6.1 - Internal stability

$\dot{x} = F(x)$, $x(t) \in \mathbb{R}^n$

x^* is an equilibrium point of $\dot{x} = F(x)$ if $x(t) \equiv x^*$ is a solution of this equation.

Lemma x^* is an equilibrium point of $\dot{x} = F(x) \Leftrightarrow \boxed{F(x) = 0}$

An equilibrium point $x^* \in \mathbb{R}$ of $\dot{x} = F(x)$ is

- **stable if $\forall U$ neighborhood of x^* $\exists V$ a neighborhood of x^* such that $x(0) \in V \Rightarrow \phi(t, 0, x(0)) \in U$.**
- **asymptotically stable if it is stable and $\exists W$ a neighborhood of x^* such that $x(0) \in W \Rightarrow \lim_{t\to\infty} \phi(t, 0, x(0)) = x^*$.**
- **unstable if it is not stable.**

(A, B, C, D) is (internally) stable if $x^* = 0 \in \mathbb{R}^n$ is an asymptotically stable point of $\dot{x} = Ax$. In this case the matrix A is also said to be stable.

Theorem

The matrix A is stable $\Leftrightarrow \boxed{\sigma(A) \subset \mathbb{C}^- := \{\lambda \in \mathbb{C} : \Re(\lambda) < 0\}}$

Proof

- The solutions of $\dot{x} = Ax$ are of the form $\phi(t, 0, x(0)) = e^{At}x(0)$.
- The system is stable if and only if sse all the entries of the matrix e^{At} converge to zero.
- Let $\lambda_l = a_l + b_l i,\ l = 1, \ldots r$ be the distinct eigenvalues of A.
- Each entry ij of e^{At} is of the form $(e^{At})_{ij} = \sum_{l=1}^{r} \alpha_{ij}^l(t)$ where

$$\alpha_{ij}^l(t) = e^{a_l t}(p_{ij}^l(t)\cos b_l t + q_{ij}^l(t)\sin b_l t)$$

and $p_{ij}^l(t)$ e $q_{ij}^l(t)$ are polynomial functions of t.

- $a_l < 0,\ l = 1, \ldots r \Rightarrow (e^{At})_{ij} \to 0,\ i, j = 1 \ldots, n$
- For each eigenvalue λ_l there is a least one $\alpha_{i^*j^*}^l(t)$ which is nonzero.
- $(e^{At})_{i^*j^*} \to 0 \Rightarrow a_l < 0$.
- The entries of the matrix e^{At} converge to zero $\Leftrightarrow \Re(\lambda) < 0, \forall \lambda \in \sigma(A)$ ♣

6.2 - External stability

Bounded Input Bounded Output - BIBO satbility

(A, B, C, D) is **BIBO stable** if

$$\forall M > 0 \;\; \exists N > 0 : x(0) = 0, \|u(t)\| < M, t \geq 0 \Rightarrow \|y(t)\| < N, t \geq 0$$

Example $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1 \;\; 1], D = 0$

$y(t) = e^{2t}x_1(0) + e^{-t}x_2(0) + \int_0^t e^{-(t-\tau)}u(\tau)d\tau$

The system is BIBO stable: for zero inicial conditions, u bounded implies y bounded.

However: for $x_1(0) \neq 0$, u bounded does not imply y bounded.

Theorem The following conditions are equivalent:

1) (A, B, C, D) is BIBO stable

2) All the poles of the system transfer function belong to $\mathbb{C}^-$

3) $\int_0^\infty \|Ce^{At}B\|dt < \infty$.

Remark If $T(s) = [\frac{n_{ij}(s)}{d_{ij}(s)}]_{ij}$, $i = 1, \ldots, p$, $j = 1, \ldots, m$, where $n_{ij}(s)$ e $d_{ij}(s)$ are coprime polynomials (i.e., are polynomial without common factors), the poles of each entry ij of $T(s)$ are the roots of the polynomial $d_{ij}(s)$. The poles of $T(s)$ are the poles of its entries.

Proof - Exercise! Suggestion:

- Show that he poles of $C(sI - A)^{-1}B + D$ coincide with the poles of $C(sI - A)^{-1}B$

- Show that for each $\lambda = a + bi$ which is a pole of the ij entry of $C(sI - A)^{-1}B$, the entry ij of $Ce^{At}B$ contains a summand of the form $e^{at}(p(t)\cos t + q(t)\sin t)$ where the polynomials $p(t)$ and $q(t)$ are not simultaneously null.

Proposition (A, B, C, D) internaly stable $\Rightarrow$ (A, B, C, D) BIBO stable.

Proof

- The eigenvalues of A are the poles of $(sI - A)^{-1}$

- The $\mathcal{P}$ of poles of $C(sI - A)^{-1}B + D$ is contained in the set $\mathcal{A}$ of poles of $(sI - A)^{-1}$

- If A is stable: $\mathcal{P} \subset \mathcal{A} = \sigma(A) \subset \mathbb{C}^-$ ♣

Remark The reciprocal implication odes not hold (see previous example).

However ...

Proposition **If (A, B, C, D) is a minimal realization then: (A, B, C, D) internally stable $\Leftrightarrow$ (A, B, C, D) BIBO stable.**

Proof: **exercise (simple, taking into saccount the relationship between poles and eigenvalues for minimal realizations.)**