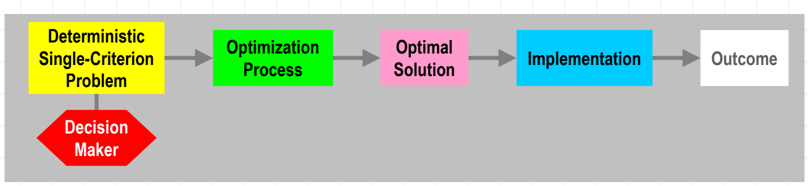
Multicriteria Decision-Aid

basic concepts and definitions

Manuel Matos
INESC Porto & FEUP
Portugal

The role of the decision maker

- Deterministic, single-criterion problems
  - The DM participates only in the problem formulation
  - The rest of the process is mainly technical, leading (hopefully) to the optimal solution
  - The decision is embedded in the problem formulation



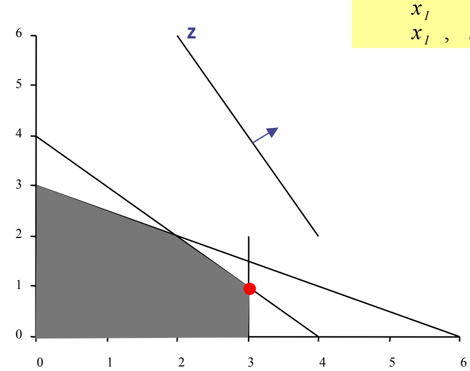
Trivial decision problems

- Minimize Cost

n	Cost
1	65
2	58
3	72
4	72
5	60
6	65
7	71
8	51
9	67
10	90
11	67
12	86
13	66
14	52
15	76

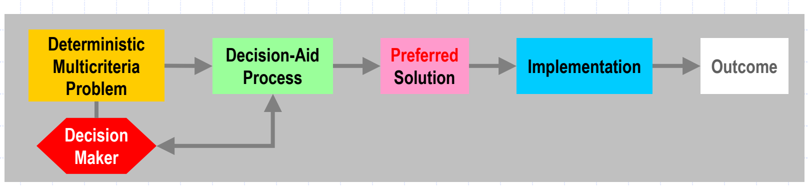
- Maximize profit z

$$\begin{aligned} &\max\ z = 2x_1 + x_2 \\ &suj:\ \ x_1 + x_2 \le 4 \\ &\qquad\ x_1 + 2x_2 \le 6 \\ &\qquad\ x_1 \le 3 \\ &\qquad\ x_1\ ,\ x_2 \ge 0 \end{aligned}$$



The role of the decision maker

- Deterministic, multicriteria problems
  - The DM participates in the problem formulation
  - The structure of preferences of the DM must be incorporated in the problem
  - The process leads to the preferred solution



Multicriteria problems

- Minimize Cost
- Maximize Reliability

n	Cost	Reliability
1	65	0.994586
2	58	0.993677
3	72	0.995333
4	72	0.995531
5	60	0.994064
6	65	0.994641
7	71	0.995954
8	51	0.992906
9	67	0.995111
10	90	0.998551
11	67	0.995425
12	86	0.997641
13	66	0.994653
14	52	0.992848
15	76	0.995913

- Maximize profit z_1
- Maximize export z_2

$$\max\ z_1 = 2x_1 + x_2$$
$$\max\ z_2 = x_2$$
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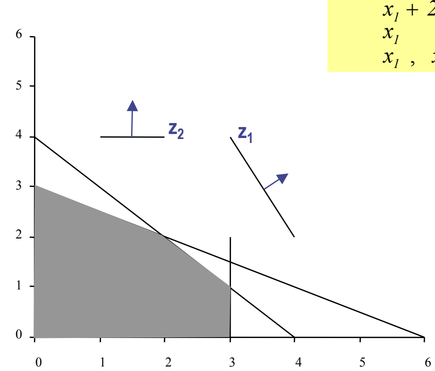
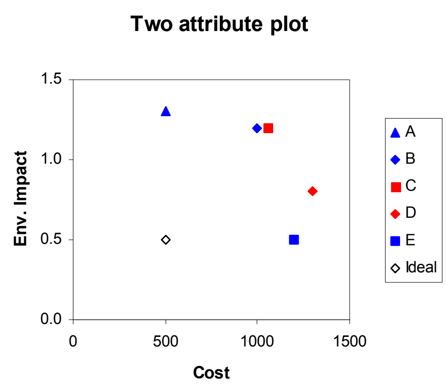


Some definitions

- Dominated (inferior) alternative
  - A solution is dominated iff there exists another one that is better in at least one criterion, without being worse in any of the remaining criteria
- Efficient (nondominated, noninferior, Pareto optimal) alternative
  - A solution is efficient iff it is not dominated by any other feasible alternative
- Ideal
  - (Non feasible) solution that joins up the individual optima
  - Defined only in the attributes' space

Example

- E dominates D
  - **E** is strictly better than **D** in both criteria
- B dominates C
  - **B** is strictly better than **C** in the **Cost** criterion
  - **B** is not worse than **C** in any criterion
- C and D are dominated
- A, B and E are efficient
  - They are not dominated by any other alternative



NB:
A possible rank: B, C, E, D, A

Examples

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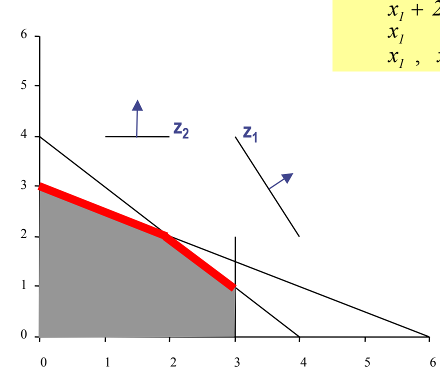

Examples

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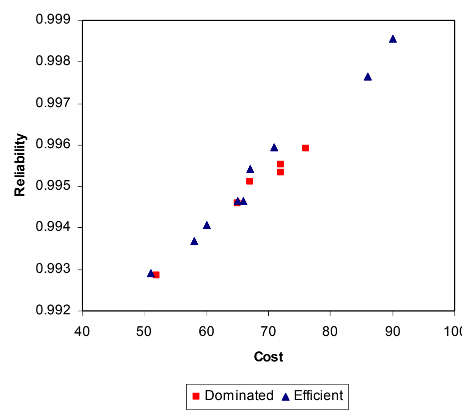
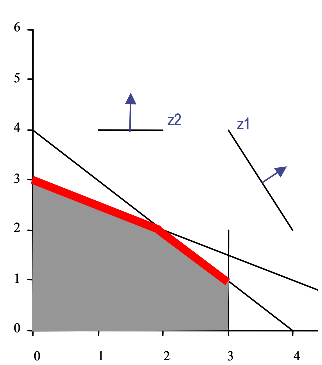


Decision space vs attribute's space

- Decision space



- Attribute's space

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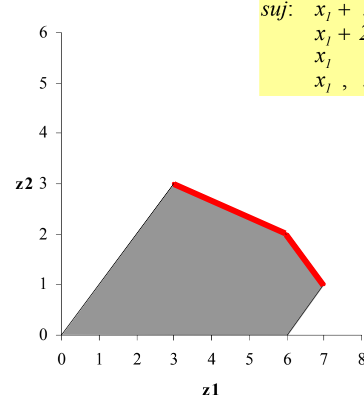
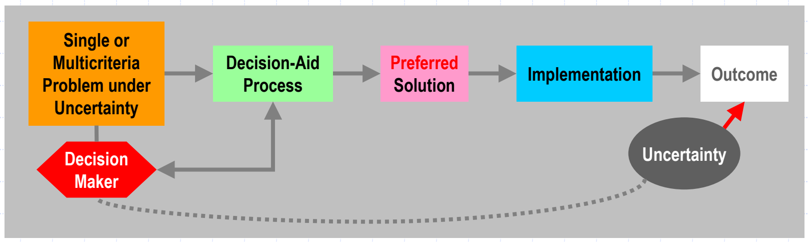


The role of the decision maker

- Single or multicriteria problems under uncertainty
  - The DM participates in the problem formulation and in the uncertainty characterization
  - The preferred solution results from the incorporation in the problem of the structure of preferences of the DM, including its risk attitude



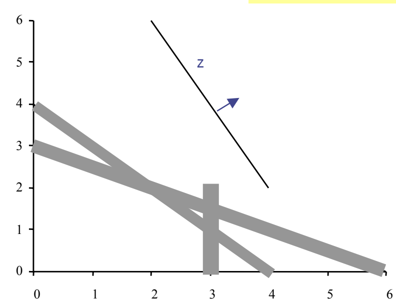
Different types of uncertainty

- Probabilistic - Different scenarios with probabilities

	Cost		
n	C1 (0.1)	C2 (0.6)	C3 (0.3)
1	59	65	75
2	50	58	71
3	68	72	60
4	69	72	62
5	53	60	63
6	51	59	65
7	68	71	77
8	56	57	75
9	62	58	80
10	62	55	70

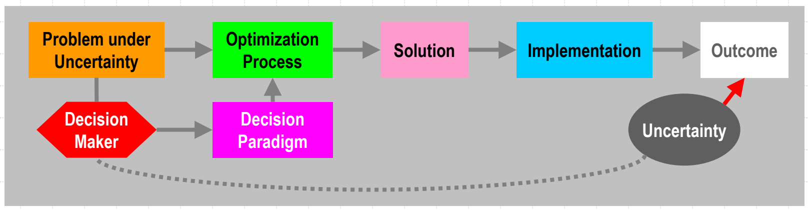
- Fuzzy - Vague or imprecise constraints

$$
\begin{aligned}
&\max\ z = 2x_1 + x_2 \\
&suj:\quad x_1 + x_2 \ \tilde{\leq}\ 4 \\
&\qquad\quad x_1 + 2x_2 \ \tilde{\leq}\ 6 \\
&\qquad\quad x_1 \ \tilde{\leq}\ 3 \\
&\qquad\quad x_1\ ,\ x_2 \geq 0
\end{aligned}
$$



The role of the decision maker

- Problems under uncertainty
  - Sometimes, the risk attitude of the DM is incorporated in the form of a pre-defined decision paradigm (expected value, regret, etc.)
  - This leads generally to an optimization process



Use of decision paradigms (or rules)

- Original problem
  - Dominated solutions shown

n	Cost C1 (0.3)	Cost C2 (0.6)	Cost C3 (0.1)
1	59	65	75
2	**50**	58	71
3	68	72	**60**
4	69	72	62
5	53	60	63
6	51	59	65
7	68	71	77
8	56	57	75
9	62	58	80
10	62	**55**	70

- Min E(Cost)

n	Expected Cost
1	64.2
2	**56.9**
3	69.6
4	70.1
5	58.2
6	57.2
7	70.7
8	58.5
9	61.4
10	58.6

- Minimax Cost

n	Minimax Cost
1	75
2	71
3	72
4	72
5	**63**
6	65
7	77
8	75
9	80
10	70

Modeling

- Identification of
  - Agents (DM, regulators, competitors, consumers, etc)
  - Relevant criteria (how to compare the outcomes of two alternatives)
  - Main uncertainties
  - Alternatives
    - in the case of multiattribute problems
- Formulation of
  - Decision variables
  - External variables and parameters
  - Coherent family of **criteria**
  - Attributes
    - How to measure the satisfaction in each criterion
    - (e.g. **Criterion** – Minimize environmental impact. **Attribute** - %CO_2

Modeling

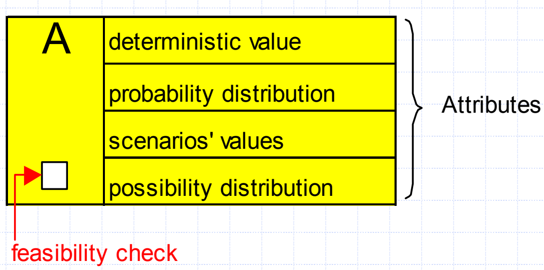
- A coherent family of criteria must be:
  - Exhaustive – All important points of view must be included
  - Consistent – If two alternatives A and B are equivalent except in criterion k, and A_k is better than B_k, then A must be at least as good as B
  - Non-redundant - Eliminating a criterion leads to the violation of one of the preceding axioms
- Other desirable proprieties
  - Legibility - The number of criteria used must be relatively low
  - Operationality - The family of criteria must be accepted by the stakeholders and the decision makers

Modeling

- Impact
  - Outcome of each particular decision (e.g. objective functions)
- Physical model
  - How to evaluate feasibility (e.g. mathematical constraints)
- Forecasting and estimation
  - Traditional (expected consumptions, wind power, etc)
  - Agents’ behavior (demand curves, offer curves, criteria, etc)
- Uncertainty
  - Probability distributions
  - Scenarios (with or without probabilities)
  - Possibility distributions (fuzzy sets)

Alternatives

- Alternatives may be explicit (MA) or implicit (MO)
- To be a candidate, an alternative must be feasible
  - or *almost* feasible
- Decisions are made based on the attributes of each alternative



Multiattribute problems

Alternatives	C_1	C_2	Criteria	C_m
A_1	a_{11}	a_{12}	$\cdots$	a_{1m}
A_2	a_{21}	a_{22}	$\cdots$	a_{2m}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
A_n	a_{n1}	a_{n2}	$\cdots$	a_{nm}

Attributes
may be
real numbers, intervals,
probability distributions,
possibility distributions,
qualitative labels

- Main characteristics
  - The alternatives are completely defined and assumed feasible
  - Attributes may be determinist, probabilistic, fuzzy (or mixed)
  - The problem may be: choice, ranking or sorting

Multiobjective problems

$$\begin{array}{ll} \textit{min} & \mathbf{f}(\mathbf{x}) \\ \textit{st:} & \mathbf{g}(\mathbf{x}) = \mathbf{0} \\ & \mathbf{h}(\mathbf{x}) \leq \mathbf{0} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

$\mathbf{x}$	vector of decision variables *(may include integer or binary variables)*
$\mathbf{f}(\mathbf{x})$	vector of objective functions
$\mathbf{g}(\mathbf{x})$	set of equality constraints
$\mathbf{h}(\mathbf{x})$	set of inequality constraints

- Main characteristics
  - Alternatives are not known in advance
  - Optimization procedures are always needed
  - May have a big number of constraints and decision variables
  - May not be completely described by the mathematical formulation
  - Planning problems are generally combinatory

Multicriteria analysis - main approaches

- Ensure that the DM follows a "rational" behavior (Normative option)
- Give some advice based on reasonable (but not indisputable) rules
- Find the preferred solution from partial decisions about decision hypothesis
- Prepare decision sets

- *Value functions, Utility theory, distance to the Ideal*
- *The French School*
- *Interactive methods*
- *Generation methods*
  Filtering of efficient solutions

from Benjamin Franklin to the President

Dear Sir,

In the affair of so much importance to you, wherein you ask my advice, I cannot, for want of sufficient premises, advise you what to determine, but if you please I will tell you how. When those difficult cases occur, they are difficult, chiefly because while we have them under consideration, all the reasons pro and con are not present to the mind at the same time; but sometimes one set present themselves, and at other times another, the first being out of sight. Hence the various purposes or informations that alternatively prevail, and the uncertainty that perplexes us. To get over this, my way is to divide half a sheet of paper by a line into two columns; writing over the one Pro, and over the other Con. Then, during three or four days consideration, I put down under the different heads short hints of the different motives, that at different times occur to me, for or against the measure. When I have thus got them all together in one view, I endeavor to estimate their respective weights; and where I find two one on each side, that seem equal. I strike them both out. If I find a reason pro equal to some two reasons con, I strike out the three. If I judge some two reasons con, equal to three reasons pro, I strike out the five; and thus proceeding I find at length where the balance lies; and if, after a day or two of further consideration, nothing new that is of importance occurs on either side, I come to a determination accordingly. And, though the weight of the reasons cannot be taken with the precision of algebraic quantities, yet when each is thus considered, separately and comparatively, and the whole lies before me, I think I can judge better, and am less liable to make a rash step, and in fact I have found great advantage from this kind of equation, and what might be called moral or prudential algebra.

Wishing sincerely that you may determine for the best, I am ever, my dear friend, yours most affectionately.

B. Franklin

Multiobjective problems

Manuel Matos
INESC Porto & FEUP
Portugal

Multiobjective problems

$$\begin{array}{ll} min & \mathbf{f}(\mathbf{x}) \\ st: & \mathbf{g}(\mathbf{x}) = \mathbf{0} \\ & \mathbf{h}(\mathbf{x}) \leq \mathbf{0} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

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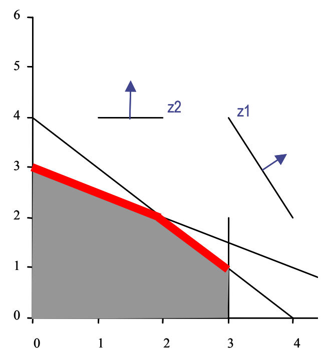
- Main characteristics
  - Alternatives are not known in advance
  - Optimization procedures are always needed
  - May have a big number of constraints and decision variables
  - May not be completely described by the mathematical formulation
  - Planning problems are generally combinatory
  - Sometimes interpreted as optimization problems with more than one objective function (vector optimization)

Decision space vs attribute's space

- Decision space



- Attribute's space

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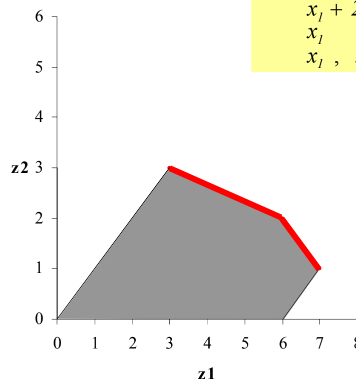
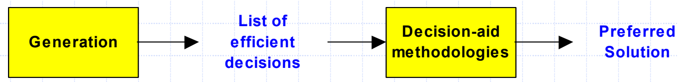


MO problems – basic strategies

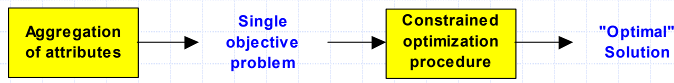
- Use of a value function
  - Transforms the problem into an optimization one
- Interactive methods
  - Based on an implicit value function (never explicitly known!)
    - Geoffrion-Dyer-Feinberg, Surrogate Worth Trade-off, Zionts-Wallenius
  - Without special conditions
    - STEM, Trimap
- Generation methods
- Goal programming

Multiobjective approaches

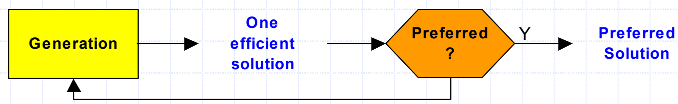
- Two phase (or generation)



- Aggregation



- Interactive



Some arguments

Strategy	Pro	Con
Value Function	Leads to optimization Induces a total order No further intervention of the DM	Difficulties in building the VF Some arbitrariness Tendency to predefinitions and confusion between OF and VF
Interactive	Reduces information overload Easier calculations (in general) Induces learning	Loss of holistic vision Produces only a final solution May need many judgments
Generation	Doesn't have parameters Gives the global picture Doesn't require the DM's presence	Doesn't produce a solution or an order Risk of generating to many solutions Heavy calculations
Goal Prog.	Well established in OR Easy to apply Adequate to large dimension problems	Only linear problems Needs goal definition Requires a lexicographic order of the criteria (no compensation)

Use of value functions

$$
\begin{aligned}
&\textit{min} && \mathbf{f}(\mathbf{x}) \\
&\textit{st:} && \mathbf{g}(\mathbf{x}) = \mathbf{0} \\
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$$v(\mathbf{x}) = v[f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})]$$

$$
\begin{aligned}
&\textit{max} && v(\mathbf{x}) \\
&\textit{st:} && \mathbf{g}(\mathbf{x}) = \mathbf{0} \\
& && \mathbf{h}(\mathbf{x}) \leq \mathbf{0} \\
& && \mathbf{x} \geq \mathbf{0}
\end{aligned}
$$

Interactive approaches

(typically, only for MO linear problems)

- General procedure
  1. Find an initial solution (efficient)
  2. Ask the DM if he is satisfied → if he is, this is the preferred solution. STOP
  3. Ask the DM which criteria he wants to improve and which criteria he accepts to worsen
  4. Use the precedent information to find a new solution
  5. Return to 2

- Some classics
  - STEM
    - STRANGE
  - Zionts-Wallenius
  - Interval Criterion Weights
  - Pareto Race
  - Trimap

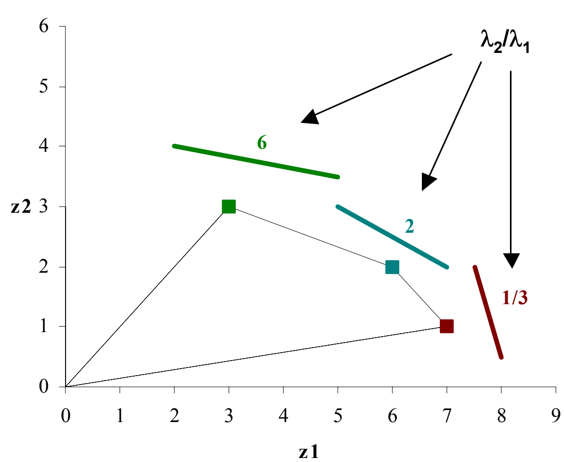
Generation methods

- Generate a set of efficient solutions
  - Parametric variation of $\lambda>0$ in

$$\min f(\mathbf{x}) = \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x})$$

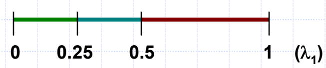
    - The optimal solution of this auxiliary problem is an efficient solution of the original multiobjective problem
    - The parameters λ are only instrumental (not judgments of the DM)
  - Constrained optimization
    - Define additional constraints in n-1 objective functions
    - Optimize the remaining objective function
    - Repeat for different RHS values of the additional constraints
  - Multiobjective simplex

Parametric variation

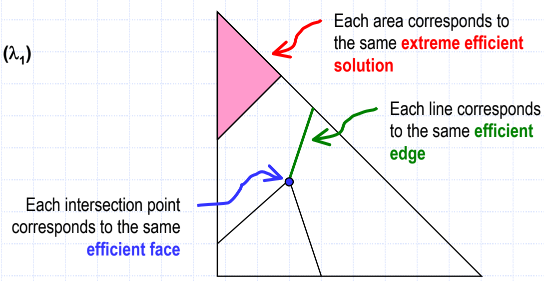


Parametric variation

- In MO linear problems, post-optimization (parametric analysis) can be used to find all the efficient solutions
  - e.g. (previous problem)

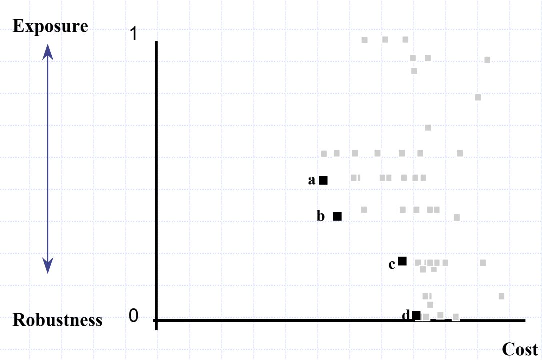


e.g. (tricriteria problem)



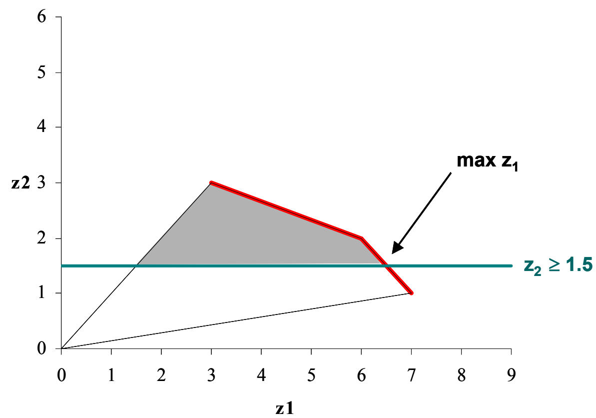
Parametric variation

- Difficulties in discrete problems
  - Some efficient solutions are never selected



a,b,c,d - Efficient solutions
c - "Convex dominated" **but not dominated**

Constrained optimization (ε - constraint)



Compound and emergent strategies

- Generation > Filtering
  - Use aspiration levels and elimination rules
  - Reduces the number of alternatives to consider
  - Still doesn't produces a solution or order
- Generation > Multiattribute method
  - Constitutes a complete approach
  - Opens the way to the use of less prescriptive methodologies
- Meta-heuristics e multiobjective genetic algorithms
  - Adequate for MO problems with integer or binary variables
  - Explore the efficient zone (or part of it)
  - May include interactivity

Multiattribute problems

Manuel Matos

INESC Porto & FEUP

Portugal

Multiattribute problems

Alternatives	C_1	C_2	Criteria	C_m
A_1	a_{11}	a_{12}	$\cdots$	a_{1m}
A_2	a_{21}	a_{22}	$\cdots$	a_{2m}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
A_n	a_{n1}	a_{n2}	$\cdots$	a_{nm}

Attributes

may be

real numbers, intervals,
probability distributions,
possibility distributions,
qualitative labels

- Main characteristics
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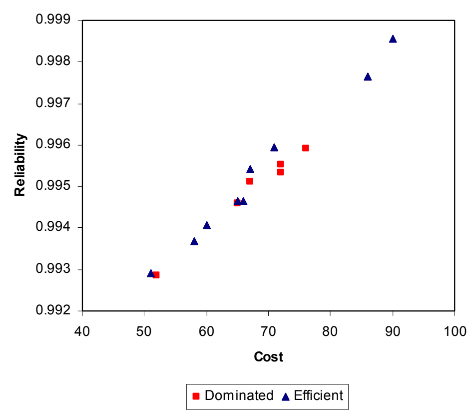
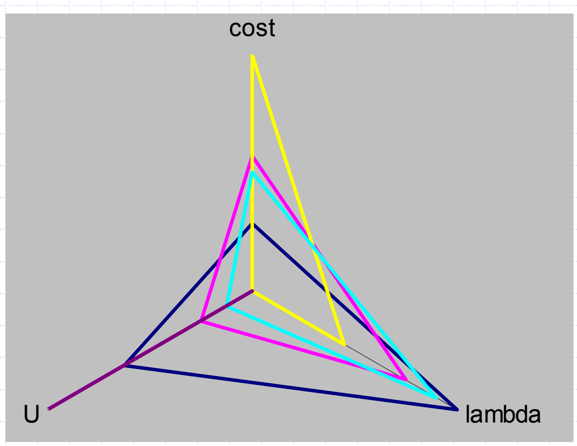


Example

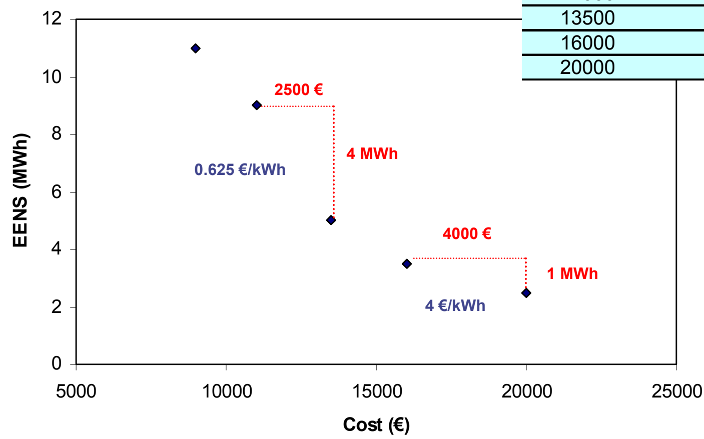
- 5 alternatives in 3 criteria



Trade-off analysis

- 5 possible investment plans

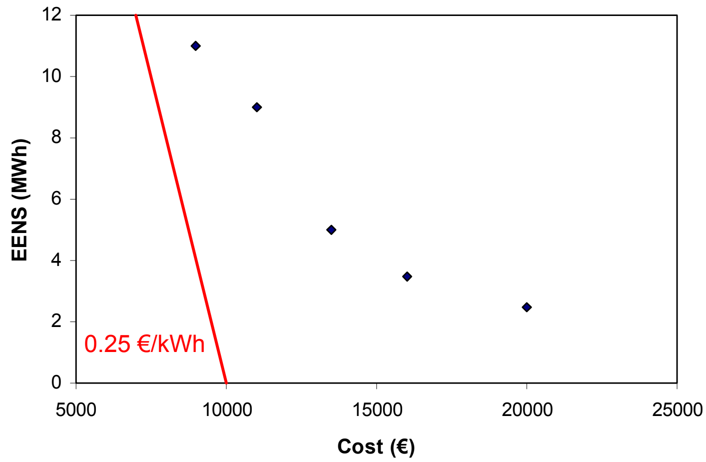
Cost (€)	EENS (MWh)
9000	11
11000	9
13500	5
16000	3.5
20000	2.5



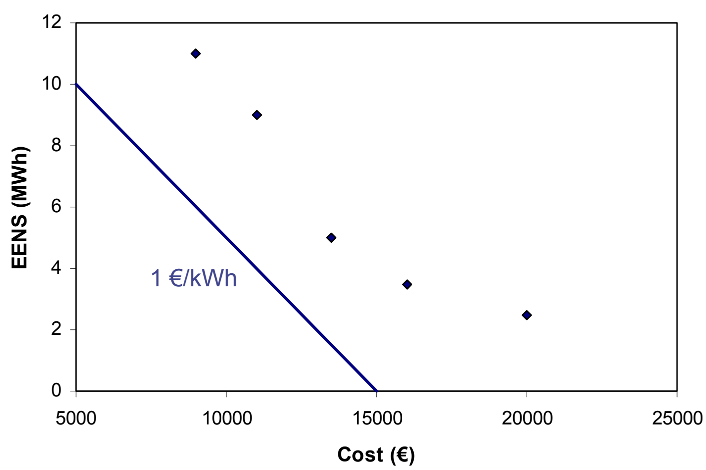
Trade-off analysis

- When comparing B to A (two efficient alternatives)
  - We gain something in one criterion
  - We loose something in another criterion
- If we have a reference value for the trade-off
  - We know immediately if we prefer A or B
  - It's easy to select the preferred alternative

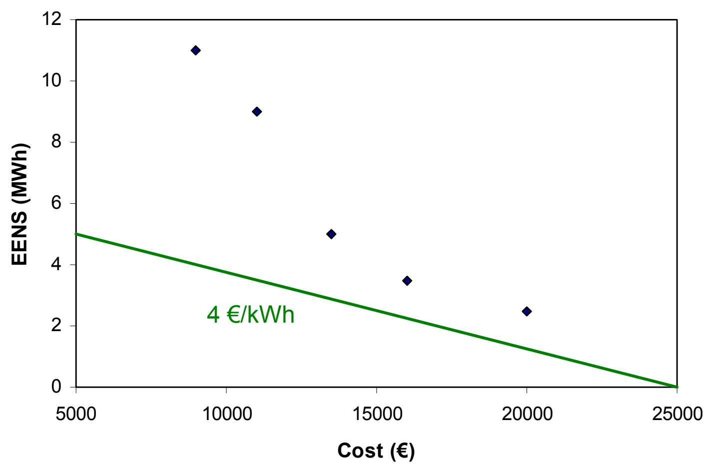
Trade-off analysis



Trade-off analysis

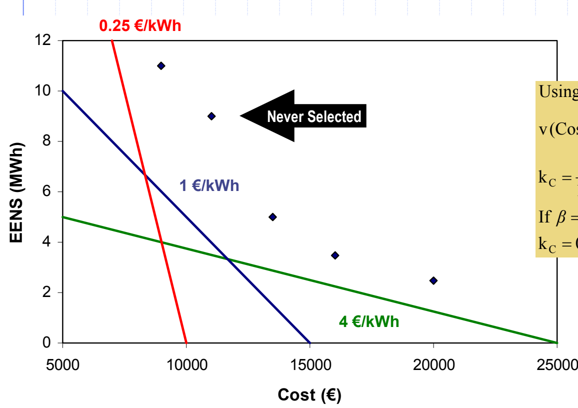


Trade-off analysis



Trade-off analysis

- Each trade-off β defines a family of indifference lines
  - f(Cost, EENS) = Cost + β. EENS β in €/MWh



Using a normalized **value function** :

$$v(\text{Cost}, \text{EENS}) = k_C \frac{20000 - \text{Cost}}{20000 - 9000} + k_E \frac{11 - \text{EENS}}{11 - 2.5}$$

$$k_C = \frac{11000}{11000 + 8.5\beta} \qquad k_E = \frac{8.5\beta}{11000 + 8.5\beta}$$

If $\beta = \mathbf{1000\ €/MWh}$

$k_C = 0.564 \quad k_E = 0.436$

Trade-off analysis

- Conclusions:
  - Constant trade-offs lead to linear indifference curves
  - … and to linear value functions
  - … with constant weights
    - that have no special meaning as indicators of the relative importance of the criteria in general

- Important issues
  - The process may be extended to more than two criteria
  - Trade-offs are not always constant
    - e.g. beyond a certain level, your willingness to pay for extra reliability decreases
  - … leading to non-linear indifference curves

Summarizing

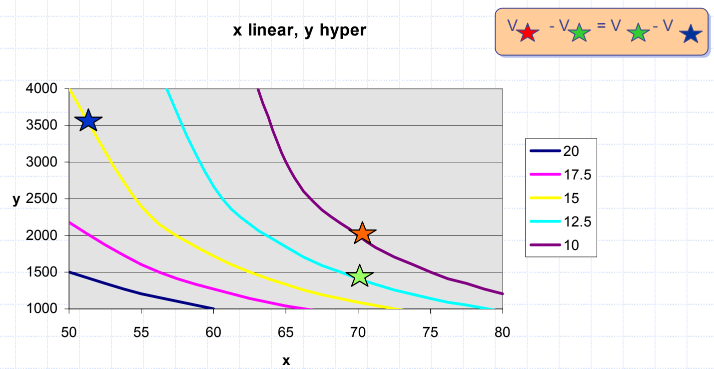
- Indifference curve (attribute space)
  - Set of the alternatives that are valued the same way by the Decision Maker
  - The indifference curves completely describe the structure of preferences of the Decision Maker

- Trade-off between two attributes X and Y
  - What you must lose in X to increase one unit in Y, without leaving the indifference curve (slope of the curve)

- Weights
  - If and only if the trade-offs are constant, weights are constant

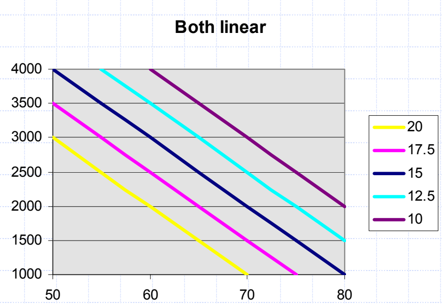
Indifference curves

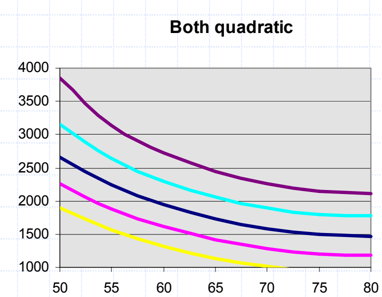
- Indifference curves join all the points with the same *global value*
- The DM is indifferent between two points in the same curve



Indifference curves

- Other (additive) value functions...





Minimization in both criteria
Value scale (20≻17.5, etc)

Value functions

- A formal way to address multiattribute problems
  - Sometimes also called deterministic utility functions
- Requires
  - Verifying assumptions
  - Construction of the individual value functions
  - Indifference judgments to build the multiattribute value function
- Difficulties
  - Building individual value functions
- Problems
  - Tendency to use naïve weights asked directly to the DM

BIG MISTAKE!

Value functions - existence

- If Z is a subset of R_m
  - i.e. if each alternative A is described by m attributes (A_1, A_2, ..., A_m)
- and
  - ($A \geq B$ and $A \neq B$) $\Rightarrow A \succ B$ for all A, B $\in$ Z
  - For all A, B, C $\in$ Z such that $A \succ B \succ C$, it exists exactly one $\lambda \in (0, 1)$ such that $B \sim [\lambda.A + (1-\lambda).C]$
    (Archimedean Condition)
- Then, it exists a real value function v() such that:
  - $A \succ B \Leftrightarrow v(A) > v(B)$
  - $A \sim B \Leftrightarrow v(A) = v(B)$

Independence and additivity (m>2)

- Given a set of attributes K, a subset X of K is said to be preferentially independent (p.i.) from its complement Y=K-X iff, for a particular value P_Y

$$(A_X, P_Y) \succeq (B_X, P_Y) \Rightarrow (A_X, Q_Y) \succeq (B_X, Q_Y)$$

  - stands for all Q_Y, A and B being arbitrary.

- A set K is mutually preferentially independent (m.p.i.) if every subset X of K is p.i. from its complement K-X

- For three or more criteria (m>2), this is a sufficient condition to additivity: $A \succeq B \Rightarrow v_1(A_1)+\ldots+v_m(A_m) \geq v_1(B_1)+\ldots+v_m(B_m)$

Additivity (m=2)

- For two criteria, an additional condition is necessary for additivity
  - For instance, the Thomsen condition:

$$\begin{gathered}\text{For all } P, Q, A \\ (P_1, A_2) \sim (A_1, Q_2) \text{ and } (A_1, P_2) \sim (Q_1, A_2) \Rightarrow (P_1, P_2) \sim (Q_1, Q_2)\end{gathered}$$

  - or the cancellation condition
    - also guaranties that K is m.p.i.

$$\begin{gathered}\text{For all } P, Q, A \\ (P_1, A_2) \succeq (A_1, Q_2) \text{ and } (A_1, P_2) \succeq (Q_1, A_2) \Rightarrow (P_1, P_2) \succeq (Q_1, Q_2)\end{gathered}$$

- More weak conditions exist for difficult cases

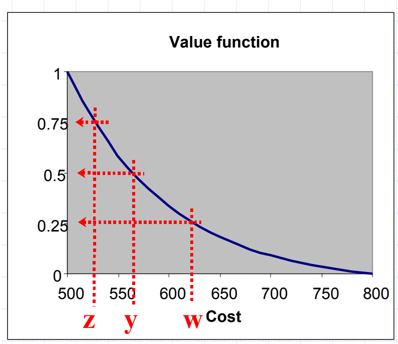
Building value functions

- Direct construction
  - Too complicated
- Verify preferential independence conditions
  - Then: $v(A) = \Psi(v_1(A_1), \ldots, v_m(A_m))$
- Check for additivity conditions...
  - If they hold: $v(A) = k_1 v_1(A_1) + k_2 v_2(A_2) + \ldots + k_m v_m(A_m)$
- ...or less restrictive conditions
  - That let you use (eg two normalized individual value functions)

$$v(A) = k_1 v_1(A_1) + k_2 v_2(A_2) + k_{12} v_1(A_1) v_2(A_2)$$

Building individual value functions

- Fix $v(x_{min})=1$, $v(x_{max})=0$
- Find *y* such that
  - $x_{max} \rightarrow y$ or $y \rightarrow x_{min}$
  - is indifferent to the DM
- Then, $v(y)=0.5$
- Repeat to find w (interval $[y, x_{max}]$)
  - *v(w)=0.25*
- and z (interval $[x_{min}, y]$)
  - *v(z)=0.75*
- ... (trace the curve)



- Verify! *The DM should be indifferent between $w \rightarrow y$ and $y \rightarrow z$*

MA value functions - parameters

- Assess the parameters k_1 and k_2 $v(A) = k_1 v_1(A_1) + k_2 v_2(A_2)$
  - Build "extreme" alternatives:

Ideal : best A_1, best A_2	P : best A_1, worst A_2	Q : worst A_1, best A_2
$v = 1, v_1 = 1, v_2 = 1$	$v_1 = 1, v_2 = 0$	$v_1 = 0, v_2 = 1$
$k_1 + k_2 = 1$	$v(P) = k_1$	$v(Q) = k_2$

  - Ask for a judgment (eg: P $\succeq$ Q, that implies $k_1 \geq k_2$)
  - Find M = (z, worst A_2) **~** Q
    - Then:

$$v(M) = v(Q) \Rightarrow k_1 v_1(z) = k_2$$

$$k_1 = \frac{1}{1 + v_1(z)} \qquad k_2 = 1 - k_1$$

    - **This is very different from asking directly for weights!**

Example

Cost (€)	EENS (MWh)
9000	11
11000	9
13500	5
16000	3.5
20000	2.5

$$v(\text{Cost}, \text{EENS}) = k_C \frac{20000 - \text{Cost}}{20000 - 9000} + k_E \frac{11 - \text{EENS}}{11 - 2.5}$$

- Build "extreme" alternatives:
  - P=(9000, 11), Q=(20000, 2.5)
- Search for an indifference
  - *P or Q?*
    - The DM says P $\succeq$ Q
  - *P'=(11000, 11) or Q?*
    - P' $\succeq$ Q
  - *P"=(12000, 11) or Q?*
    - Q $\succeq$ P"
  - M=(11500, 11) ~ Q=(20000, 2.5)
  - $v_C(11500)=0.773$
  - $k_C=0.564$ $k_E=0.436$

NB:
8500€ compensates 8.5 MWh
Trade-off = 1 €/kWh

$$v(M) = v(Q) \Rightarrow k_1 v_1(z) = k_2$$

$$k_1 = \frac{1}{1 + v_1(z)} \qquad k_2 = 1 - k_1$$

Minimum distance to the Ideal

- A possible decision paradigm for deterministic multiattribute problems
  - Induces an order in the set of the alternatives
  - May also be used in multiobjective problems
- Ideal (Zeleny)
  - (Non feasible) solution, defined only in the attributes' space, that joins up the individual optima
- Distance to the Ideal
  - w_k are scale factors
  - Choice of p is a decision problem!

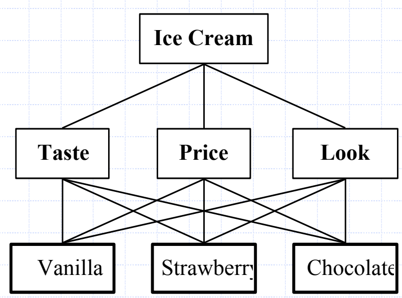
$$d_p(A, Ideal) = \sum_{k=1}^{m} w_k . \left(\left|A_k - Ideal_k\right|\right)^p$$

$$d_1(A, Ideal) = \sum_{k=1}^{m} w_k . \left|A_k - Ideal_k\right|$$

$$d_\infty(A, Ideal) = \max_k \left\{w_k . \left|A_k - Ideal_k\right|\right\}$$

AHP

- Analytic Hierarchy Process
  - Thomas Saaty
- Hierarchical organization of the criteria
- Comparison matrices
  - Between sub-criteria, regarding the parent criterion
  - Between alternatives, regarding a level 1 criterion
- Calculation of a final order of priorities



AHP: input and calculations (1)

- Input - Judgments about the relative preference of the alternatives, regarding each attribute
  - May be expressed by linguistic labels
  - Converted then to numbers (the Saaty scale)
  - Form a matrix of comparisons
  - Inconsistencies are allowed (to a certain degree)

- Calculations – The priorities on an attribute correspond to the greatest eigen-vector of its matrix
  - May be approximated by the average of the normalized columns

AHP: example

Taste

	Vanilla	Strawberry	Chocolate	
Vanilla	**1**	**3/2**	**5**	0.540
Strawberry	**2/3**	**1**	**3**	0.348
Chocolate	**1/5**	**1/3**	**1**	0.112

Price

	Vanilla	Strawberry	Chocolate	
Vanilla	**1**	**1/3**	**1**	0.185
Strawberry	**3**	**1**	**5**	0.659
Chocolate	**1**	**1/5**	**1**	0.156

Look

	Vanilla	Strawberry	Chocolate	
Vanilla	**1**	**1/5**	**1**	0.149
Strawberry	**5**	**1**	**4**	0.691
Chocolate	**1**	**1/4**	**1**	0.160

AHP: input and calculations (2)

- The process is repeated with the relative importance of the attributes
  - Or the relative importance of sub-attribute of an attribute

	Taste	Price	Look	
Taste	**1**	**5**	**7**	0.731
Price	**1/5**	**1**	**3**	0.188
Look	**1/7**	**1/3**	**1**	0.081

- Conclusion - global priorities of the alternatives

	Taste	Price	Look				
Vanilla	0.540	0.185	0.149		0.731		**0.442**
Strawberry	0.348	0.659	0.691	X	0.188	=	**0.434**
Chocolate	0.112	0.156	0.160		0.081		**0.124**

AHP: a surprise...

- Eliminating CHOCOLATE, but keeping the remaining judgments...

	Taste	Price	Look
Vanilla	3/5	1/4	1/6
Strawberry	2/5	3/4	5/6

- ... The following new global priorities are obtained !

Vanilla	**0.499**
Strawberry	**0.501**

AHP - comments

- Strong points
  - Easy to use and understand
    - Accepts linguistic labels
  - Flexible - allow small inconsistencies
  - Judgments substitute unavailable information
    - The attributes' values are not used in the calculations

- Weak points
  - Uses value ratio evaluations instead of value difference evaluations
    - *"How many times is alternative A preferred to B?"*
  - Rank reversal problems
  - Most of the work and conclusions are specific of the problem in hand

Decision-aid methodologies

- The French School of decision-aid proposes a number of methods that try to better model the structure of preferences of the DM, without prescribing a total order

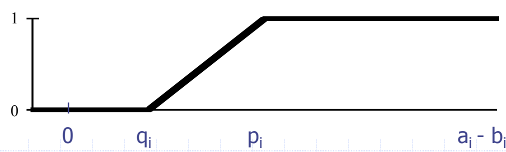
- The methodologies include
  - indifference thresholds
  - hesitations between strict preference and indifference (weak preference)
  - veto thresholds
  - incomparability situations
  - the complementary concepts of concordance and discordance

- Aggregation of preferences mainly by rules
  - as opposed to formulas

- Members of the family
  - ELECTRE I, IS, II, III, IV, Tri, PROMETHEE, GAIA

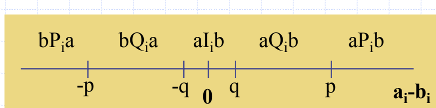
The French School

- Extension of the classic paradigm (P, I) by considering two additional situations:
  - Q - weak preference R – incomparability
- Definition, in each criterion i, of indifference limits q(i) and preference limits p(i), used to define intervals of indifference, weak preference and strict preference



Electre IV - basic ideas

- The method is based on pairwise comparisons between alternatives
- In each criterion i, some thresholds are defined:
  - q - indifference threshold
  - p - strict preference threshold
  - v - veto threshold
- We may have (alternatives **a** and **b**, maximization):



I - indifference
P - strict preference
Q - weak preference

Electre IV - procedure

- Aggregation rules
  - Comparison between alternatives **a** and **b** may lead to different types of dominance (quasi, canonic, pseudo, sub, veto) of **a** over **b** (or vice-versa), or to no dominance
    - Each alternative has a **qualification** (# situations where it dominates - # situations where it is dominated) for each type of dominance
- Distillation
  - Descending: begins with the alternatives with greater qualification
  - Ascending: begins with the alternatives with lesser qualification
    - In both cases, the effect of the selected alternatives is annulled on the remaining ones
- Final preoder
  - Combination of the two distillations

Electre IV – binary relations

- Quasi-dominance **-** The couple (b, a) verifies the relation of quasi-dominance if and only if:
  - for every criterion, b is either preferred or indifferent to a,
  - and if the number of criterion for which the performance of a is better than the one of b (a staying indifferent to b) is strictly inferior to the number of criteria for which the performance of b is better than the one of a.
- Canonic-dominance - The couple (b, a) verifies the relation of canonic-dominance if and only if:
  - for no criterion, a is strictly preferred to b,
  - and if the number of criteria for which a is weakly preferred to b is inferior or equal to the number of criteria for which b is strictly preferred to a,
  - and if the number of criteria for which the performance of a is better than the one of b is strictly inferior to the number of criteria for which the performance of b is better than the one of a.

Electre IV – binary relations

- Pseudo-dominance - The couple (b, a) verifies the relation of pseudo-dominance if and only if:
  - for no criterion, a is strictly preferred to b,
  - and if the number of criteria for which a is weakly preferred to b is inferior or equal to the number of criteria for which b is strictly or weakly preferred to a.
- Sub-dominance - The couple (b, a) verifies the relation of sub-dominance if and only if:
  - for no criterion, a is strictly preferred to b.
- Veto-dominance - The couple (b, a) verifies the relation of veto-dominance if and only if:
  - either for no criterion, a is strictly preferred to b,
  - or a is strictly preferred to b for only one criterion but this criterion not vetoing the outranking of a by b and furthermore, b is strictly preferred to a for at least half of the criteria.

Electre IV - illustration

- A small distribution planning problem

alternative	cost	lambda	U
A	1000	0.10	7
B	800	0.15	10
C	500	0.21	12
D	850	0.12	11
E	1200	0.30	4

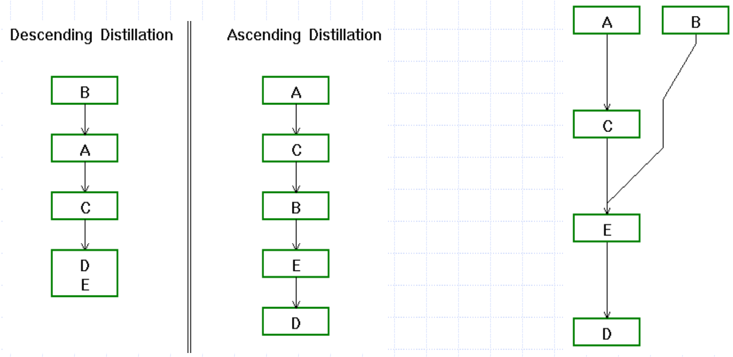
threshold	cost	lambda	U
q	50	0.05	0
p	150	0.1	2
v	500		6

1.0 - quasi
0.8 - canonic
0.6 - pseudo
0.4 - sub
0.2 - veto

	A	B	C	D	E
A	1	0	0.2	0.8	0.8
B	0	1	0	1	0
C	0	0.4	1	0.4	0
D	0	0.4	0	1	0
E	0	0	0	0	1

Electre IV - illustration

- Distillations and final preorder



Final remarks

- In deterministic multiattribute problems, the main issue is preference modeling
- Building correctly a value function may be a good approach, namely if automatic decisions are needed
  - Trade-off analysis is just a particular case
- Decision-aid methods are an interesting alternative when the DM desires a more detailed representation of his preferences
  - Very adequate when a large number of criteria exist