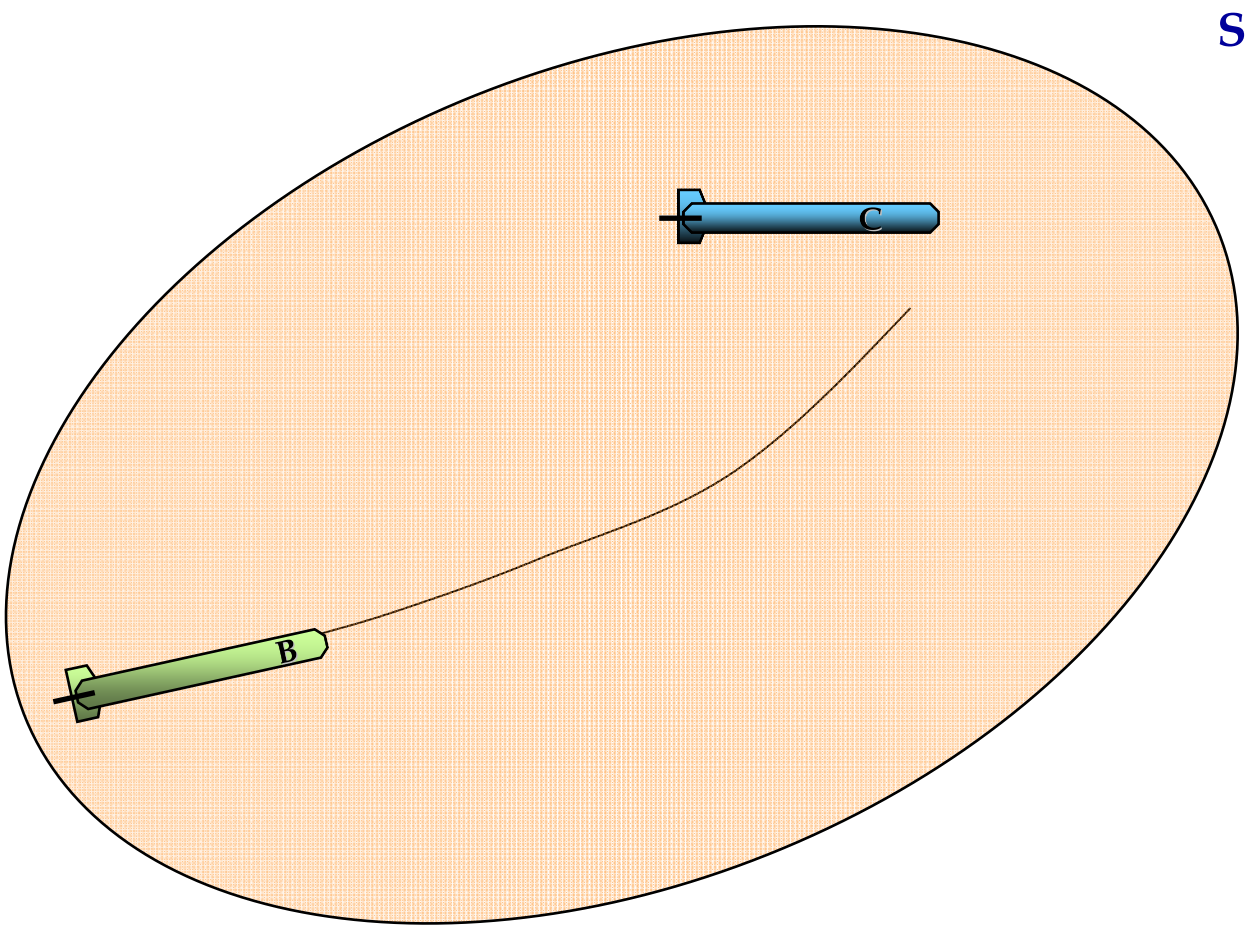


Practical problem: avoiding obstacle



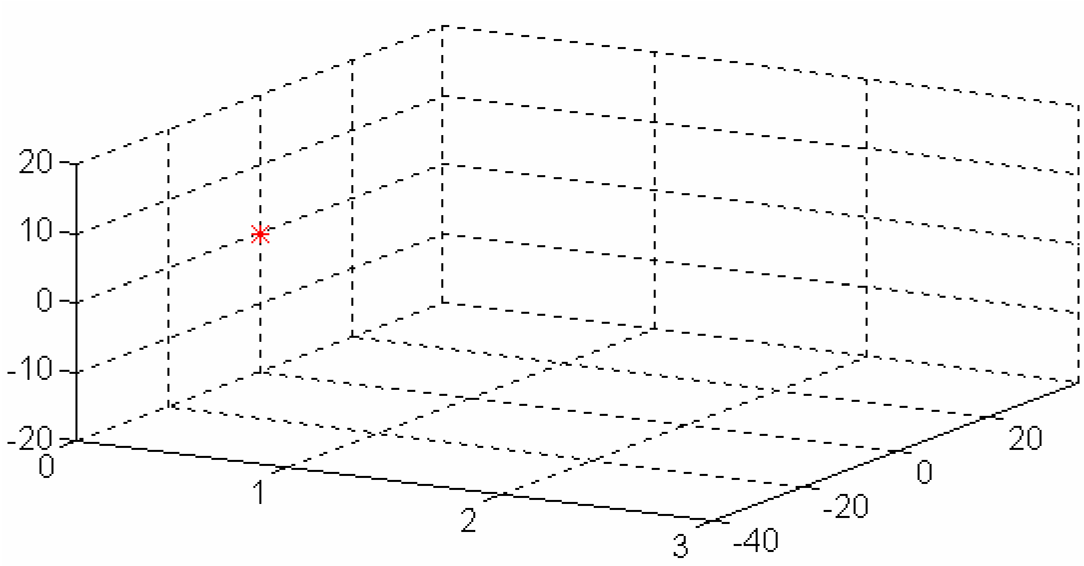


Practical problem: staying inside a set



Examples

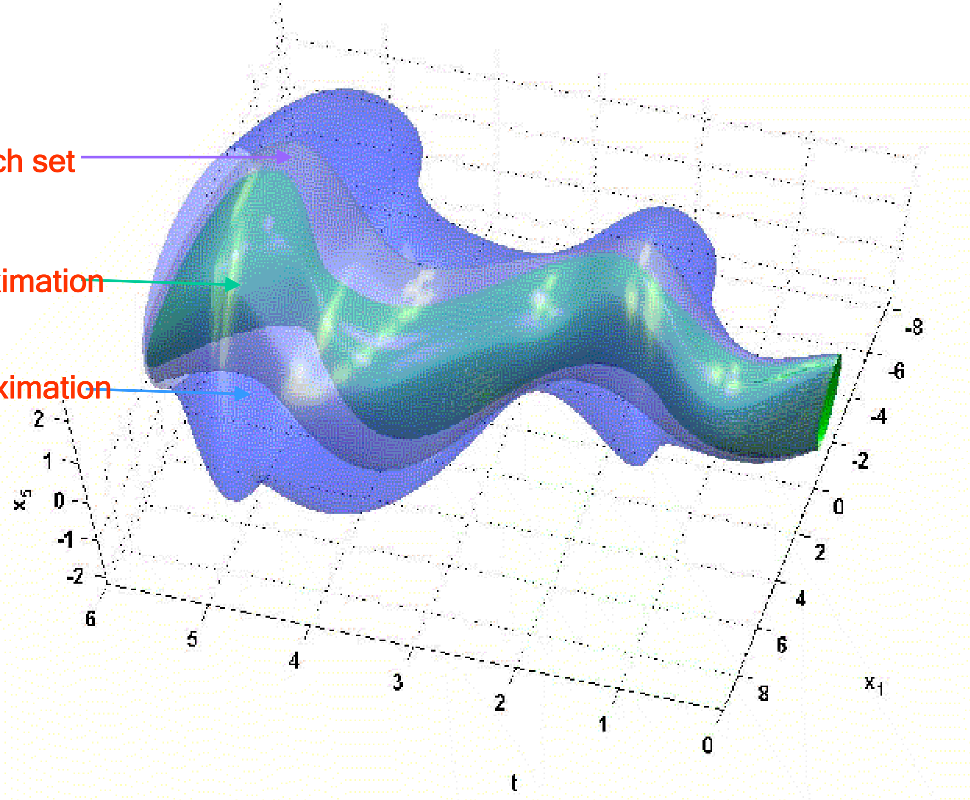
Reach set of a linear system

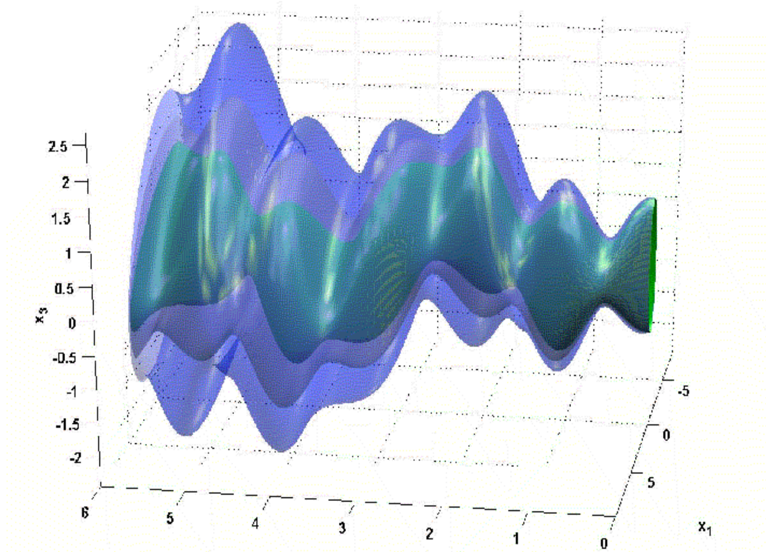


J. Almeida and F. Lobo Pereira, Reach set computation for linear systems

8th order system reach set

- Techniques
  - Ellipsoidal calculus and HJB
- 8th order system reach set
  - Projections on 2D spaces
  - No disturbance

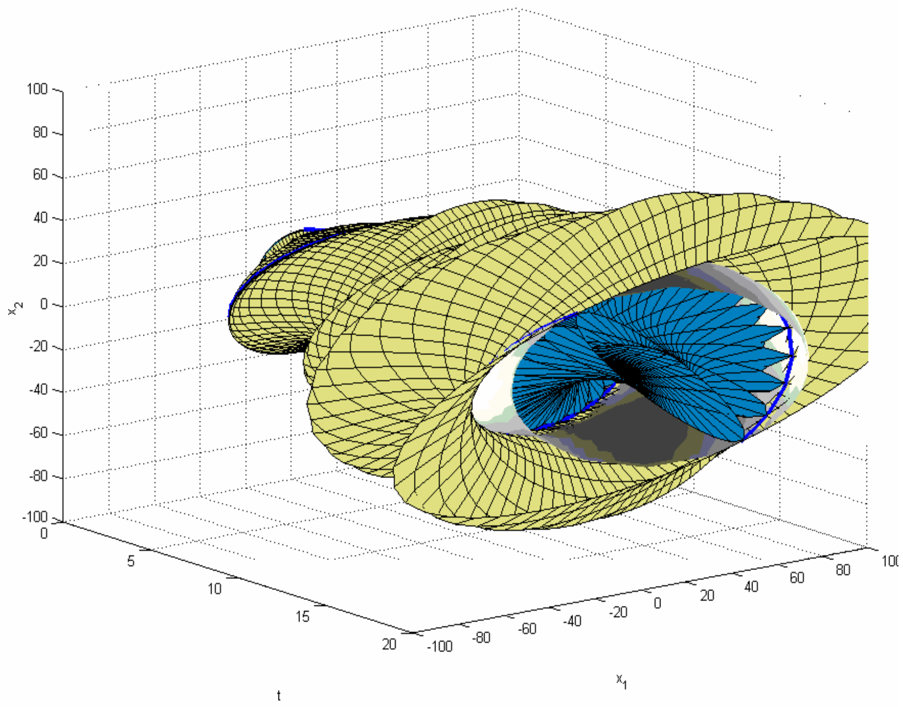




A. Kurzhanskii and P. Varaiya, **Ellipsoidal techniques for reachability analysis**, LNCS, 2000

Reach set under disturbances

- Techniques
  - Ellipsoidal calculus
- Reach set computation under uncertainty



System:

$$\begin{aligned}
\dot{x}_1 &= x_2 + u_1 + v_1 \\
\dot{x}_2 &= -\omega_1^2 x_1 + u_2 + v_2 \\
\dot{x}_3 &= x_4 + u_3 + v_3 \\
\dot{x}_4 &= -\omega_2^2 x_3 + u_4 + v_4
\end{aligned}$$

The time interval: $t \in [0, 20]$
the initial set:

$$\mathcal{X}^0 = \mathcal{E}(x^0, X^0) = \{x \in \Re^4 : (x_1^2 + x_2^2 + x_3^2 + x_4^2)/20^2 \leq 1\}$$

The restriction on control:

$$u(t) \in \mathcal{P}(t) = \mathcal{E}(p(t), P(t)) = \{u \in \Re^4 : \varepsilon^2 u_1^2 + \eta_1^2 u_2^2 + \varepsilon^2 u_3^2 + \eta_1^2 u_4^2 \leq 1\}$$

The restriction on disturbance:

$$v(t) \in \mathcal{Q}(t) = \mathcal{E}(q(t), Q(t)) = \{v \in \Re^4 : \nu_1^2 v_1^2 + \varepsilon^2 v_2^2 + \nu_2^2 v_3^2 + \varepsilon^2 v_4^2 \leq 1\}$$

A. Kurzhanskii and P. Varaiya, Ellipsoidal techniques for reachability analysis, LNCS, 2000

Why are we interested in reach sets?

Properties and reachability

- Selected properties
  - **Safety:** the system does not enter some set
  - **Attainability:** the system enters some set
  - **Invariance:** the system does not leave some set

The knowledge of the reach set makes controller synthesis easier

- Checking those amounts to solving reachability problems
- Hybrid systems in which a problem can be solved algorithmically in a finite number of steps are called decidable
- The general problem of reachability for hybrid systems is undecidable.

How to compute reach sets?

Reach set computation

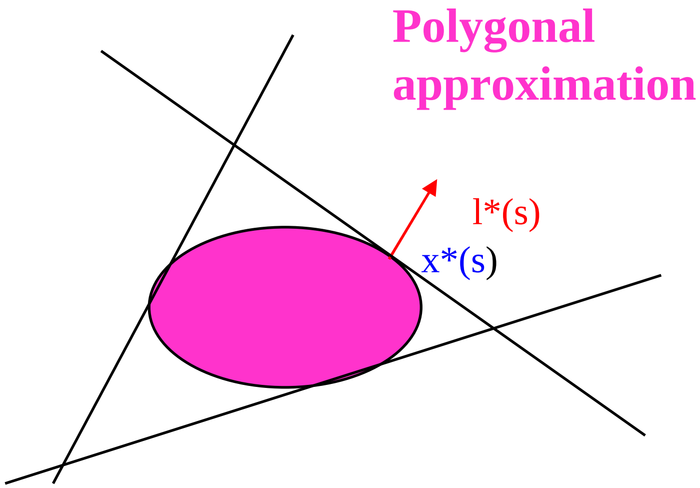
- Purely symbolic methods
  1. based on the existence of analytic solutions to the differential equations
  2. the representation of the state space in a decidable theory of the real numbers.

- Methods that combine
  1. numeric integration of differential equations
  2. symbolic representations of approximations of reach set typically using polyhedra or ellipsoids

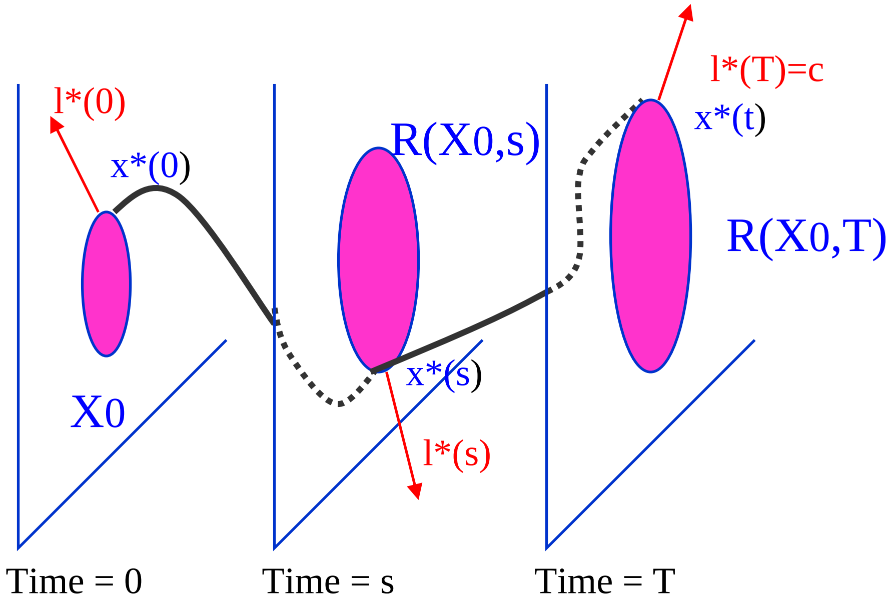
Reach sets and the maximum principle

Interpretation

- $R(X_0,s,t_0)$ Reach set at time s starting from X_0 at time t_0



- Maximum principle



Optimal control problem

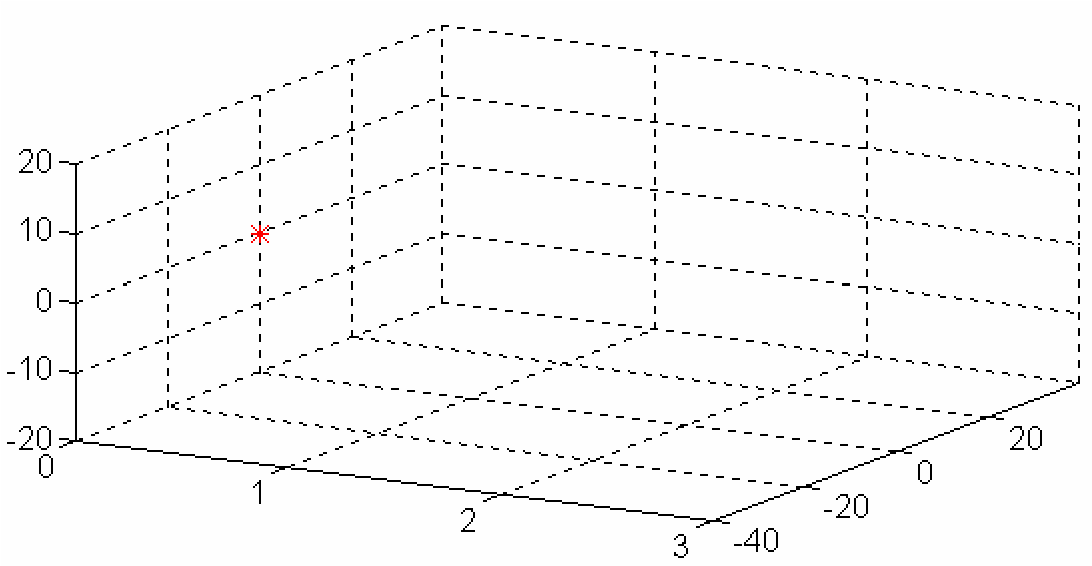
Max cx(T)

dx/dt(t) = A x(t) + u(t), t in [0,T]

u(t) belongs to U

P. Varaiya, Reach set computation using optimal control, Proceedings of the KIT Workshop on Verification of hybrid systems, Grenoble, 1998

Evolution of the reach set



J. Almeida and F. Lobo Pereira, Reach set computation for linear systems

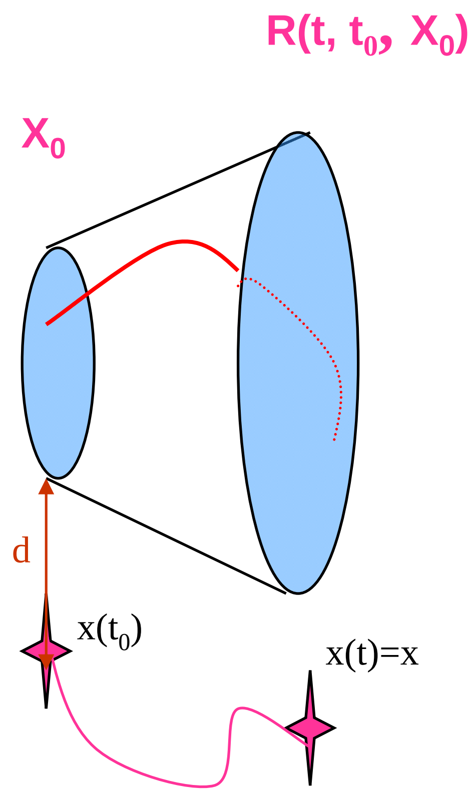
Reach sets and the value function

Interpretation

- System
  - $dx/dt(t) = f(t,x,u)$, $u(t)$ in $U(t)$
  - $x(t_0) = x_0$ in X_0

- Value function
  - $V(t,x) = \min_u\{d^2\ (x(t_0),X_0) \mid x(t)=x\}$

- Interpretation
  - $R(X_0,t,t_0) = \{x: V(t,x) <= 0\}$

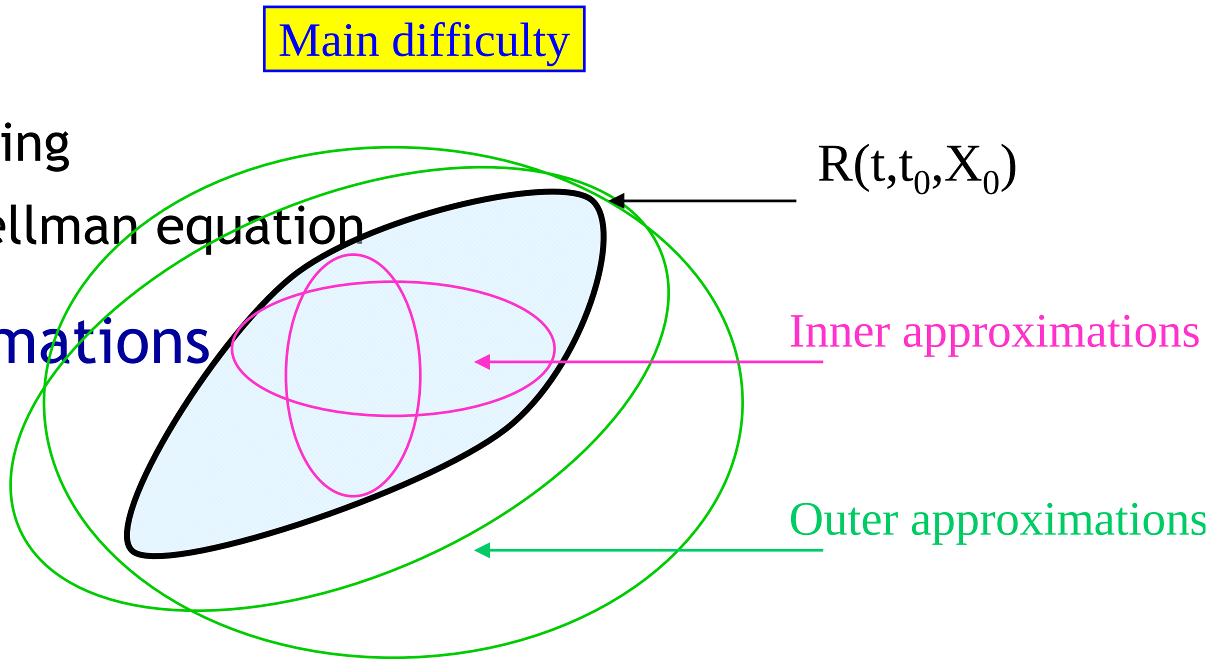


The reach set is given by the level sets of this value function

A. Kurzhanskii and P. Varaiya, Ellipsoidal techniques for reachability analysis, LNCS, 2000

Approximation

- Value functions: applicable to the general reachability problem
  - Under closed and open loop
  - Under uncertainty
- Solution
  - Dynamic programming
  - Hamilton-Jacobi-Bellman equation
- Ellipsoidal approximations



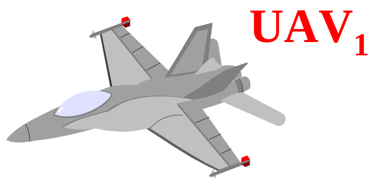
A. Kurzhanskii and P. Varaiya, On reachability under uncertainty

To appear in Siam Journal of Control and Optimization

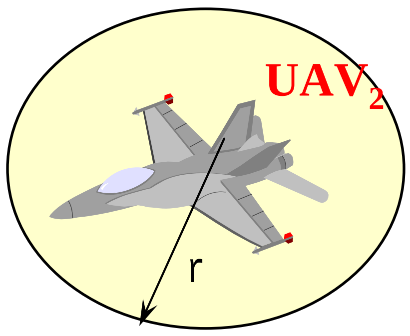
A constructive procedure

Example

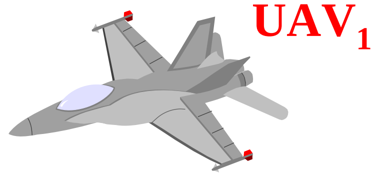
- Informal specifications
  - UAV_1 is required to reach a ball of radius r centered at UAV_2 at time θ
  - UAV_1 does not know what UAV_2 is doing
  - No communications
  - UAV_1 knows the position of UAV_2

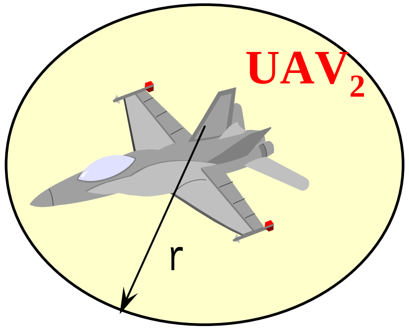


- Worst case scenario
  - UAV_2 tries to escape "capture"
- Pursuit-evation game



- 2 D dynamics with acceleration inputs
  - $dx_1/dt = A_1x_1 + B_1u$ $\quad |u(t)| \leq \beta_1$
  - $dx_2/dt = A_2x_2 + B_2 v$ $\quad |v(t)| \leq \beta_2$
  - $x_1 = [x_{11}, x_{12}, x_{13}, x_{14}]^T$
  - $x = [x_1, x_2]^T$ in R^8
  - $dx/dt(t) = Ax + Bu + Cv$





- Formal specifications
  - Capture set
    - $M = \{(\theta, x): (x_{11}-x_{21})^2 + (x_{12}-x_{22})^2 \leq r\}$
  - Time interval
    - $T = (0, \theta]$
  - State space
    - $N = (0, \theta] \times R^8$

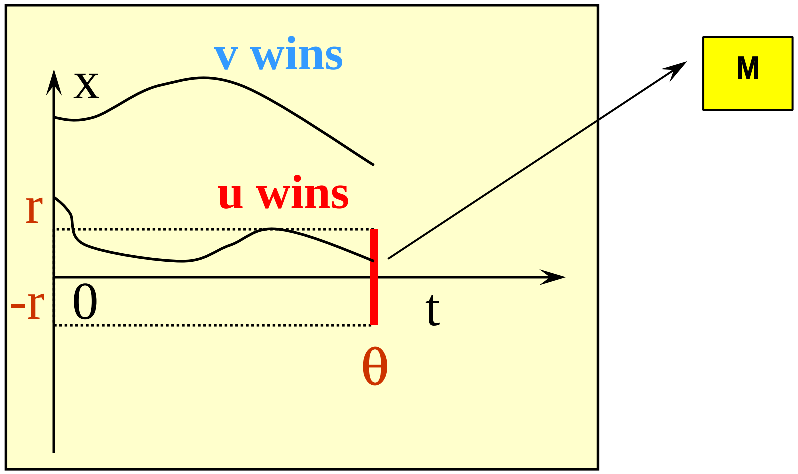
UAV$_1$

$$\dot{x}_1 = x_3$$
$$\dot{x}_3 = (-\alpha_1 x_3 + u_1)/m_1$$
$$\dot{x}_2 = x_4$$
$$\dot{x}_4 = (-\alpha_1 x_4 + u_2)/m_1$$

Background (1-dim dynamics)

- Simple example
  - $dx/dt(t) = r_1(t)u(t) + r_2(t)v(t)$, $|u(t)| \le \beta_1$, $|v(t)| \le \beta_2$, $T = [0, \theta]$
- Antagonistic players
  - Consider

    $M = \{(t,x): t = \theta, |x| \le r\}$
  - u wins if x reaches M
  - v wins if x does not reach M
  - No communication
  - Each vehicle knows the position of the other one



- Problem
  - What is the set W of positions (t,x) such that there is a winning strategy for u (safe set for u)?

Differential games formulation

Consider the motion of $x \in \Re$ under adversary control inputs (u, v):

$$\dot{x} = f(t, x, u, v), u \in P, v \in Q$$

Consider the set M:

$$M = \{(t, x) \in T \times \Re : t = \theta, l_1 \leq x \leq l_2\}$$

Cost functional:

$$\gamma(x(t_0, x_0, U, V)) = \begin{cases} 1 & \text{if } x(.) \text{ intersects M} \\ 0 & \text{otherwise} \end{cases}$$

U is a control function for the first player and V is a control function for the second player.

Under some hypotheses this game has a **value**. Formally:

$$\inf_U \sup_V \gamma(x(t_0, x_0, U, V)) = \sup_V \inf_U \gamma(x(t_0, x_0, U, V))$$

The game has a **saddle point**, or equivalently there exist strategies U^*, V^* such that:

$$\gamma(x(t_0, x_0, U^*, V)) \leq \gamma(x(t_0, x_0, U^*, V^*)) \leq \gamma(x(t_0, x_0, U, V^*))$$

Theorem For any closed set M and for any initial position (t_0, x_0), one and only one of the following assertions is valid:

- The value of the game is 0. Any (U^*, V) is a saddle point (V is any feedback strategy).
- The value of the game is 1.

N.N. Krasovskii and A.I. Subbotin, **Game-theoretical control problems**, Springer-Verlag, 1988

Solution methodology

- (1-dim case)
  - dx/dt(t) = r_1(t)u(t) + r_2(t)v(t), |u(t)| ≤ β_1, |v(t)| ≤ β_2, T = [0,θ]

$$M = \{(t,x) \in T \times \mathbb{R} : t = \theta, l_1 \leq x \leq l_2\}$$

In this problem setup it is possible to derive a closed form for the u-stable bridge. First set:

$$f_1(t,x) := \max_{u \in P} \min_{v \in Q} f(t,x,u,v) \quad (52)$$

$$f_2(t,x) := \min_{u \in P} \max_{v \in Q} f(t,x,u,v) \quad (53)$$

Consider solutions w_1 and w_2 to the following ODEs:

$$\dot{w}_1(t) = f_1(t, w_1(t)), w_1(\theta) = l_1 \quad (54)$$

$$\dot{w}_2(t) = f_1(t, w_2(t)), w_2(\theta) = l_2 \quad (55)$$

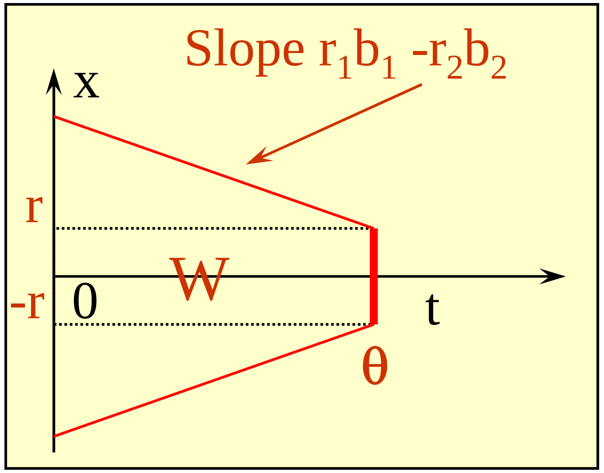
The u-stable bridge is the set:

$$W_0 := \{(t,x) \in T \times \mathbb{R} : t \in T_*, x \in [w_1(t), w_2(t)]\}$$

where $T_* = [\tau_*, \theta], \tau_* = sup\{t \in T : w_2(t) > w_1(t)\}$.

- (1-dim case)
  - $dx/dt(t) = r_1(t)u(t) + r_2(t)v(t)$, $|u(t)| \le \beta_1$, $|v(t)| \le \beta_2$, $T = [0,\theta]$
  - $M = \{(t,x): t = \theta, |x| \le r\}$
- Define

$$\rho(t) = r + \int_t^\theta (b_1|r_1(\tau)| - b_2|r_2(\tau)|d\tau$$



  - Symmetry in dx/dt
  - Limit case: both u and v do their best
  - ftss consider $|r_1(\tau)| = r_1$ and $|r_2(\tau)| = r_2$
- W set of states from which v looses the game (u wins)
  - $W = \{(t,x): t \text{ in } T^*, |x| <= \rho(t)\}$ where
    - $T^* = T$ if $\rho(t) >= 0$, for all t in T
    - $T^* = [\tau^*, \theta]$ where $\tau^* = \sup \{t \text{ in } T: \rho(t) < 0\}$

Transformation for n-dim case

$$\dot{x}(t) = Ax(t) + Bu(t) + Cv(t) \Rightarrow \dot{y}(t) = B_* u_*(t) + C_* V_*(t)$$

The transformation is applied using:

$$y(t) = F(t)x(t)$$

where

$$F(t) = F\Phi_*(t)$$

F = (I, O, -I, O)

I and O are the 2-by-2 identity and zero matrices.

$\Phi_*(t)(i = 1, 2)$ is given by: $\begin{bmatrix} 1 & 0 & \phi_i(t) & 0 \\ 0 & 1 & 0 & \phi_i(t) \\ 0 & 0 & \psi_i(t) & 0 \\ 0 & 0 & 0 & \psi_i(t) \end{bmatrix}$

with

$$\phi_i(t) = [1 - e^{-\frac{\alpha_i}{m_i}t}]\frac{m_i}{\alpha_i}$$

$$\psi_i(t) = e^{-\frac{\alpha_i}{m_i}t}$$

Transform the sets M and N as well:

$$M_* = \{(0, y) : y \in \Re^2, \|y\| \leq r_*\}$$

$$N_* = (-\infty, 0] \times \Re^2$$

M is defined by a norm (1 dim problem).

Apply 1-dim result.

The maximal u-stable bridge is given by:

$$W_0 = \{(t, y) \in T \times \Re^2, \|y\| \leq \rho_0(t)\}$$

ρ_0 is given by the equation :

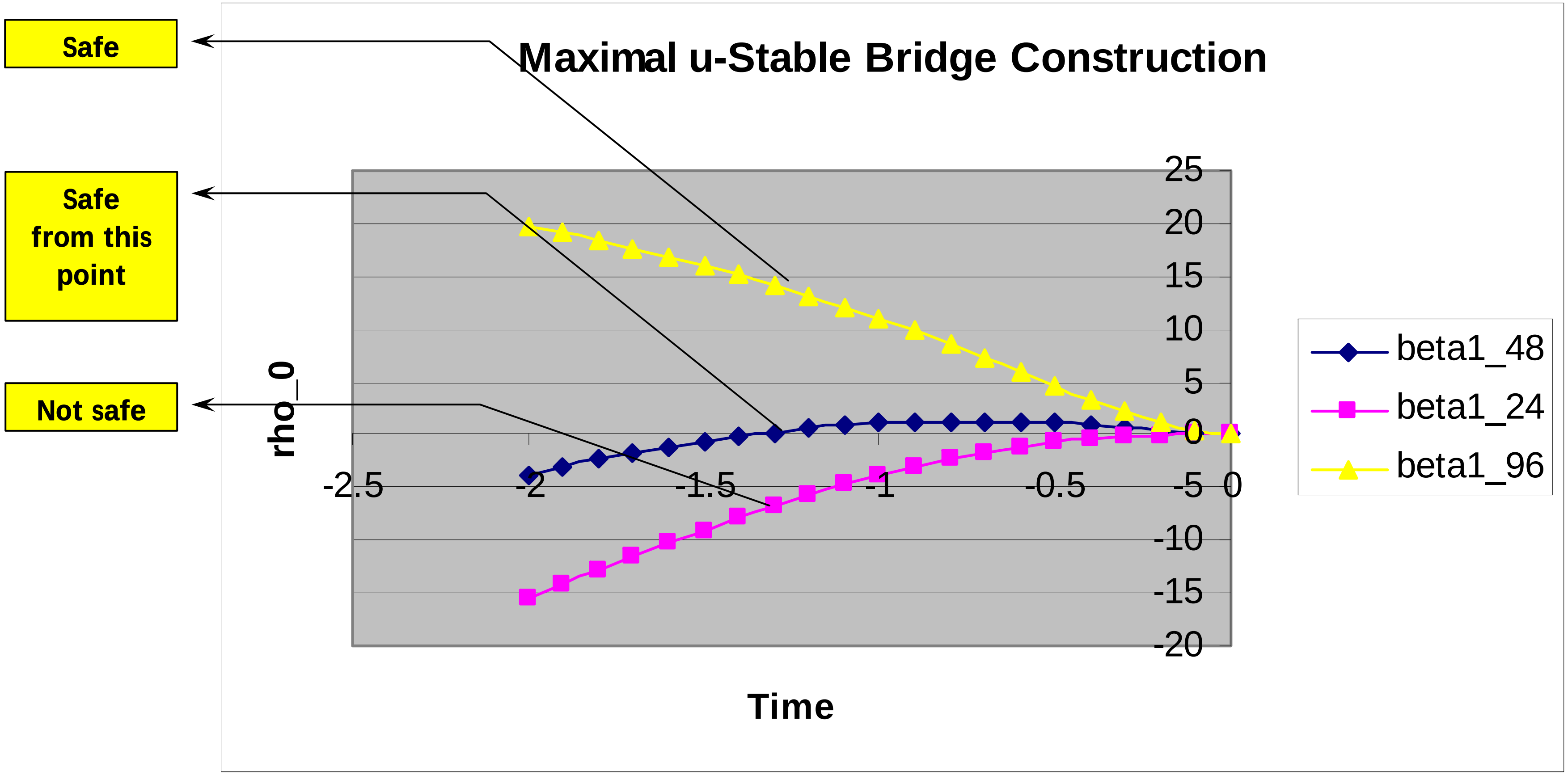
$$\rho_0(t) = r_* + \int_t^0 (\beta_1 |r_1(\tau)| + \beta_2 |r_2(\tau)|) d\tau$$

Note that $\rho_0(t)$ is the diameter of the W_0 at time t.

Example

- Calculate profiles of ρ_0 as a function of the parameters
- $r_*=0$, $\beta_2=24$, $\alpha_1=3.5$, $\alpha_2=1$ (follower has more friction)
- Plot $\rho(t)$ for different values of β_1 (= 24, 48, 96).
- For β_1 = 24, not possible.
- For β_1 = 48, possible for small set of IC.
- For β_1 = 96, possible for much larger set.

Example


Safe
Safe from this point
Not safe
Maximal u-Stable Bridge Construction
rho_0
Time
25
20
15
10
5
0
-5
-10
-15
-20
-2.5
-2
-1.5
-1
-0.5
0
beta1_48
beta1_24
beta1_96