INTRODUCTION TO OPTIMAL CONTROL

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Organization

1. Introduction.

   General considerations. Motivation. Problem Formulation. Classes of problems. Issues in optimal control theory

2. Necessary Conditions of Optimality - Linear Systems

   Linear Systems Without and with state constraints. Minimum time. Linear quadratic regulator.

3. Necessary Conditions of Optimality - Nonlinear Systems.

   Basic Problem. Perturbations of ODEs. Problems with state constraints.

4. Additional issues

   Nonsmoothness. Degeneracy. Nondegenerate conditions with and without a priori normality assumptions.

5. Conditions of the Hamilton-Jacobi-Bellman Type.

   Verification functions. Principle of Optimality Result on Reachable sets. Introduction to Dynamic Programming

6. A Framework for Coordinated Control

   An example of motion coordination of multiple AUVs

Key References

- Pravin Varaiya,“Notes on Optimization”, Van Nostrand Reinhold Company, 1972.
- Francis Clarke, ”Optimization and Nonsmooth Analysis”, John Wiley, 1983.
- Richard Vinter, “Optimal Control”, Birkhauser, 2000.
- João Sousa, Fernando Pereira, “A set-valued framework for coordinated motion control of networked vehicles”, J. Comp. & Syst. Sci. Inter., Springer, 45, 2006, p.824-830.
- David Luenberger, “Optimization by Vector Space Methods”, Wiley, 1969.
- Aram Arutyunov, “Optimality Conditions: Abnormal and Degenerate Problems”, Kluwer Academic Publishers, 2000
- Martino Bardi, Italo Capuzzo-Dolcetta, “Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations”, Birkhauser, 1997.
- Vladimir Girsanov, “Lectures on Mathematical Theory of Extremum Problems”, Lect. Notes in Econom. and Math. Syst., 67, Springer Verlag, 1972.
- Francesco Borrelli, http://www.me.berkeley.edu/ frborrel/Courses.html

INTRODUCTION

Ingredients of the Optimal Control Problem

o Objective functional - Criterium to quantify the performance of the system.

o Systems dynamics - The state variable characterizes the evolution of the system overtime. Physically, the attribute "dynamic" is due to the existence of energy storages which affect the behavior of the system.

Once specified a control strategy and the initial state, this equation fully determines the time evolution of the state variable.

o Control Constraints - The control variable represents the possibility of intervening in order to change the behavior of the system so that its performance is optimized.

o Constraints on the state variable - The satisfaction of these constraints affect the evolution of the system, and restricts the admissible controls.

Applications

Useful for optimization problems with inter-temporal constraints.

o Management of renewable and non-renewable resources

o Investment strategies,

o Management of financial resources,

o Resources allocation,

o Planning and control of productive systems (manufacturing, chemical processes,..),

o Planning and control of populations (cells, species),

o Definition of therapy protocols,

o Motion planning and control in autonomous mobile robotics

o Aerospace Navigation,

o Synthesis in decision support systems,

o Etc . . .

The General Problem

$$\begin{array}{rll} (P) \quad \text{Minimize} & g(x(1)) & \\ \text{by choosing} & (x,u) : [0,1] \longrightarrow I\!R^n \times I\!R^m & \\ \text{satisfying :} & \dot{x}(t) = f(t, x(t), u(t)), \qquad [0,1]\ \mathcal{L}\text{-a.e.}, & (1) \\ & x(0) = x_0, & (2) \\ & u(t) \in \Omega(t), \qquad [0,1]\ \mathcal{L}\text{-a.e.}. & (3) \end{array}$$

$$(P') \text{ Minimize } \{g(z) :\ z \in \mathcal{A}(1; (x_0, 0))\}$$

$\mathcal{A}(1; (x_0, 0))$ - the set in $I\!R^n$ that can be reached at time 1 from x_0 at time 0.

General definitions

Dynamic System - System whose state variable conveys its past history. Its future evolution depends not only on the future (“inputs”) but also on the current value of the state variable.

Trajectory - Solution of the differential equation (1) with the boundary condition (2) and for a given controlo function satisfying (3).

Admissible Control Process - A (x, u) satisfying the constraints (1,2,3).

Attainable set - $\mathcal{A}(1; (x_0, 0))$ is the set of state space points that can be reached from x_0 with admissible control strategies

$$\mathcal{A}(1; (x_0, 0)) := \{x(1) : \text{ for all admissible control processes } (x, u)\}$$

Boundary process - Control process whose trajectory (or a given function of it) remains in the boundary of the attainable set (or a given function of it).

Local/global minimum - Point for which the value of the objective function is lower than that associated with any other/other within a neighborhood feasible point.

Types of Problems

o Bolza - $g(x(1)) + \int_0^1 L(s, x(s), u(s))ds$.

o Lagrange - $\int_0^1 L(s, x(s), u(s))ds$.

o Mayer - $g(x(1))$.

Other types of constraints besides the above:

o Mixed constraints - $g(t, x(t), u(t)) \leq 0$, $\forall t \in [0, 1]$.

o Isoperimetric constraints, $\int_0^1 h(s, x(s), u(s))ds = a$.

o Endpoints and intermediate state constraints, $y(1) \in S$.

o State constraints, $h_i(t, x(t)) \leq 0$ para todo o $t \in [0, 1]$, $i = 1, \ldots, s$.

Overview of Main Issues - Necessary Conditions of Optimality

Let x^* be an optimal trajectory for (P). Then, $\exists$ a.c. $p : [0,1] \to I\!R^n$, satisfying

$$-\dot{p}^T(t) \;=\; p^T(t) D_x f(t, x^*(t), u^*(t)), \qquad [0,1]\ \mathcal{L}\text{-a.e.}, \tag{4}$$

$$-p^T(1) \;=\; \nabla_x g(x^*(1)). \tag{5}$$

where $u^* : [0,1] \to I\!R^m$ is a control strategy s.t. $u^*(t)$ maximizes

$$v \to p^T(t) f(t, x^*(t), v) \quad \text{in} \quad \Omega(t), \qquad [0,1]\ \mathcal{L}\text{-a.e.}. \tag{6}$$

(6) eliminates the control as it defines implicitly

$$u^*(t) = \bar{u}(x^*(t), p(t)).$$

Then, solving (P) amounts to solve

$$\begin{aligned} -\dot{p}^T(t) &= p^T(t) D_x f(t, x^*(t), \bar{u}(x^*(t), p(t))), \; p(1) = -\nabla_x g(x^*(1)), \\ \dot{x}^*(t) &= f(t, x^*(t), \bar{u}(x^*(t), p(t))), \qquad x(0) = x_0. \end{aligned}$$

Algorithms

Step 1 – Select an initial control strategy u.

Step 2 – Compute a pair (x, p), by using $(1, 2, 4, 5)$.

Step 3 – Check if $u(t)$ ssatisfies (6).
If positive, the algorithm terminates.
Otherwise, proceed to Step 4.

Step 4 – Update the control in order to lower the cost function.

Step 5 – Prepare the new iteration and goto Step 2.

Overview of Main Issues - Existence of Solution

Necessary for the consistency of the optimality conditions.

o Let $\mathbf{N} \in I\!\!N$ be the largest natural number, then $\mathbf{N} \geq n$, $\forall n \in I\!\!N$.

o In particular, the inequality should hold for $n = \mathbf{N}^2$.

o Dividing both sides of $\mathbf{N}^2 \leq \mathbf{N}$ by $\mathbf{N}$, one gets $\mathbf{N} \leq 1$.

Existence conditions: Lower semi-continuity of the objective function and the corresponding compactness of the attainable set.

H0 g is lower semi continuous.

H1 $\Omega(t)$ is compact $\forall t \in [0,1]$ and $t \to \Omega(t)$ is Borel mensurable.

H2 f is continuous in all its arguments.

H3 $|f(t,x,u) - f(t,y,u)| \leq K_f \|x - y\|$.

H4 $\exists K > 0 \ : |x \cdot f(t,x,u)| \leq K(1 + \|x\|^2)$ for all the values of (t,u).

H5 $f(t,x,\Omega(t))$ is convex $\forall x \in I\!\!R^n$ e $\forall t \in [0,1]$.

Overview of Main Issues - Sufficient Conditions of Optimality

Let $V : [0,1] \times I\!R^n \to I\!R$ be a smooth function s.t. in the neighborhood of $(t, x^*(t))$,

$$V(1,z) = g(z),$$
$$V(0,x_0) \geq g(x^*(1)),$$
$$V_t(t,x^*(t)) - \sup\{V_x(t,x^*(t))f(t,x^*(t),u) : u \in \Omega(t)\} = 0, \quad [0,1]\ \mathcal{L}\text{-a.e.}, \tag{7}$$

where x^* is solution to (1) with $u = u^*$ and $x^*(0) = x_0$, then the control process (x^*, u^*) is optimal for (P).

V - solution to Hamilton-Jacobi-Bellman equation (7) - is the verification function which under certain conditions coincides with the value function.

Although these conditions have a local character, there are results giving conditions of global nature.

New types of solutions - Viscosity, Proximal, Dini, ... - generalizing the classic concept.

NECESSARY CONDITIONS OF OPTIMALITY: LINEAR SYSTEMS

An exercise on the separation principle applied to the following problems:

- Linear cost and affine dynamics.
- The above with affine endpoint state constraints.
- Minimum time with affine dynamics.
- Linear dynamics and quadratic cost.

The Basic Linear Problem

$$
\begin{array}{rl}
(P_1) \text{ Minimize} & -c^T x(1) \\
\text{by choosing} & (x,u) : [0,1] \to I\!R^n \times I\!R^m \text{ s.t.:} \\
& \dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad [0,1]\ \mathcal{L}\text{-a.e.}, \\
& x(0) = x_0 \in I\!R^n, \\
& u(t) \in \Omega(t), \qquad [0,1]\ \mathcal{L}\text{-a.e.},
\end{array}
$$

being $A \in I\!R^{n\times n}$, $B \in I\!R^{n\times m}$, and $c \in I\!R^n$.

Example of control constraint set: $\Omega(t) := \displaystyle\prod_{k=1}^{m}[\alpha_k, \beta_k]$.

Given x_0 and $u : [0,1] \to I\!R^m$, and, being $\Phi(b,a) := e^{\int_a^b A(s)ds}$, we have:

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)B(s)u(s)ds.$$

Maximum Principle

The control strategy u^* is optimal for (P_1) if and only if $u^*(t)$ maximizes

$$v \rightarrow p^T(t)B(t)v, \text{ in } \Omega(t), \quad [0,1] \; \mathcal{L}\text{-a.e.},$$

where $p : [0,1] \rightarrow I\!R^n$ is an a.c. function s.t.:

$$\begin{aligned} -\dot{p}^T(t) &= p^T(t)A(t), \qquad [0,1] \; \mathcal{L}\text{-a.e.} \\ p(1) &= c. \end{aligned}$$

For this problem, the Maximum Principle is a necessary and sufficient condition.

Geometric Interpretation:

- o Existence of a boundary control process associated with the optimal trajectory.
- o The adjoint variable vector is perpendicular to the attainable set at the optimal state value for all times.

Geometric Interpretation

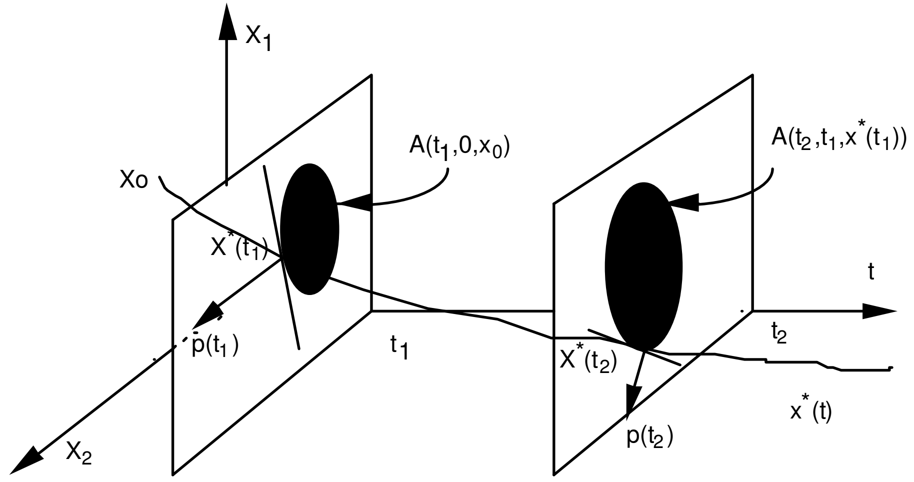

Fig. 2. Relation between the adjoint variable and the attainable set (inspired in [17])

Proposition

Let $c^T x^*(1) \geq c^T z$, $\forall z \in \mathcal{A}(1;(x_0,0))$ and $c \neq 0$, i.e.,

$$-p^T(1) = c \text{ is perpendicular to } \mathcal{A}(1;(x_0,0)) \text{ at } x^*(1) \in \partial\mathcal{A}(1;(x_0,0)).$$

Then, $\forall t \in [0,1)$,

o $x^*(t) \in \partial\mathcal{A}(t;(x_0,0))$,

o $-p^T(t)$ is perpendicular to $\mathcal{A}(t;(x_0,0))$ at $x^*(t)$.

Analytic Interpretation

It consists in showing that the switching function

$$\sigma : [0,1] \rightarrow I\!\!R^m := -p^T(t)B(t)$$

is the gradient of the objective functional $J(u) := -c^T x(1)$ relatively to the value of the control function at time t, $u(t)$.

By computing the directional derivative and using the time response formula for the dynamic linear system, we have:

$$J'(u;w) = \int_0^1 \sigma(t)w(t)dt = < \nabla_u J(u), w > .$$

Here, $\nabla_u J(u) : [0,1] \rightarrow I\!\!R^m$ is the gradient of the cost functional w.r.t. to control, and $< \cdot, \cdot >$ is the inner product, in the functional space.

Deduction of the Maximum Principle

Exercise

- Express the optimality conditions as a function of the state variable at the final time.

  Check that $\{x^*(1)\}$ and $\mathcal{A}(1;(x_0,0))$ fulfill the conditions to apply a Separation Theorem.

  After showing the equivalence between the trajectory optimality and the fact of being a boundary process

  $$\text{Observe that } (c^T x^*(1), x^*(1)) \in \partial\{(z,y) : z \geq c^T y, y \in \mathcal{A}(1;(x_0,0))\}$$

  write the condition of perpendicularity of the vector c to $\mathcal{A}(1;(x_0,0))$.

- Express the conditions obtained above in terms of the control variable at each instant in the given time interval by using the time response formula.

  In this step, the control maximum condition, the o.d.e. and the boundary conditions satisfied by the adjoint variable are jointly obtained.

Example

Let $t \in [0,1]$, $u(t) \in [-1,1]$, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, e $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

By writing $e^{A\tau} = \alpha_0(\tau)I + \alpha_1(\tau)A + \alpha_2(\tau)A^2$, where $\tau = 1-t$, we get

$$\sigma(t) := p^T(t)B = Ce^{(A(1-t))}B = \alpha_2(1-t).$$

The eigenvalues of A - roots of the characteristic polynomial de A, $p(\lambda) = \det(\lambda I - A) = 0$. By Cayley-Hamilton theorem,

$$\begin{aligned} \alpha_0(\tau) + \alpha_1(\tau) + \alpha_2(\tau) &= e\tau \\ \alpha_0(\tau) + 2\alpha_1(\tau) + 4\alpha_2(\tau) &= e^{2\tau} \\ \alpha_0(\tau) + 3\alpha_1(\tau) + 9\alpha_2(\tau) &= e^{3\tau}. \end{aligned}$$

Thus,

$$\alpha_2(\tau) = \frac{e^{3\tau} - 2e^{2\tau} + e^{\tau}}{2}.$$

Since $\sigma(t) > 0$, $\forall t \in [0,1]$, we have $u^*(t) = 1$, $\forall t \in [0,1]$.

The Linear problem with Affine Endpoint State Constraints

$$(P_2) \qquad \text{Minimize } -c^T x(1)$$

by choosing $(x,u):[0,1] \to I\!R^n \times I\!R^m$ s.t.:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \qquad [0,1]\ \mathcal{L}\text{-a.e.}, \\ x(0) &\in X_0 \subset I\!R^n, \\ x(1) &\in X_1 \subset I\!R^n, \\ u(t) &\in \Omega(t), \qquad [0,1]\ \mathcal{L}\text{-a.e.}, \end{aligned}$$

being X_i, $i = 0,1$, given by $X_i := \{z \in I\!R^n : D_i z = e_i\}$.

The pair $(x^*(0), u^*)$ is optimal for (P_2) if

$$\begin{aligned} (x^*(0), x^*(1)) &\in X_0 \times X_1, \quad u^*(t) \in \Omega(t) \quad [0,1]\ \mathcal{L}\text{-a.e., and} \\ c^T x^*(1) &\geq c^T z \qquad \forall\, z \in \mathcal{A}(1;(X_0,0)) \cap X_1, \end{aligned}$$

being $\mathcal{A}(1;(X_0,0)) := \bigcup_{a \in X_0} \mathcal{A}(1;(a,0))$.

Maximum Principle

These conditions are necessary and sufficient.

Let (x^*, u^*) be an admissible control process for (P_2), i.e., s.t. $x^*(0) \in X_0$, $u^*(t) \in \Omega(t)$ and $x^*(1) \in X_1$. Then:

A) Necessity

If $(x^*(0), u^*)$ is optimal, then, $\exists p : [0,1] \to I\!R^n$ e $\lambda \geq 0$, s.t.:

$$\lambda + \|p(t)\| \neq 0, \tag{8}$$
$$-\dot{p}^T(t) = p^T(t)A(t), \quad [0,1]\ \mathcal{L}\text{-a.e.}, \tag{9}$$
$$p(1) - \lambda c \text{ is perpendicular to } X_1 \text{ at } x^*(1), \tag{10}$$
$$p(0) \text{ is perpendicular to } X_0 \text{ at } x^*(0), \tag{11}$$
$$u^*(t) \text{ maximizes the map } v \to p^T(t)B(t)v \text{ on } \Omega(t), \quad [0,1]\ \mathcal{L}\text{-a.e.}. \tag{12}$$

B) Sufficiency

If (8)-(12) hold with $\lambda > 0$, then $(x^*(0), u^*)$ is optimal.

Geometric Interpretation

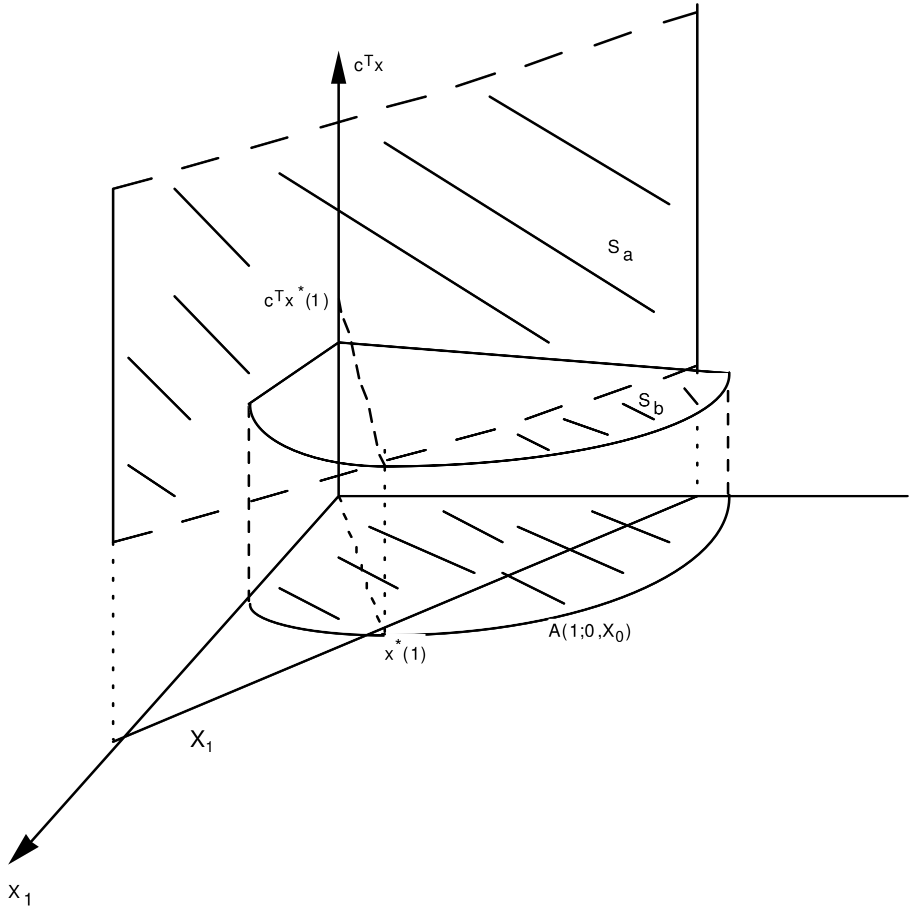

Fig. 2. Separation of the optimal state at the final time subject to affine constraints (inspired from [17]).

Proof Sketch of the Necessity

The optimality conditions applied at $(x^*(0), u^*)$ imply $\exists \lambda \geq 0$ and $q \in I\!R^n$ s.t.:

$$\lambda + \|q\| \neq 0 \tag{13}$$

q is perpendicular to X_1 at $x^*(1)$. (14)

$$(\lambda c + q)^T x^*(1) \geq (\lambda c + q)^T x \forall\, x \in \mathcal{A}(1; (X_0, 0)). \tag{15}$$

$\Phi(1,0)^T(\lambda c + q)$ is perpendicular to X_0 at $x^*(0)$. (16)

Let (λ, q) be a vector defining the hyperplane separating the sets
$S_a := \{s_a = (r_a, x_a) : \, r_a > c^T x^*(1), \, x_a \in X_1\}$,
$S_b := \{s_b = (r_b, x_b) : \, r_b = c^T x_b, \, x_b \in \mathcal{A}(1; (X_0, 0))\}$.

$$\lambda r_a + q^T x_a \geq (\lambda c + q)^T x_b, \qquad \forall x_b \in \mathcal{A}(1; (X_0, 0)), \ \forall x_a \in X_1, \ \forall r_a > c^T x^*(1).$$

- o (13) $\Longleftarrow$ non-triviality of the separator, r_a arbitrary and $x_a = x^*(1)$.
- o (15) $\Longleftarrow$ choice of $x_a = x^*(1)$ and the arbitrary approximation of r_a a $c^T x^*(1)$.
- o (14) $\Longleftarrow$ r_a arbitrarily close to $c^T x^*(1)$, x_a arbitrary in X_1 and $x_b = x^*(1)$.
- o (16) $\Longleftarrow$ (15) with $\{\Phi(1,0)[z - x^*(0)] + x^*(1) : \, z \in X_0\} \subset \mathcal{A}(1; (X_0, 0))$.

Proof Sketch of the Sufficiency

For all $x \in \mathcal{A}(1; (X_0, 0))$, we have, $\forall z \in X_0$, $\forall v \in \mathcal{A}(1; (0, 0))$:

$$\begin{aligned}
(\lambda c + q)^T x &= p^T(1)x \\
&= p^T(1)[\Phi(1,0)z + v] \\
&= p^T(1)\Phi(1,0)[z - x^*(0)] + p^T(1)\Phi(1,0)x^*(0) + p^T(1)v \\
&= p^T(0)[z - x^*(0)] + p^T(1)[\Phi(1,0)x^*(0) + v],
\end{aligned}$$

Note that the first parcel is null and that the second one is in $\mathcal{A}(1; (x^*(0), 0)) \subset \mathcal{A}(1; (X_0, 0))$.
Thus, $(\lambda c + q)^T x \leq p^T(1)x^*(1) = (\lambda c + q)^T x^*(1)$.

Since $x \in \mathcal{A}(1; (X_0, 0)) \cap X_1$, q is perpendicular to X_1 at $x^*(1)$.
Hence, the sufficiency.

Example - Formulation

$$
\begin{aligned}
\text{Minimize} \quad & c^T x(1) + \alpha_0 \int_0^1 u(t)dt \\
\text{where} \quad & \dot{x}(t) = Ax(t) + Bu(t), \ [0,1] \ \mathcal{L}\text{-a.e.} \\
& x_1(0) + x_2(0) = 0 \\
& x_1(1) + 3x_2(1) = 1 \\
& u(t) \in [0,1], \ [0,1] \ \mathcal{L}\text{-a.e.},
\end{aligned}
$$

being $\alpha_0 > 0$, $\quad A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

a) Determine the values of α_0 for which there exist optimal control switches within the time interval $[0, 1]$.

b) Determine the switching function as a function of α_0.

Example - solution clues

For a given $\lambda \in \{0, 1\}$, the system of equations $\bar{p} = p(1) - \lambda c$ is perpendicular to X_1, $p(0)$ is perpendicular to X_0, and $p^T(0) = p^T(1)e^A$ fully determine the adjoint variable.

Let $\lambda = 1$. Thus, we have

$$e^{At} = \begin{bmatrix} 2e^t - e^{2t} & e^{2t} - e^t \\ -2(e^{2t} - e^t) & 2e^{2t} - e^t \end{bmatrix}, \quad \text{and} \quad p(1) = \begin{bmatrix} 1 + \frac{1}{3}p_1 \\ 1 + p_1 \end{bmatrix} \text{ e } p(0) = \begin{bmatrix} p_0 \\ p_0 \end{bmatrix}.$$

Note that these last two relations determine p_0 e p_1.

To put the problem in the canonical form, add a component to the state variable, and the maximum condition becomes:

$$u^*(t) \text{ maximizes, in } [0, 1], \text{ the map } v \rightarrow [p^T(1)e^{A(1-t)}B - \alpha_0]v.$$

There exists an interval of values of α_0 for which the switching point is in $(0, 1)$.

The Minimum Time Problem

$$(P_3) \qquad \text{Minimize } T$$

by choosing $(x, u) : [0, T] \to I\!R^n \times I\!R^m$ such that:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \qquad [0,T]\ \mathcal{L}\text{-a.e.}, \\ x(0) &= x_0 \in I\!R^n, \\ x(T) &\in O(T) \subset I\!R^n, \\ u(t) &\in \Omega(t), \qquad [0,T]\ \mathcal{L}\text{-a.e.}, \end{aligned}$$

being T the final time and the multifunction $O : [0, T] \hookrightarrow \mathcal{P}(I\!R^n)$ define the target to be attained in minimum time, being $\mathcal{P}(I\!R^n)$ the set of subsets in $I\!R^n$.

Typically, this multi-function is continuous and takes compact sets as values. For example, $O(t) = \{z(t)\}$, being $z : [0, 1] \to I\!R^n$ a continuous function.

Generalization: Objective function defined by $g(t_0, x(t_0), t_1, x(t_1))$; Terminal Constraints given by $(t_0, x(t_0), t_1, x(t_1)) \in O \subset I\!R^{2(n+1)}$.

Geometric Interpretation

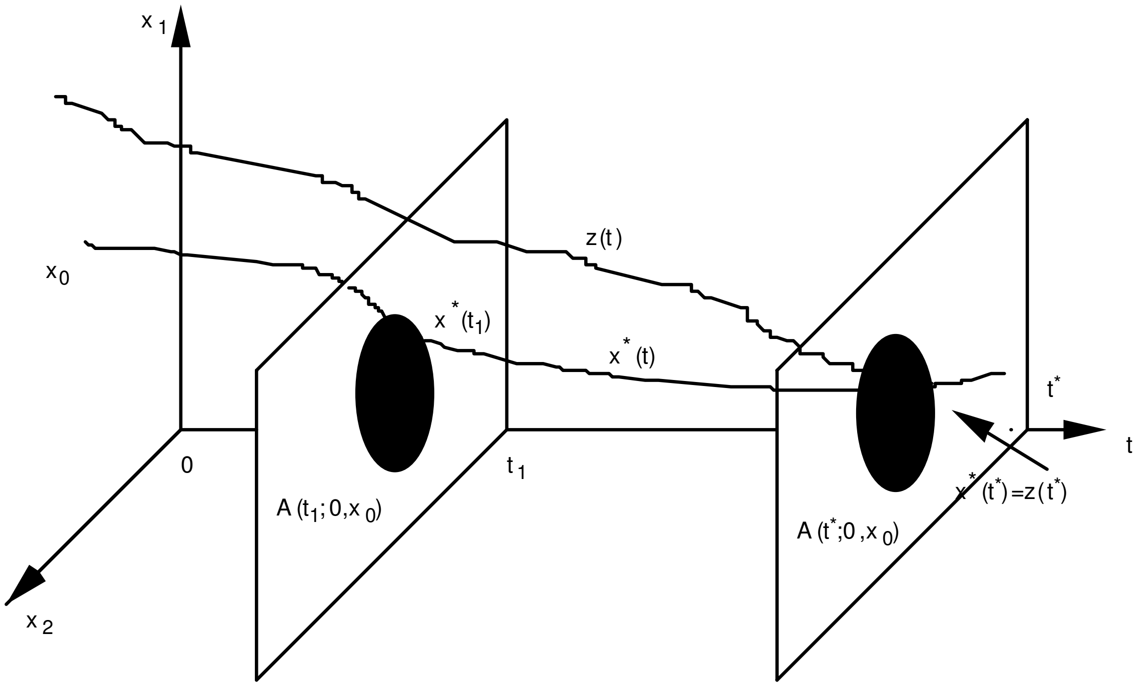

Fig. 3. Determination of minimum time for problem (P3) (inspired on [17]).

The optimal state at t^* is the intersection of sets $O(t^*)$ and $\mathcal{A}(t^*; (x_0, 0))$, and, thus, necessarily in the boundary of both sets.
Time t^* is given by

$$\inf\{\, t > 0 : \, O(t) \cap \mathcal{A}(t; (x_0, 0)) = \{x^*(t)\}\}.$$

Maximum Principle

Let (t^*, u^*) be optimal.
Then, there exists $h \in I\!\!R^n$ e $p : [0, t^*] \to I\!\!R^n$ a.c. s.t.

$$\begin{aligned} \|p(t)\| &\neq 0, && (17)\\ -\dot{p}^T(t) &= p^T(t)A(t), \quad [0, t^*]\ \mathcal{L}\text{-a.e.}, && (18)\\ p(t^*) &= q. && (19)\\ u^*(t) &\quad \text{maximizes} \quad v \to p^T(t)B(t)v \text{ em } \Omega(t), \quad [0, t^*]\ \mathcal{L}\text{-a.e.}, && (20)\\ x^*(t^*) &\quad \text{minimizes} \quad z \to h^T z \text{ in } O(t^*). && (21) \end{aligned}$$

Deduction: By geometric considerations, $\exists h \in I\!\!R^n$, $h \neq 0$, simultaneously perpendicular to $\mathcal{A}(t^*; (x_0, 0))$ and to $O(t^*)$ em $x^*(t^*)$, i.e., $\forall z \in O(t^*)$ and $\forall x \in \mathcal{A}(t^*; (x_0, 0))$,

$$h^T z \leq h^T x^*(t^*) \leq h^T x.$$

From here, we have (21) and, by writing x and $x^*(t^*)$, as the state at t^* as response of the system, respectively, to arbitrary admissible control and the optimal control, (20), we obtain the optimality conditions.

The Linear Quadratic Regulator Problem

$$(P_4) \quad \text{Minimize } \frac{1}{2}x^T(1)Sx(1) + \frac{1}{2}\int_0^1 \left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right] dt,$$

by choosing $(x, u) : [0, 1] \to I\!R^n \times I\!R^m$ such that:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad [0,1]\ \mathcal{L}\text{-a.e.}, \\ x(0) &= x_0 \in I\!R^n. \end{aligned}$$

$S \in I\!R^{n\times n}$ and $Q(t) \in I\!R^{n\times n}$ are positive semi-definite, $\forall t \in [0, 1]$, and $R(t) \in I\!R^{m\times m}$ is positive definite, $\forall t \in [0, 1]$.

Optimality Conditions.

The solution to (P_4) is given by

$$u^*(t) = -R^{-1}(t)B^T(t)S(t)x^*(t)$$

where $S(\cdot)$ is solution to the Riccati equation:

$$\begin{aligned} -\dot{S}(t) &= A^T(t)S(t) + S(t)A(t) - S(t)B(t)R^{-1}(t)B^T(t)S(t) + Q(t), \ \forall t \in [0,1], && (22) \\ S(1) &= S. && (23) \end{aligned}$$

Observations:

(a) The optimal control is defined as a linear state feedback law.

(b) The Kalman gain, $K(t) := R^{-1}(t)B^T(t)S(t)$, can be computed a priori.

Exercise: Given $\|a\|_P = a^T Pa$, show that the cost function on $[t, 1]$ is:

$$\frac{1}{2}x^T(t)S(t)x(t) + \frac{1}{2}\int_t^1 \|R^{-1}(s)B^T(s)S(s)x(s) + u(s)\|_R^2 ds.$$

Obviously that, for u^*, the above integrand becomes zero, and the optimal cost on $[0, 1]$ is equal to $\frac{1}{2}x_0^T S(0)x_0$.

Sketch of the Proof

Consider the pseudo-Hamiltonian (or Pontryagin function) to be optimized by the optimal control value

$$H(t,x,p,u) := p^T[Ax+Bu] + \frac{1}{2}[x^TQ(t)x + u^TR(t)u],$$

where p is the adjoint variable s.t.

$$\begin{aligned} -\dot{p}(t) &= \nabla_x H(t,x^*(t),p(t),u^*(t)) \;=\; A^T(t)p(t) + Q^T(t)x^*(t), \quad [0,1]\ \mathcal{L}\text{-a.e.}, \\ p(1) &= Sx^*(1). \end{aligned}$$

Thus, from $\nabla_u H(t,x^*(t),p(t),u)|_{u=u^*(t)} = 0$, we have $u^*(t) = -R^{-1}(t)B^T(t)p(t)$, and this enables the elimination of the control from the dynamics and the adjoint equation.

We have a system of linear differential equations in x and p. This, with $p(1) = Sx(1)$, implies the linear dependence of p in x, $\forall t$, i.e.,

$$\exists S : [0,1] \to \mathbb{R}^{n\times n} \text{ s.t. } p(t) = S(t)x(t).$$

After some simple algebraic operations, we conclude that $S(\cdot)$ satisfies (22) with the boundary condition. (23).

Example - Formulation

Let us consider the following dynamic system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad [0,1]\ \mathcal{L}\text{-a.e.}, \\ x(0) &= x_0, \\ y(t) &= Cx(t), \end{aligned}$$

with the following objective function

$$\frac{1}{2}x^T(T)Sx(T) + \frac{1}{2}\int_0^T \left[\|y(t) - y_r(t)\|^2 + u(t)^T R(t)u(t)\right] dt,$$

where S and $R(t)$ are symmetric, positive definite matrices.

a) Write down the maximum principle conditions for this problem.

b) Derive the optimal control in a state feedback form when $y_r(t) = Cx_r(t)$, $x_r(0) = x_0$ and $\dot{x}_r(t) = A_r x_r(t)$.

Example - Solution

a) Let

$$H(t, x(t), u(t), p(t)) := p^T(t)\,[Ax(t) + Bu(t)] + \left[\|y(t) - y_r(t)\|^2 + u^T(t)R(t)u(t)\right]$$

where $p : [0, 1] \to I\!R^n$ satisfies:

$$\begin{aligned} p(T) &= Sx^*(T), \\ -\dot{p}(t) &= A^T(t)p(t) + C^T C[x^*(t) - x_r(t)]. \end{aligned}$$

The optimal control u^* maximizes the map

$$v \to p^T(t)Bv + v^T R(t)v, \qquad \forall t \in [0, T].$$

From here, we conclude that

$$u^*(t) = -R^{-1}(t)B^T p(t).$$

Example - Solution (cont.)

b) By considering

$$z := \begin{bmatrix} x \\ x_r \end{bmatrix}, \; \bar{A} := \begin{bmatrix} A & 0 \\ 0 & A_r \end{bmatrix}, \; \bar{B} := \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}, \; \bar{S} := \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}, \; \bar{Q} := \begin{bmatrix} C^TC & -C^TC \\ -C^TC & C^TC \end{bmatrix},$$

we obtain the following auxiliary problem:

$$\text{Minimize} \quad \frac{1}{2} z^T(T)\bar{S}z(T) + \frac{1}{2}\int_0^T \left[z^T(t)\bar{Q}z(t) + u^T(t)R(t)u(t) \right] dt,$$

$$\text{such that} \quad z(0) = \begin{bmatrix} x_0 \\ x_0 \end{bmatrix}, \quad \text{and} \quad \dot{z}(t) \;=\; \bar{A}z(t) + \bar{B}u(t), \quad [0,T] \; \mathcal{L}\text{-a.e. .}$$

From the optimality conditions, we have $u^*(t) = -K_1(t)x^*(t) - K_2(t)x^r(t)$, where

$$K_1(t) = R^{-1}(t)B^T S_1(t), \; K_2(t) = R^{-1}(t)B^T S_2(t),$$

and $S_1(\cdot)$ e $S_2(\cdot)$ satisfy, respectively,

$$-\dot{S}_1(t) = A^T S_1(t) + S_1(t)A - S_1(t)BR^{-1}(t)B^T S_1(t) + C^TC \;\; \text{and}$$
$$-\dot{S}_2(t) = A^T S_2(t) + S_2(t)A - S_1(t)BR^{-1}(t)B^T S_2(t) - C^TC$$

in the interval $[0,T]$, with $S_1(T) = S$ and $S_2(T) = 0$.

Additional Bibliography

- Pontryagin, L., Boltyanskii, V., Gamkrelidze, R., Mishchenko, E., “The Mathematical Theory of Optimal Processes”, Pergamon-Macmillan, 1964.
- Anderson, A., Moore, B., ”Linear Optimal Control”, Prentice-Hall, 1971.
- Athans, M., Falb, P., ”Optimal Control”, McGraw Hill, 1966.
- Aubin , J., Cellina, A., ”Differential Inclusions: Set-Valued Maps and Viability Theory”, Springer-Verlag, 1984.
- Bryson, A., Ho, Y., ”Applied Optimal Control”, Hemisphere, 1975.
- Clarke F.H., Ledyaev Yu.S., Stern R.J., Wolenski P.r., “Nonsmooth Analysis and Control Theory”, Springer, 1998.
- Grace, A., ”Optimization Toolbox, User’s Guide”, The Math. Works Inc., 1992.
- Lewis, F., ”Optimal Control”, John Wiley Sons, 1987.
- Macki, J., Strauss, A., ”Introduction to Optimal Control Theory”, Springer-Verlag, 1981.
- Monk, J. et al., ”Control Engineering, Unit 15 - Optimal Control”, 1978.
- Neustadt, L., ”Optimization, A Theory of Necessary Conditions”, Princeton University Press, 1976.
- Pesch, H., Bulirsch, R., ”The Maximum Principle, Bellman’s Equation and Carathéodory’s Work”, Historical Paper in J. of Optimization Theory and Applications, Vol. 80, No. 2, pp. 199-225, 1994.
- Tu, P., “Introductory Optimization Dynamics”, Springer-Verlag, 1984.