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ORIGINAL ARTICLE

Stabilization of discrete 2D behaviors by regular partial interconnection

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Abstract In this work, we consider linear, shift-invariant and complete two-dimensional (2D) discrete systems from a behavioral point of view. In particular, we examine behaviors with two types of variables: the variables that we are interested to control (the *to-be-controlled variables*) and the variables on which we are allowed to enforce restrictions (the *control variables*). The main purpose of this contribution is to derive necessary and sufficient conditions for the stabilization of the to-be-controlled variables by 'attaching' a controller to the control variables. This problem turns out to be related to the decomposition of a given behavior into the sum of two sub-behaviors. Moreover, we show that under certain conditions, it is possible to obtain a constructive solution and characterize the structure of the to-be-controlled behavior.

Keywords Two-dimensional behaviors · Partial regular interconnection · Stabilization · Finite-dimensional autonomous behaviors

Mathematics Subject Classification (2000) 93B05 · 93B07 · 93B25 · 93C05 · 93C35 · 93C65 · 93D15

1 Introduction

A behavior, denoted by $\mathfrak{B}$, is a set of trajectories that obey certain laws described by a mathematical model. In the case of 2D discrete systems, where the trajectories

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evolve over a two-dimensional domain, this model typically consists of a set of partial difference equations and the behavior is the corresponding solution set. In this context, control is viewed as the ability to impose adequate additional restrictions to the variables of the behavior to obtain a desired overall functioning pattern.

Most of the literature on behavioral control is concerned with the situation in which all variables of $\mathfrak{B}$ are available for control, i.e., it is allowed to impose extra restrictions on all the variables of $\mathfrak{B}$. We refer to this situation as *full* control or *full* interconnection [12,23].

Another important case considered in the literature is when the system variables are divided into two sets: the variables that we are interested to control (called *to-be-controlled variables*) and the variables on which we are allowed to enforce restrictions (called *control variables*). This situation is known as *partial* control or *partial* interconnection [1,7,13,16,19,24]. In this more involved situation, although we cannot act directly upon the to-be-controlled variables, we can nevertheless influence their dynamics by imposing restrictions on the control variables.

We confine ourselves to interconnections where the restrictions imposed by the controller do not overlap with the restrictions already active in the behavior. These type of interconnections are called *regular interconnections* and they are closely related to the notion of feedback control in the classical state–space systems, see [15,29].

In this paper, the control problem of interest is the problem of stabilization. In contrast with 1D behaviors, higher dimensional behaviors admit several notions of stability. Here, we adopt a notion of stability defined with respect to a stability cone as considered in [11,17].

The problem of stabilization is well understood for 1D behaviors in both contexts of full and partial control, see for instance [1,12,24]. However, stabilization of 2D and nD behaviors has only been studied in the context of full control, see for instance [11,17,20,27], with different underlying notions of stability. The main purpose of this contribution is to investigate the problem of stabilization of 2D behaviors by regular partial interconnection.

The structure of the paper is as follows. Section 2 is devoted to the introduction of the essential material so as to tackle the problem of stabilization by regular partial interconnections. In Subsect. 2.1, we recall the basic background from the field of 2D (discrete) behavioral theory, centering around concepts such as controllability, autonomy, primeness of 2D polynomial matrices, etc. In Subsect. 2.2, we present the underlying notion of stability considered in the paper. Subsection 2.3 sets up the notation and terminology that will be used for behaviors with two different types of variables. In Sect. 2.4, we give a brief exposition of the theory of partial control in the behavioral framework. Finally, Sect. 3 addresses the problem of stabilization by partial interconnection and our main results are stated and proved. A short version of these results has been presented in [8].

2 Preliminaries

In this section, we introduce the necessary material and notation on behavioral theory for 2D systems. In the first two subsections, we introduce 2D behaviors and their stability. The last two subsections are concerned with the theory of behaviors with two different types of variables of which the partition into to-be-controlled variables and control variables is a particular case.

2.1 2D (kernel) behaviors

Throughout the paper $\mathbb{R}[\underline{s}, \underline{s}^{-1}] := \mathbb{R}[s_1, s_1^{-1}, s_2, s_2^{-1}]$ denotes the ring of Laurent polynomials, in the indeterminates s_1 and s_2, with coefficients in $\mathbb{R}$. We consider 2D behaviors $\mathfrak{B}$ defined over $\mathbb{Z}^2$ that can be described by a set of linear partial difference equations, i.e.,

$$\mathfrak{B} = \ker R(\underline{\sigma}, \underline{\sigma}^{-1}) := \{z \in \mathcal{U}^q \mid R(\underline{\sigma}, \underline{\sigma}^{-1})w \equiv 0\} \subset \mathcal{U}^q,$$

where $\mathcal{U}$ is the trajectory universe, here taken to be $(\mathbb{R})^{\mathbb{Z}^2}$, $\underline{\sigma} = (\sigma_1, \sigma_2)$, $\underline{\sigma}^{-1} = (\sigma_1^{-1}, \sigma_2^{-1})$, the σ_i's are the elementary 2D shift operators (defined by $\sigma_i w(\underline{k}) = w(\underline{k} + e_i)$, for $\underline{k} \in \mathbb{Z}^2$, where e_i is the ith element of the canonical basis of $\mathbb{R}^2$) and $R(\underline{s}, \underline{s}^{-1})$ is a 2D Laurent-polynomial (or in short, L-polynomial) matrix known as *representation* of $\mathfrak{B}$. Throughout this paper, these behaviors are simply referred to as *behaviors*. If no confusion arises, given an L-polynomial matrix $A(\underline{\sigma}, \underline{\sigma}^{-1})$, we sometimes write A instead of $A(\underline{\sigma}, \underline{\sigma}^{-1})$ and $A(\underline{s}, \underline{s}^{-1})$. In particular, we use the simplified notation $\ker R$ to refer always to $\ker R(\underline{\sigma}, \underline{\sigma}^{-1}) \subset \mathcal{U}^q$.

Instead of characterizing $\mathfrak{B}$ by means of a representation matrix R, it is also possible to characterize it by means of its *orthogonal module* $\mathrm{Mod}(\mathfrak{B})$, which consists of all the 2D L-polynomial rows $r(\underline{s}, \underline{s}^{-1}) \in \mathbb{R}^{1\times q}[\underline{s}, \underline{s}^{-1}]$ such that $\mathfrak{B} \subset \ker r(\underline{\sigma}, \underline{\sigma}^{-1})$, and can be shown to coincide with the $\mathbb{R}[\underline{s}, \underline{s}^{-1}]$-module $\mathrm{RM}(R)$ generated by the rows of R, i.e., $\mathrm{Mod}(\mathfrak{B}) = \mathrm{RM}(R(\underline{s}, \underline{s}^{-1}))$, see [25] for details.

It turns out that sums, intersections and inclusions of behaviors can be formulated in terms of the corresponding modules.

Theorem 1 [29, pag. 1074] *Let $\mathfrak{B}^1$ and $\mathfrak{B}^2$ be two behaviors. Then, $\mathfrak{B}^1 + \mathfrak{B}^2$ and $\mathfrak{B}^1 \cap \mathfrak{B}_2$ are also behaviors and*

1. $\mathrm{Mod}(\mathfrak{B}^1 + \mathfrak{B}^2) = \mathrm{Mod}(\mathfrak{B}^1) \cap \mathrm{Mod}(\mathfrak{B}^2)$.
2. $\mathrm{Mod}(\mathfrak{B}^1 \cap \mathfrak{B}^2) = \mathrm{Mod}(\mathfrak{B}^1) + \mathrm{Mod}(\mathfrak{B}^2)$.
3. $\mathfrak{B}^1 \subset \mathfrak{B}^2 \;\Leftrightarrow\; \mathrm{Mod}(\mathfrak{B}^2) \subset \mathrm{Mod}(\mathfrak{B}^1)$.

Note that part 3 in Theorem 1 implies that if $\mathfrak{B}^1 = \ker R_1 \subset \mathfrak{B}^2 = \ker R_2$, then there exists an L-polynomial matrix S such that $R_2 = SR_1$.

For a full column rank L-polynomial matrix $R \in \mathbb{R}^{p\times q}[\underline{s}, \underline{s}^{-1}]$ define its Laurent variety (or zeros) as

$$\mathcal{V}(R) = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 \mid \mathrm{rank}(R(\lambda_1, \lambda_2)) < \mathrm{rank}(R),\ \lambda_1\lambda_2 \neq 0\},$$

where the first rank is taken over $\mathbb{C}$ and the second one over $\mathbb{R}[\underline{s}, \underline{s}^{-1}]$. Note that $\mathcal{V}(R)$ is equal to the set of common zeros of the $q \times q$ minors of R.

Definition 2 A full column rank L-polynomial matrix $R \in \mathbb{R}^{p\times q}[\underline{s}, \underline{s}^{-1}]$ is said to be *right minor prime* (rMP) if $\mathcal{V}(R)$ is finite and *right zero prime* (rZP) if $\mathcal{V}(R)$ is empty. A full row rank L-polynomial matrix $R \in \mathbb{R}^{p\times q}[\underline{s}, \underline{s}^{-1}]$ is said to be left minor/zero prime (ℓMP/ℓZP) if R^T is right minor/zero prime, respectively. An L-polynomial matrix L is called a *minimal left annihilator* (MLA) of R if it has full row rank, $LR = 0$, and for any other L-polynomial matrix S such that $SR = 0$ we have that $S = AL$ for some L-polynomial matrix A. We define minimal right annihilators in a similar way, with the obvious adaptations.

Note that L is an MLA of R if it has full row rank and $\operatorname{im} R = \ker L$. It can be shown that an MLA always exists, it is ℓMP and it is uniquely determined modulo a unimodular matrix.

We next review the notions of controllability and autonomy in the context of the behavioral approach.

Definition 3 A behavior $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}^2}$ is said to be *controllable* if for all $z_1, z_2 \in \mathfrak{B}$ there exists $\delta > 0$ such that for all subsets $U_1, U_2 \subset \mathbb{Z}^2$ with $d(U_1, U_2) > \delta$, there exists a $z \in \mathfrak{B}$ such that $z\,|_{U_1} = z_1\,|_{U_1}$ and $z\,|_{U_2} = z_2\,|_{U_2}$.

In the above definition, $d(\cdot,\cdot)$ denotes the Euclidean metric on $\mathbb{Z}^n$ and $z\,|_U$, for some $U \subset \mathbb{Z}^n$, denotes the trajectory z restricted to the domain U.

In contrast with the one dimensional case, 2D behaviors admit a stronger notion of controllability called rectifiability. Whereas controllable behaviors are the ones that can be represented by a ℓMP L-polynomial matrix, or in other words by an MLA of some L-polynomial matrix, rectifiable behaviors are the ones that can be represented by ℓZP matrices, i.e., are of the form $\ker R$ with R an ℓZP L-polynomial matrix.

On the other hand, we shall say that a behavior is autonomous if it has no free variables, i.e., no "inputs", we refer to [4] for a formal definition. It can be shown [4] that $\mathfrak{B} = \ker R$ is autonomous if and only if R has full column rank. In the 1D case, all autonomous behaviors are finite dimensional vector spaces but in the 2D case this is no longer true. Indeed, as it is well known, for instance for the classical state-space systems, the initial conditions constitute the only freedom of an autonomous behavior. Whereas for discrete 1D systems, such conditions are given in a finite number of points, for discrete 2D systems initial conditions may as well be given in an infinite number of grid points (for instance, in a discrete line). Therefore, contrary to the 1D case, 2D autonomous behaviors may be infinite dimensional. It turns out that a 2D autonomous behavior $\mathfrak{B} = \ker R$ has finite dimension if and only if R is rMP [4].

Autonomous behaviors play an important role in this paper since, as we will see, autonomy is a necessary condition for stability. It is worth pointing out that although the representation of an autonomous behavior $\mathfrak{B}$ is highly non-unique, any two different representations of $\mathfrak{B}$ share the same Laurent variety, i.e., if $\mathfrak{B} = \ker R_1 = \ker R_2$, then $\mathcal{V}(R_1) = \mathcal{V}(R_2)$.

Every 2D behavior $\mathfrak{B}$ can be decomposed into the sum $\mathfrak{B} = \mathfrak{B}^c + \mathfrak{B}^a$, where $\mathfrak{B}^c$ is the *controllable part* of $\mathfrak{B}$ (defined as the largest controllable sub-behavior of $\mathfrak{B}$) and $\mathfrak{B}^a$ is a (non-unique) autonomous sub-behavior. This sum can be chosen to be direct for 1D behaviors, but this is not always possible for multidimensional behaviors, see [28].

2.2 Stability

A discrete 1D behavior $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$ is said to be *stable* if all its trajectories tend to the origin as time goes to infinity. In the 2D case, although in some applications one of the independent variables is time, in many other situations (as for instance in image processing where both variables are spatial), this is not the case. Therefore there is not a priori fixed way of progressing in the independent variable domain, and one needs to specify which are the relevant directions. This is here made by defining a cone of directions corresponding to the region of the plane which is relevant for each particular problem. In this context we shall define stability with respect to a specified stability region, as in [17], by adapting the ideas in [11] to the discrete case. For this purpose we identify a *direction* in $\mathbb{Z}^2$ with an element $\underline{d} = (d_1, d_2) \in \mathbb{Z}^2$ whose components are coprime integers, and define a *stability cone* in $\mathbb{Z}^2$ as the set of all positive integer linear combinations of 2 linearly independent directions.

By a *half-line* associated with a direction $\underline{d} \in \mathbb{Z}^2$ we mean the set of all points of the form $\alpha\underline{d}$ where α is a nonnegative integer; clearly, the half-lines in a stability cone S are the ones associated with the directions $\underline{d} \in S$.

Given a stability cone $S \subset \mathbb{Z}^2$, a trajectory $z \in \mathcal{U}^q$ is said to be S-*stable* if it tends to zero along every half line in S. A behavior $\mathfrak{B}$ is S-*stable* if all its trajectories are S-*stable*. It turns out that stable behaviors on $\mathcal{U}^q$ (with respect a stability cone S) must be finite dimensional, and therefore autonomous behaviors.

Lemma 4 ([17, Lemma 2]) *Every 2D behavior $\mathfrak{B} \subset \mathcal{U}^q$ which is stable with respect to some stability cone S is a finite dimensional linear subspace of the trajectory universe, $\mathcal{U}^q = (\mathbb{R}^q)^{\mathbb{Z}^2}$.*

To characterize stability, we introduce some preliminary notation. Given two elements $\underline{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $\underline{k} = (k_1, k_2) \in \mathbb{Z}^2$, we define

$$\underline{\lambda}^{\underline{k}} := \lambda_1^{k_1}\lambda_2^{k_2}.$$

With this notation a 2D q-vector polynomial function $p(\underline{k})$ of $\underline{k}$ is such that

$$p(\underline{k}) = \sum_{\underline{i} \in I} \alpha_{\underline{i}} \underline{k}^{\underline{i}},$$

where $I \subset (\mathbb{Z}_+)^2$ is a finite bi-index set and $\alpha_{\underline{i}} \in \mathbb{R}^q$.

Definition 5 We say that z is *pure polynomial exponential* with *frequency* $\underline{\lambda}$ if $z(\underline{k}) = p(\underline{k})\underline{\lambda}^{\underline{k}}$, with $p(\underline{k})$ a q-vector polynomial function. If z is a linear combination of pure polynomial exponential we say that z is a *polynomial exponential*. The frequencies of a polynomial exponential $z = \sum \alpha_i z_i$, where z_i are pure polynomial exponential, are defined as all the frequencies of the pure polynomial exponential z_i. A frequency $\underline{\lambda} \in \mathbb{R}^2$ is said to be S-stable if for every direction $\underline{d} \in S$,

$$|\underline{\lambda}^{\underline{d}}| < 1.$$

Theorem 6 ([17, Theorem 8], [26, Theorem 4.4]) *Let $\mathfrak{B} = \ker R \subset \mathcal{U}^q$ be a behavior, and let S be a stability cone. The following are equivalent:*

1. *$\mathfrak{B}$ is S-stable.*
2. *$\mathcal{V}(R)$ is finite and every $\underline{\lambda} \in \mathcal{V}(R)$ is S-stable.*
3. *Every $z \in \mathfrak{B}$ is a polynomial exponential with S-stable frequencies.*

2.3 Behaviors with two types of variables

Since in this paper we are interested in considering different types of variables in a behavior (the to-be-controlled variables and the control variables), we introduce the notation $\mathfrak{B}_{(w,c)}$ for a behavior whose variable z is partitioned into two sub-variables w and c. Partitioning the corresponding representation matrix as $[R \;\; M]$, we can write

$$\mathfrak{B}_{(w,c)} = \{(w, c) \in \mathcal{U}^{\mathrm{w}+\mathrm{c}} \mid R(\underline{\sigma}, \underline{\sigma}^{-1})w + M(\underline{\sigma}, \underline{\sigma}^{-1})c = 0\} = \ker[R \; M].$$

In the case, one is only interested in analyzing the evolution of one of the sub-variables, say, w, it is useful to eliminate the other one (c) and consider the projection of the behavior $\mathfrak{B}_{(w,c)}$ into $\mathcal{U}^{\mathrm{w}}$, defined as

$$\pi_w(\mathfrak{B}_{(w,c)}) = \{w \mid \exists\, c \text{ such that } (w, c) \in \mathfrak{B}_{(w,c)}\}.$$

The elimination Theorem [14] guarantees that $\pi_w(\mathfrak{B}_{(w,c)})$ is also a (kernel) behavior, for which a representation can be constructed as follows: take a minimal left annihilator (MLA) E of M. Then $\pi_w(\mathfrak{B}_{(w,c)}) = \ker(ER)$, see [10, Cor. 2.38].

On the other hand given a behavior $\mathfrak{B} = \ker\, R \subset \mathcal{U}^{\mathrm{w}}$ we define the lifting of $\mathfrak{B}$ into $\mathcal{U}^{\mathrm{w}+\mathrm{c}}$ as

$$\mathfrak{B}^*_{(w,c)} := \{(w, c) \in \mathcal{U}^{\mathrm{w}+\mathrm{c}} \mid \; c \text{ is free and } w \in \mathfrak{B}\}. \tag{1}$$

Obviously $\mathfrak{B}^*_{(w,c)} = \ker[R \; 0]$. Analogous definitions can be given if the roles of w and c are interchanged. For the sake of brevity, if no confusion arises, we identify $\mathfrak{B}$ and $\mathfrak{B}^*_{(w,c)}$ and denote $\mathfrak{B}_w := \pi_w(\mathfrak{B}_{(w,c)})$ and $\mathfrak{B}_c := \pi_c(\mathfrak{B}_{(w,c)})$.

Definition 7 Given a behavior $\mathfrak{B}_{(w,c)} \subset \mathcal{U}^{w+c}$ we say that c is *observable* from w if $(w, c_1), (w, c_2) \in \mathfrak{B}_{(w,c)}$ implies $c_1 = c_2$. The weaker notion of detectability is defined along the same lines. Let S be a stability cone. We say that c is *S-detectable* from w if $(w, c_1), (w, c_2) \in \mathfrak{B}_{(w,c)}$ implies $c_1 - c_2$ tends to zero along every half line in S.

Usually, in control problems involving behaviors with two types of variables, it is important to consider the set of variables that are not observable or *hidden* from the remaining set of variables, see [19,22,23]. Hence, given a behavior $\mathfrak{B}_{(w,c)}$ we shall define

$$\mathfrak{B}_{(0,c)} := \{c \in \mathcal{U}^{\mathrm{c}} \mid (0, c) \in \mathfrak{B}_{(w,c)}\},$$

as the behavior of the variables c that are not observable or "hidden" from w. Clearly, $\mathfrak{B}_{(0,c)} = \ker M$. Similarly we define $\mathfrak{B}_{(w,0)}$ as the set of w variables that are hidden from the variables c.

Remark 8 The definition of observability and detectability can be reformulated in terms of the hidden behaviors. Indeed, taking into consideration that we are dealing with linear behaviors, it is not difficult to verify that c is observable from w if and only if $\mathfrak{B}_{(0,c)}$ is the zero behavior. Moreover, c is S-detectable from w if and only if $\mathfrak{B}_{(0,c)}$ is S-stable. Similarly, w is observable (S-detectable) from c if and only if $\mathfrak{B}_{(w,0)}$ is the zero behavior (S-stable).

2.4 Control by regular partial interconnection

The behavioral approach to control rests on the basic idea that to control a system is to impose appropriate additional restrictions to its variables in order to obtain a new desired behavior. These additional restrictions are achieved by interconnecting the given system with another system called the controller. From the mathematical point of view, system interconnection corresponds to the intersection of the behavior to be controlled with the controller behavior.

Two situations have been considered in the literature. The first one is known as *full interconnection* and corresponds to the case where the controller is allowed to impose restrictions on all the system variables. In this case, the interconnection of a behavior to be controlled, $\mathfrak{B} \subset \mathcal{U}^{\mathrm{w}}$, with a controller behavior, $\mathcal{C} \subset \mathcal{U}^{\mathrm{w}}$, yields a controlled behavior given by

$$\mathcal{K} = \mathfrak{B} \cap \mathcal{C}, \tag{2}$$

or alternatively, in module terms, by $\mathrm{Mod}(\mathcal{K}) = \mathrm{Mod}(\mathfrak{B}) + \mathrm{Mod}(\mathcal{C})$. If (2) holds, we say that $\mathcal{K}$ is *implementable* by full interconnection from $\mathfrak{B}$.

A particular interesting type of interconnection corresponds to the case where the restrictions imposed by the controller do not overlap with the restrictions already active for the behavior to be controlled. Recalling that the elements of the modules associated with a behavior represent the corresponding equations (or restrictions), this means, in terms of the corresponding modules that

$$\mathrm{Mod}(\mathfrak{B}) \cap \mathrm{Mod}(\mathcal{C}) = \{0\},$$

(or, equivalently, that $\mathfrak{B} + \mathcal{C} = \mathcal{U}^{\mathrm{w}}$) and therefore

$$\mathrm{Mod}(\mathcal{K}) = \mathrm{Mod}(\mathfrak{B}) \oplus \mathrm{Mod}(\mathcal{C}).$$

In this case we say that the interconnection of $\mathfrak{B}$ and $\mathcal{C}$ is a *regular interconnection* and denote it by $\mathfrak{B} \cap_{\mathrm{reg}} \mathcal{C}$.

The second situation corresponds to the case where the system variables are divided into two disjoint sets: the set of to-be-controlled variables, whose behavior we want to shape, and the set of control variables, on which the controller is allowed to act in order to achieve the desired result. With the purpose of making the notion of partial control more precise we introduce the following notation.

Consider a behavior $\mathfrak{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}+\mathtt{c}}$ (the plant), where the w is the (vector of) *to-be-controlled variable(s)*, and c is the (vector of) *control variable*(s). To interpret the interconnection of the plant $\mathfrak{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}+\mathtt{c}}$ with the controller $\mathcal{C} \subset \mathcal{U}^{\mathtt{c}}$ in terms of behavior intersection, we first have to lift the controller behavior $\mathcal{C}$ and regard it as a behavior $\mathcal{C}^*_{(w,c)}$ in the extended variable (w, c). This yields the "extended" controlled behavior

$$\mathfrak{B}_{(w,c)} \cap \mathcal{C}^*_{(w,c)} = \{(w, c) \in \mathcal{U}^{\mathtt{w}+\mathtt{c}} \mid (w, c) \in \mathfrak{B}_{(w,c)},\ c \in \mathcal{C}\}.$$

For the sake of simplicity, whenever no confusion arises, we shall simply write $\mathfrak{B}_{(w,c)} \cap \mathcal{C}$ instead of $\mathfrak{B}_{(w,c)} \cap \mathcal{C}^*_{(w,c)}$.

The behavior of interest is now

$$\mathcal{K} = \pi_w(\mathfrak{B}_{(w,c)} \cap \mathcal{C}).$$

In contrast with the situation in which all variables are available for control, the *full* interconnection case, we refer to this situation as *partial* interconnection or *partial* control, see [7] for preliminary results in this context.

Also in the context of partial interconnections, regularity plays an important role. Given two behaviors $\mathfrak{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{C} \subset \mathcal{U}^{\mathtt{c}}$, we say that the interconnection $\mathfrak{B}_{(w,c)} \cap \mathcal{C}$ is *regular* if

$$\mathrm{Mod}(\mathfrak{B}_{(w,c)}) \cap \mathrm{Mod}(\mathcal{C}^*_{(w,c)}) = \{0\},$$

or equivalently if $\mathfrak{B}_{(w,c)} + \mathcal{C}^*_{(w,c)} = \mathcal{U}^{\mathtt{w}+\mathtt{c}}$. In this case, we denote the interconnection by $\mathfrak{B}_{(w,c)} \cap_{\mathrm{reg}} \mathcal{C}^*_{(w,c)}$ or (in simplified notation) by $\mathfrak{B}_{(w,c)} \cap_{\mathrm{reg}} \mathcal{C}$.

The following lemma presents some results about partial interconnections and hidden behaviors that will be used in the sequel.

Lemma 9 *Let $\mathfrak{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{C} \subset \mathcal{U}^{\mathtt{c}}$ be two behaviors. Then, the following hold true.*

1. $\pi_w(\mathfrak{B}_{(w,c)} \cap \mathcal{C}) = \pi_w(\mathfrak{B}_{(w,c)} \cap (\mathcal{C} + \mathfrak{B}_{(0,c)}))$.
2. $\mathfrak{B}_{(w,c)} \cap_{\mathrm{reg}} \mathcal{C}$ *if and only if* $\mathfrak{B}_{(w,c)} \cap_{\mathrm{reg}} (\mathcal{C} + \mathfrak{B}_{(0,c)})$.
3. $\mathfrak{B}_{(w,c)} \cap_{\mathrm{reg}} \mathcal{C}$ *if and only if* $\mathfrak{B}_c \cap_{\mathrm{reg}} \mathcal{C}$.

Proof Let $\mathfrak{B}_{(w,c)} = \ker[R \quad M]$ and $\mathcal{C} = \ker C$. Note that $\mathfrak{B}_{(0,c)} = \ker M \subset \mathcal{U}^c$ and since $\mathfrak{B}_{(0,c)} \subset \mathcal{C} + \mathfrak{B}_{(0,c)}$, then $\mathcal{C} + \mathfrak{B}_{(0,c)} = \ker KM$ for some L-polynomial matrix K.

1. It is enough to show that $\pi_w(\mathfrak{B}_{(w,c)} \cap (\mathcal{C} + \mathfrak{B}_{(0,c)})) \subset \pi_w(\mathfrak{B}_{(w,c)} \cap \mathcal{C})$ since the other inclusion is trivial. Let $w \in \pi_w(\mathfrak{B}_{(w,c)} \cap (\mathcal{C} + \mathfrak{B}_{(0,c)}))$. Then, by definition of π_w there exists a c such that $(w, c) \in \mathfrak{B}_{(w,c)} \cap (\mathcal{C} + \mathfrak{B}_{(0,c)}) = \ker \begin{bmatrix} R & M \\ 0 & KM \end{bmatrix}$. Clearly, c must satisfy $KMc = 0$, i.e., $c \in \mathcal{C} + \mathfrak{B}_{(0,c)} = \ker KM$ and therefore $c = c^* + c^{**}$, where $c^* \in \mathcal{C}$ and $c^{**} \in \mathfrak{B}_{(0,c)} = \ker M$. Hence, as

$(w, c) \in \ker[R\ \ M]$, $(w, c^*) \in \ker[R\ \ M]$ which implies that $(w, c^*) \in \ker \begin{bmatrix} R & M \\ 0 & C \end{bmatrix} = \mathfrak{B}_{(w,c)} \cap \mathcal{C}$, and therefore $w \in \pi_w(\mathfrak{B}_{(w,c)} \cap \mathcal{C})$.

2. In terms of the corresponding modules we need to show that

$$\mathrm{RM}([R\ \ M]) \cap \mathrm{RM}([0\ \ C]) = \{0\} \text{ if and only if } \mathrm{RM}([R\ \ M]) \cap \mathrm{RM}([0\ \ KM]) = \{0\}.$$

As $\ker C = \mathcal{C} \subset \mathcal{C} + \mathfrak{B}_{(0,c)} = \ker KM$, $\mathrm{RM}(KM) \subset \mathrm{RM}(C)$ and the "only if" part is obvious. For the converse, let $(0, 0) \neq (r, m) \in \mathrm{RM}([R\ \ M]) \cap \mathrm{RM}([0\ \ C])$. Clearly r must be zero and then there exists an L-polynomial row s such that $s[R\ \ M] = (0, m) \neq (0, 0)$, which implies $sM = m \in \mathrm{RM}(C) \cap \mathrm{RM}(M) = \mathrm{RM}(KM)$. Thus, $(0, m) \in \mathrm{RM}([R\ \ M]) \cap \mathrm{RM}([0\ \ KM])$.

3. By Theorem 1, the proof of 3 amounts to showing that

$$\mathrm{RM}([R\ \ M]) \cap \mathrm{RM}([0\ \ C]) = \{0\} \text{ if and only if } \mathrm{RM}(LM) \cap \mathrm{RM}(C) = \{0\},$$

where L is an MLA of R. In order to prove the "if" part, let $(0, 0) \neq (r, m) \in \mathrm{RM}([R\ \ M]) \cap \mathrm{RM}([0\ \ C])$. It is easy to see that r must be zero and therefore there exists $s \in L$ such that $s[R\ \ M] = (0, m)$. Thus, $0 \neq sM = m \in \mathrm{RM}(LM) \cap \mathrm{RM}(C)$. To prove the converse implication suppose that $0 \neq m \in \mathrm{RM}(LM) \cap \mathrm{RM}(C)$. Then, $m = \alpha LM = \beta C$ for some L-polynomial rows α and β. This implies that $(0, m) = \alpha L[R\ \ M] = \beta[0\ \ C]$ and therefore $(0, 0) \neq (0, m) \in \mathrm{RM}([R\ \ M]) \cap \mathrm{RM}([0\ \ C])$. $\square$

Remark 10 Obviously, a behavior $\mathcal{K} \subset \mathcal{U}^{\mathrm{w}}$ is implementable from a given behavior $\mathfrak{B} \subset \mathcal{U}^{\mathrm{w}}$ by full (not necessarily regular) interconnection if and only if $\mathcal{K} \subset \mathfrak{B}$. This condition is however not enough in the partial interconnection case. Indeed, it was proven in [1,19,22] that $\mathcal{K}$ is implementable by partial (not necessarily regular) interconnection from $\mathfrak{B}_{(w,c)}$ if and only if

$$\mathfrak{B}_{(w,0)} \subset \mathcal{K} \subset \mathfrak{B}_w.$$

It is immediately apparent that the study of partial control problems requires additional tools with respect to full control problems. One such a tool is the notion of the *canonical controller* which has proved to be a key concept for solving many implementation problems by partial control, see for instance [5,6,16,19,22]. For a given control objective $\mathcal{K} \subset \mathcal{U}^{\mathrm{w}}$, the canonical controller associate with $\mathcal{K}$ is defined as follows:

$$\mathcal{C}^{can}(\mathcal{K}) := \{c \mid \exists\ w \text{ such that } (w, c) \in \mathfrak{B}_{(w,c)} \text{ and } w \in \mathcal{K}\}.$$

For the problem of stabilization, since the control objective is not unique (as we are interested in stability, but not require a specific behavior to be achieved), we shall define the set of all canonical controllers associate to the set of implementable S-stable behaviors:

$$\begin{aligned}\mathcal{C}_s^{can} :&= \{\mathcal{C}^{can}(\mathfrak{B}_w^s) \mid \mathfrak{B}_w^s \text{ is an } S\text{-stable behavior and } \mathfrak{B}_{(w,0)} \subset \mathfrak{B}_w^s \subset \mathfrak{B}_w\}\\ &= \{\mathcal{C}^{can}(\mathfrak{B}_w^s) \mid \mathfrak{B}_w^s \text{ is an implementable } S\text{-stable behavior}\}.\end{aligned}$$

3 Stabilization by regular partial interconnection

In this section we establish necessary and sufficient conditions for the solvability of the problem of stabilizing a given behavior by a regular partial interconnection. Moreover, we show that, under certain conditions, we can derive a constructive solution to the problem and characterize the structure of the to-be-controlled behavior.

The problem of stabilization by regular partial interconnection can be formally stated as follows: Given a behavior $\mathfrak{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}+\mathtt{c}}$ and a stability cone S, find conditions for the existence of a controller behavior $\mathcal{C} \subset \mathcal{U}^{\mathtt{c}}$ such that

$$\pi_w(\mathfrak{B}_{(w,c)} \cap_{\text{reg}} \mathcal{C}) \text{ is an } S\text{-stable behavior.}$$

Assumption: We assume in the sequel that $\mathfrak{B}_{(w,0)}$ is an S-stable behavior. This entails no loss of generality since, as follows from Remark 10 it is a necessary condition for the stabilization of $\mathfrak{B}_{(w,c)}$ by regular partial interconnection. Note that this means that w is S-detectable from c in $\mathfrak{B}_{(w,c)}$, a condition that already appears in [1] for the 1D case.

Next, we present a result that characterizes the situation in which the to-be-controlled variables of a given behavior $\mathfrak{B}_{(w,c)}$ are stable with respect to a stability cone.

Lemma 11 *Let S be a stability cone and $\mathfrak{B}_{(w,c)} = \ker[R\ M]$ a behavior. Let L and E be an MLA of R and M respectively. Then, the following are equivalent:*

1. *$\mathfrak{B}_w$ is S-stable.*
2. *$M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c$ is S-stable.*
3. $\ker\begin{bmatrix} L \\ E \end{bmatrix}$ *is S-stable.*
4. *There exists an S-stable behavior $\mathfrak{B}^s$ such that $\mathfrak{B}_c = \mathfrak{B}^s + \mathfrak{B}_{(0,c)}$.*

Proof (1) $\Rightarrow$ (2) Let (w, c) such that $R(\underline{\sigma}, \underline{\sigma}^{-1})w = -M(\underline{\sigma}, \underline{\sigma}^{-1})c$. It is easy to see that if w is S-stable, then $R(\underline{\sigma}, \underline{\sigma}^{-1})w$, and therefore also $M(\underline{\sigma}, \underline{\sigma}^{-1})c$, is S-stable.
(2) $\Rightarrow$ (1) By definition $\mathfrak{B}_w = \{w \mid \exists\ c \text{ such that } Rw = -Mc\} = \{w \mid \exists\ v \in M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c\ :\ Rw = v\}$. Since $M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c$ is S-stable, v is a polynomial exponential trajectory with S-stable frequencies. Thus, if w is such that $Rw = v$, then

$$w = w^* + w^0,$$

with $w^0 \in \ker R$ and w^* a polynomial exponential trajectory whose frequencies are contained in the frequencies of v [26]. Thus, the frequencies of w are among those of w^* and w^0. Since $\ker R = \mathfrak{B}_{(w,0)}$ is assumed to be S-stable, together with the condition that v is S-stable, this implies that w is S-stable, i.e., $\mathfrak{B}_w$ is S-stable.

($2 \Leftrightarrow 3$) It follows from the fact that applying [18, Lemma 2.13] we obtain

$$\ker \begin{bmatrix} L \\ E \end{bmatrix} = M(\underline{\sigma}, \underline{\sigma}^{-1}) \ker LM = M(\underline{\sigma}, \underline{\sigma}^{-1}) \mathfrak{B}_c.$$

(2) $\Rightarrow$ (4) By Lemma 4, $M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c$ is finite dimensional and let $\{w_1, \ldots, w_r\}$ be a basis for $M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c$ where each w_i is an S-stable polynomial exponential trajectory. For each w_i there exists an S-stable polynomial exponential trajectory c_i such that

$$w_i = M(\underline{\sigma}, \underline{\sigma}^{-1}) c_i,$$

where $c_i \in \mathfrak{B}_c$, $i = 1, \ldots, r$. Define $\mathfrak{B}^s$:=span$<c_1, \ldots, c_r>$, where the span is considered over $\mathbb{R}[\underline{\sigma}, \underline{\sigma}^{-1}]$. Note that $\mathfrak{B}^s$ is an S-stable (kernel) behavior (thus linear and shift invariant) contained in $\mathfrak{B}_c$. For all $w = \sum \alpha_i w_i \in M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c$, $\alpha_i \in \mathbb{R}$, we have that $c = \sum \alpha_i c_i$ satisfies $w = M(\underline{\sigma}, \underline{\sigma}^{-1})c$. This implies that $M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c \subset M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}^s$. The reciprocal is obvious, taking into consideration that $\mathfrak{B}^s \subset \mathfrak{B}^c$. So,

$$M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}_c = M(\underline{\sigma}, \underline{\sigma}^{-1})\mathfrak{B}^s.$$

This implies that

$$\mathfrak{B}_c = \mathfrak{B}^s + \ker M.$$

Indeed, if $c \in \mathfrak{B}_c$, then $M(\underline{\sigma}, \underline{\sigma}^{-1})c = M(\underline{\sigma}, \underline{\sigma}^{-1})c^s$ for some $c^s \in \mathfrak{B}^s$. Thus, $c - c^s \in \ker M$, i.e., $c \in c^s + \ker M$ and therefore $c \in \mathfrak{B}^s + \ker M$, proving that $\mathfrak{B}_c \subset \mathfrak{B}^s + \ker M$. On the other hand, both $\mathfrak{B}^s$ and $\ker M$ are contained in $\mathfrak{B}_c$ (the former by construction and the latter since $\mathfrak{B}_c = \ker LM$), and so $\mathfrak{B}_c \supset \mathfrak{B}^s + \ker M$.

(4) $\Rightarrow$ (2) Obvious. □

The following Theorem provides necessary and sufficient conditions for the solvability of the problem of stabilization by regular partial interconnection.

Theorem 12 *Let S be a stability cone and $\mathfrak{B}_{(w,c)} = \ker[R\ M]$ a behavior. Let L, E and $[F_1\ F_2]$ be an MLA of R, M and $\begin{bmatrix} L \\ E \end{bmatrix}$ respectively and denote $\mathfrak{B}_1 := \ker \begin{bmatrix} L \\ E \end{bmatrix}$.*

Then, the following are equivalent:

1. *$\mathfrak{B}_{(w,c)}$ is S-stabilizable by regular partial interconnection.*
2. *There exists a controller behavior $\mathfrak{C}$ such that $M(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}_c \cap_{\mathrm{reg}} \mathfrak{C})$ is S-stable.*
3. *There exists a behavior $\mathfrak{B}_2$ such that*

$$\mathfrak{B}_1 + \mathfrak{B}_2 = \ker E \quad \textit{and} \quad \mathfrak{B}_1 \cap \mathfrak{B}_2 \textit{ is } S\textit{-stable.}$$

4. *There exist matrices $\overline{A}$ and $\overline{K}$ such that*

$$\left[\begin{array}{cc|c} 0 & I & \\ F_1 & F_2 & \overline{A} \end{array}\right] \quad \textit{is an MLA of} \begin{bmatrix} L \\ E \\ \hline K \end{bmatrix} \quad \textit{and} \quad \ker \begin{bmatrix} L \\ E \\ \hline K \end{bmatrix} \quad \textit{is } S\textit{-stable.}$$

5. *There exists a* $\mathfrak{C} \in \mathfrak{C}_s^{can}$ *that is implementable by regular full interconnection from* $\mathfrak{B}_c$*.*

Proof (1) $\Rightarrow$ (2) Let $\mathfrak{C} = \ker K$ be the controller that stabilizes $\mathfrak{B}_{(w,c)}$ by regular partial interconnection. Denote $\widetilde{\mathfrak{B}}_{(w,c)} := \mathfrak{B}_{(w,c)} \cap_{\text{reg}} \mathfrak{C} = \ker \begin{bmatrix} R & M \\ 0 & K \end{bmatrix}$. Clearly, $\pi_c(\widetilde{\mathfrak{B}}_{(w,c)}) = \ker \begin{bmatrix} LM \\ K \end{bmatrix} = \mathfrak{B}_c \cap_{\text{reg}} \mathfrak{C}$ and then

$$\begin{bmatrix} M(\underline{\sigma}, \underline{\sigma}^{-1}) \\ K(\underline{\sigma}, \underline{\sigma}^{-1}) \end{bmatrix} \pi_c(\widetilde{\mathfrak{B}}_{(w,c)}) = \begin{bmatrix} M(\underline{\sigma}, \underline{\sigma}^{-1}) \\ K(\underline{\sigma}, \underline{\sigma}^{-1}) \end{bmatrix} \ker \begin{bmatrix} LM \\ K \end{bmatrix} = M(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}_c \cap \mathfrak{C}).$$

Next, since $\pi_w(\widetilde{\mathfrak{B}}_{(w,c)})$ is S-stable we apply (1) $\Rightarrow$ (2) of Lemma 11 to conclude that $\begin{bmatrix} M(\underline{\sigma}, \underline{\sigma}^{-1}) \\ K(\underline{\sigma}, \underline{\sigma}^{-1}) \end{bmatrix} \pi_c(\widetilde{\mathfrak{B}}_{(w,c)})$ is S-stable and therefore so it is $M(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}_c \cap \mathfrak{C})$.

(2) $\Rightarrow$ (1) As $\pi_c(\mathfrak{B}_{(w,c)} \cap_{\text{reg}} \mathfrak{C}) = \mathfrak{B}_c \cap_{\text{reg}} \mathfrak{C}$ it follows from (2) $\Rightarrow$ (1) of Lemma 11 that $M(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}_c \cap_{\text{reg}} \mathfrak{C})$ S-stable implies $\pi_w(\mathfrak{B}_{(w,c)} \cap_{\text{reg}} \mathfrak{C})$ S-stable, which is equivalent to say that $\mathfrak{C}$ stabilizes $\mathfrak{B}_{(w,c)}$ by regular partial interconnection.

(2) $\Rightarrow$ (3) Since $\mathfrak{B}_{(0,c)} = \ker M \subset \mathfrak{C} + \mathfrak{B}_{(0,c)}$, there exists an L-polynomial matrix K such that $\mathfrak{C} + \mathfrak{B}_{(0,c)} = \ker KM$. We prove that $\mathfrak{B}_2 := \ker \begin{bmatrix} K \\ E \end{bmatrix}$ satisfies statement 3.

It is not difficult to check that

$$M(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}_c \cap \mathfrak{C}) = M(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}_c \cap (\mathfrak{C} + \mathfrak{B}_{(0,c)})).$$

Further, we can construct a kernel representation of $M(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}_c \cap (\mathfrak{C} + \mathfrak{B}_{(0,c)})) = M(\underline{\sigma}, \underline{\sigma}^{-1})(\ker LM \cap \ker KM)$ using [18, Lemma 2.13]. Indeed, we have that

$$M(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}_c \cap (\mathfrak{C} + \mathfrak{B}_{(0,c)})) = M(\underline{\sigma}, \underline{\sigma}^{-1})(\ker \begin{bmatrix} LM \\ KM \end{bmatrix}) = \ker \begin{bmatrix} L \\ E \\ K \end{bmatrix},$$

which is S-stable as $M(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}_c \cap \mathfrak{C})$ is S-stable. But

$$\ker \begin{bmatrix} L \\ E \\ K \end{bmatrix} = \ker \begin{bmatrix} L \\ E \\ K \\ E \end{bmatrix} = \mathfrak{B}_1 \cap \mathfrak{B}_2 \text{ and so, } \mathfrak{B}_1 \cap \mathfrak{B}_2 \text{ is } S - \text{stable.}$$

It remains to prove that $\mathfrak{B}_1 + \mathfrak{B}_2 = \ker E$. By assumption and part 1 of Lemma 9 we have that $\mathfrak{B}_c \cap_{\text{reg}} (\mathfrak{C} + \mathfrak{B}_{(0,c)})$ or, in terms of the corresponding $\mathbb{R}[\underline{s}, \underline{s}^{-1}]$-modules,

$$\mathrm{RM}(LM) \cap \mathrm{RM}(KM) = \{0\}.$$

We now claim that

$$\mathrm{RM}(LM) \cap \mathrm{RM}(KM) = \{0\} \quad \text{if and only if } \mathrm{RM}(\begin{bmatrix} L \\ E \end{bmatrix}) \cap \mathrm{RM}(\begin{bmatrix} K \\ E \end{bmatrix}) = \mathrm{RM}(E). \tag{3}$$

For the "if" part, let $r \in \mathrm{RM}(LM) \cap \mathrm{RM}(KM)$, i.e., $r = \ell M = kM$ with $\ell \in \mathrm{RM}(L)$ and $k \in \mathrm{RM}(K)$. Then, $(\ell - k)M = 0$, or in other words $(\ell - k) \in \mathrm{RM}(E)$. This implies in particular that $\ell \in \mathrm{RM}(\begin{bmatrix} K \\ E \end{bmatrix})$ and therefore $\ell \in \mathrm{RM}(\begin{bmatrix} L \\ E \end{bmatrix}) \cap \mathrm{RM}(\begin{bmatrix} K \\ E \end{bmatrix})$ which is equal, by assumption, to $\mathrm{RM}(E)$. Hence, $r = \ell M = 0$.

For the "only if" part, it is enough to show that $\mathrm{RM}(\begin{bmatrix} L \\ E \end{bmatrix}) \cap \mathrm{RM}(\begin{bmatrix} K \\ E \end{bmatrix}) \subset \mathrm{RM}(E)$; the reciprocal inclusion is obvious. Let $r \in \mathrm{RM}(\begin{bmatrix} L \\ E \end{bmatrix}) \cap \mathrm{RM}(\begin{bmatrix} K \\ E \end{bmatrix})$. Then, $r = \ell + r^* = k + r^{**}$ for some $\ell \in \mathrm{RM}(L)$, $k \in \mathrm{RM}(K)$, $r^*, r^{**} \in \mathrm{RM}(E)$ and therefore $rM = \ell M = kM$. Using the assumption that $\mathrm{RM}(LM) \cap \mathrm{RM}(KM) = \{0\}$ it follows that $\ell M = kM = rM = 0$ which implies that $r \in \mathrm{RM}(E)$. This concludes the proof of the claim. Finally, by Theorem 1 we have that $\mathrm{RM}(\begin{bmatrix} L \\ E \end{bmatrix}) \cap \mathrm{RM}(\begin{bmatrix} K \\ E \end{bmatrix}) = \mathrm{RM}(E)$ if and only if $\mathfrak{B}_1 + \mathfrak{B}_2 = \ker E$.

(3) $\Rightarrow$ (2) As $\mathfrak{B}_2 \subset \ker E$, there exists an L-polynomial matrix K such that $\mathfrak{B}_2 = \ker \begin{bmatrix} K \\ E \end{bmatrix}$. By assumption and Theorem 1, $\mathrm{RM}(\begin{bmatrix} L \\ E \end{bmatrix}) \cap \mathrm{RM}(\begin{bmatrix} K \\ E \end{bmatrix}) = \mathrm{RM}(E)$. According to (3), we obtain

$$\mathrm{RM}(LM) \cap \mathrm{RM}(KM) = \{0\}.$$

Thus, defining $\mathfrak{C} := \ker KM$, we have that $\mathfrak{B}_c \cap_{\text{reg}} \mathfrak{C}$.

To show that $M(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}_c \cap \mathfrak{C})$ is S-stable, note that a kernel representation of $M(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}_c \cap \mathfrak{C}) = M(\underline{\sigma}, \underline{\sigma}^{-1})(\ker LM \cap \ker KM)$ is $\ker \begin{bmatrix} L \\ E \\ K \end{bmatrix}$. Moreover

$$\ker \begin{bmatrix} L \\ E \\ K \end{bmatrix} = \ker \begin{bmatrix} L \\ E \\ K \\ E \end{bmatrix} = \mathfrak{B}_1 \cap \mathfrak{B}_2 \text{ which is } S\text{-stable by assumption.}$$

(3) $\Rightarrow$ (4) As $\mathfrak{B}_2 \subset \ker E$, there exists an L-polynomial matrix K such that $\mathfrak{B}_2 = \ker \begin{bmatrix} K \\ E \end{bmatrix}$. Let $[X_1\ X_2\ Y_1\ Y_2]$ be an MLA of $\begin{bmatrix} L \\ E \\ K \\ E \end{bmatrix}$. Then by [18, Lemma 2.14]

we have that $\mathfrak{B}_1 + \mathfrak{B}_2 = \ker([X_1\ X_2]\begin{bmatrix} L \\ E \end{bmatrix})$. By assumption, $\ker([X_1\ X_2]\begin{bmatrix} L \\ E \end{bmatrix}) = \ker E = \ker([0\ \ I]\begin{bmatrix} L \\ E \end{bmatrix})$. Hence, in particular $\ker([0\ \ I]\begin{bmatrix} L \\ E \end{bmatrix}) \subset \ker([X_1\ \ X_2]\begin{bmatrix} L \\ E \end{bmatrix})$ and then it follows that there exists an L-polynomial matrix S such that $[X_1\ X_2]\begin{bmatrix} L \\ E \end{bmatrix} = S[0\ \ I]\begin{bmatrix} L \\ E \end{bmatrix}$. Moreover, $[[X_1\ X_2] - [S[0\ \ I]]]\begin{bmatrix} L \\ E \end{bmatrix} = 0$, which implies $[X_1\ \ X_2] - [S[0\ \ I]] = T[F_1\ \ F_2]$ for some L-polynomial matrix T since $[F_1\ \ F_2]$ is an MLA of $\begin{bmatrix} L \\ E \end{bmatrix}$. Hence,

$$[X_1\ \ X_2] = [S\ \ T]\begin{bmatrix} 0 & I \\ F_1 & F_2 \end{bmatrix}. \tag{4}$$

Clearly,

$$\begin{bmatrix} X_1 & X_2 & Y_1 & Y_2 \\ 0 & I & 0 & -I \\ F_1 & F_2 & 0 & 0 \end{bmatrix}\begin{bmatrix} L \\ E \\ K \\ E \end{bmatrix} = 0.$$

Moreover, by (4),

$$\begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & -S & -T \end{bmatrix}\begin{bmatrix} X_1 & X_2 & Y_1 & Y_2 \\ 0 & I & 0 & -I \\ F_1 & F_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & -I \\ F_1 & F_2 & 0 & 0 \\ 0 & 0 & Y_1 & Y_2 - S \end{bmatrix}.$$

Since $\begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & -S & -T \end{bmatrix}$ is unimodular and $[X_1\ X_2\ Y_1\ Y_2]$ is an MLA of $\begin{bmatrix} L \\ E \\ K \\ E \end{bmatrix}$ then

$$\ker\begin{bmatrix} 0 & I & 0 & -I \\ F_1 & F_2 & 0 & 0 \\ 0 & 0 & Y_1 & Y_2 - S \end{bmatrix} = \ker\begin{bmatrix} X_1 & X_2 & Y_1 & Y_2 \\ 0 & I & 0 & -I \\ F_1 & F_2 & 0 & 0 \end{bmatrix} = \operatorname{im}\begin{bmatrix} L \\ E \\ K \\ E \end{bmatrix}.$$

Based on the matrix, $\begin{bmatrix} 0 & I & 0 & -I \\ F_1 & F_2 & 0 & 0 \\ 0 & 0 & Y_1 & Y_2 - S \end{bmatrix}$ we shall construct an MLA of $\begin{bmatrix} L \\ E \\ K \\ E \end{bmatrix}$ of the form $\begin{bmatrix} 0 & I & 0 & -I \\ F_1 & F_2 & 0 & 0 \\ 0 & 0 & \overline{Y_1} & \overline{Y_2} \end{bmatrix}$. As a first step, we show that F_1 has full row rank. Suppose there exists r such that $rF_1 = 0$. Then, $r[F_1 \;\; F_2]\begin{bmatrix} L \\ E \end{bmatrix} = rF_2E = 0$. Since E has full row rank (being an MLA of M), $rF_2 = 0$ and so $r[F_1 \;\; F_2] = 0$. However, as $[F_1 \;\; F_2]$ has full row rank (being an MLA of $\begin{bmatrix} L \\ E \end{bmatrix}$) then $r = 0$. Hence F_1 has full row rank.

Now, let $[\overline{Y_1} \;\; \overline{Y_2}]$ be an MLA of $\begin{bmatrix} K \\ E \end{bmatrix}$. Then $[0 \;\; 0 \;\; \overline{Y_1} \;\; \overline{Y_2}]$ is a left annihilator of $\begin{bmatrix} L \\ E \\ K \\ E \end{bmatrix}$, and therefore there exist L-polynomial matrices M_1, M_2, M_3 such that

$$[0 \;\; 0 \;\; \overline{Y_1} \;\; \overline{Y_2}] = [M_1 \;\; M_2 \;\; M_3]\begin{bmatrix} 0 & I & 0 & -I \\ F_1 & F_2 & 0 & 0 \\ 0 & 0 & Y_1 & Y_2 - S \end{bmatrix}.$$

Since F_1 has full row rank, $\begin{bmatrix} 0 & I \\ F_1 & F_2 \end{bmatrix}$ has also full row rank, which implies that $M_1 = 0$ and $M_2 = 0$. Thus, $[\overline{Y_1} \;\; \overline{Y_2}] = M_3[Y_1 \;\; Y_2 - S]$. On the other hand $[Y_1 \;\; Y_2 - S] = \overline{M}[\overline{Y_1} \;\; \overline{Y_2}]$ for some L-polynomial matrix $\overline{M}$, because it is also an annihilator of $\begin{bmatrix} K \\ E \end{bmatrix}$. Therefore,

$$\ker\begin{bmatrix} 0 & I & 0 & -I \\ F_1 & F_2 & 0 & 0 \\ 0 & 0 & \overline{Y_1} & \overline{Y_2} \end{bmatrix} = \ker\begin{bmatrix} 0 & I & 0 & -I \\ F_1 & F_2 & 0 & 0 \\ 0 & 0 & Y_1 & Y_2 - S \end{bmatrix} = \operatorname{im}\begin{bmatrix} L \\ E \\ K \\ E \end{bmatrix}.$$

Now, since $D := \ker\begin{bmatrix} 0 & I & 0 & -I \\ F_1 & F_2 & 0 & 0 \\ 0 & 0 & \overline{Y_1} & \overline{Y_2} \end{bmatrix}$ has full row rank, this matrix is an MLA of $\begin{bmatrix} L \\ E \\ K \\ E \end{bmatrix}$. Moreover, as $\begin{bmatrix} L \\ E \\ K \\ E \end{bmatrix}$ is rMP (because its kernel is, by assumption, stable and therefore finite dimensional, i.e., $\mathcal{V}(\begin{bmatrix} L \\ E \\ K \\ E \end{bmatrix})$ is finite) it follows that it is also minimal right annihilator of D. Now we can apply [3, Lemma A.2] and the reasonings in its proof to conclude that there exists two matrices $\overline{A}$, $\overline{K}$, where $\ker\overline{K} \subset \ker\begin{bmatrix} K \\ E \end{bmatrix}$, such that

$$\left[\begin{array}{cc|c} 0 & I & \\ F_1 & F_2 & \overline{A} \end{array}\right] \text{ is an MLA of } \begin{bmatrix} L \\ E \\ \overline{K} \end{bmatrix}.$$

Furthermore, $\ker \begin{bmatrix} L \\ E \\ \overline{K} \end{bmatrix}$ is S-stable as $\ker \begin{bmatrix} L \\ E \\ \overline{K} \end{bmatrix} \subset \ker \begin{bmatrix} L \\ E \\ K \\ E \end{bmatrix}$ and $\ker \begin{bmatrix} L \\ E \\ K \\ E \end{bmatrix} =$ $\mathfrak{B}_1 \cap \mathfrak{B}_2$ is S-stable by assumption.

(4) $\Rightarrow$ (3) Define $\mathfrak{B}_2 := \ker \overline{K}$. Using the assumption and [18, Lemma 2.14] we obtain that

$$\mathfrak{B}_1 + \mathfrak{B}_2 = \ker \begin{bmatrix} L \\ E \end{bmatrix} + \ker \overline{K} = \ker \begin{bmatrix} 0 & I \\ F_1 & F_2 \end{bmatrix} \begin{bmatrix} L \\ E \end{bmatrix} = \ker \overline{AK} = \ker E.$$

Finally, $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \ker \begin{bmatrix} L \\ E \\ \overline{K} \end{bmatrix}$ is S-stable by assumption.

(1) $\Rightarrow$ (5) Let $\widetilde{\mathcal{K}}$ be a controller that stabilizes $\mathfrak{B}_{(w,c)}$ by regular partial interconnection. Define $\mathcal{K} := \widetilde{\mathcal{K}} + \ker M$. Clearly, $\mathcal{K}$ also stabilizes $\mathfrak{B}_{(w,c)}$ by regular partial interconnection, see Lemma 9. Let K be such that $\mathcal{K} = \ker KM$. We show that $\mathcal{C} := \mathfrak{B}_c \cap \mathcal{K}$ satisfies statement 5.

From the definition of $\mathcal{C}_s^{can}$ we have that $\mathfrak{B}_c \cap \mathcal{K} \in \mathcal{C}_s^{can}$ as $\mathcal{C}^{can}(\pi_w(\mathfrak{B}_{(w,c)} \cap \mathcal{K})) = \mathfrak{B}_c \cap \mathcal{K}$ and $\pi_w(\mathfrak{B}_{(w,c)} \cap \mathcal{K})$ is S-stable. Finally, as $\mathfrak{B}_{(w,c)} \cap_{\text{reg}} \mathcal{K}$ we thus get from 3 of Lemma 9 that $\mathfrak{B}_c \cap \mathcal{K} = \mathfrak{B}_c \cap_{\text{reg}} \mathcal{K}$. This concludes the proof that $\mathcal{K}$ implements $\mathcal{C}$ by regular full interconnection from $\mathfrak{B}_c$.

(5) $\Rightarrow$ (1) By assumption there exists a behavior $\mathcal{K}$ such that $\mathfrak{B}_c \cap_{\text{reg}} \mathcal{K} = \mathcal{C}$. It is easy to see (from the definition of $\mathcal{C}_s^{can}$) that $\mathfrak{B}_c \cap_{\text{reg}} \mathcal{K} = \mathcal{C} \in \mathcal{C}_s^{can}$ implies that $\pi_w(\mathfrak{B}_{(w,c)} \cap_{\text{reg}} \mathcal{K})$ is S-stable, i.e., $\mathcal{K}$ stabilizes $\mathfrak{B}_{(w,c)}$ by regular partial interconnection. □

It is worth pointing out that since S-stable behaviors are finite dimensional, statement 3 of Theorem 12 can be further analyzed using the results of Bisiacco and Valcher in [3] on the problem of decomposing a 2D behavior into the sum of two sub-behaviors (one of which is fixed) having finite dimensional intersection. Unfortunately, [3, Th. 5.7] shows that the conditions for the existence of such decomposition are far from being constructive. Statement 5 reduces the problem to the implementation of *a* canonical controller $\mathcal{C}$ by regular full interconnection. Although such condition is not difficult to test for a given $\mathcal{C}$ through a direct summand condition, see [2,18,21], it becomes uneasy as $\mathcal{C}_s^{can}$ contains, in general, infinite number of elements. However, under certain condition, we are able to obtain a rather simple equivalent condition for the problem solvability.

Note also that in the statement 5 of Theorem 12, the canonical controllers are used as "control objectives" rather than actual controllers. Indeed, it was shown (see [6, Theorem 16]) that already in the 1D case, canonical controllers are very seldom useful as a controllers for regular partial control problems.

Theorem 13 *Let $\mathfrak{B}_{(w,c)} = \ker[R \;\; M]$ be a behavior, S a stability cone and L, E an MLA of R and M respectively. Assume that $[R \;\; M]$ has full row rank. Then $\mathfrak{B}_{(w,c)}$*

is S-stabilizable by regular partial interconnection if and only if $\ker L$ *is rectifiable (i.e., L is ℓZP).*

Proof As $[R \quad M]$ has full row rank, it follows that $\mathrm{RM}(L) \cap \mathrm{RM}(E) = \{0\}$. This, together with the fact that both L and E have full row rank (being an MLA of R and M respectively) amounts to saying that $[F_1 \quad F_2]$, an MLA of $\begin{bmatrix} L \\ E \end{bmatrix}$, is the zero matrix.

Suppose $\mathfrak{B}_{(w,c)}$ is S-stabilizable by regular partial interconnection. By the equivalence of 1 and 4 of Theorem 12 (together with the fact that $[F_1 \quad F_2] = 0$) we have that there exist matrices $\overline{K}, \overline{A}$ such that $\ker \begin{bmatrix} L \\ E \\ \overline{K} \end{bmatrix}$ is S-stable and

$$\left[\begin{array}{cc|c} 0 & I & \\ 0 & 0 & \overline{A} \end{array}\right] \text{ is an MLA of } \begin{bmatrix} L \\ E \\ \overline{K} \end{bmatrix}.$$

In particular this implies that $\mathrm{RM}(L) \cap \mathrm{RM}(\begin{bmatrix} E \\ \overline{K} \end{bmatrix}) = 0$. This gives $\ker L \cap_{\mathrm{reg}} \ker \begin{bmatrix} E \\ \overline{K} \end{bmatrix}$ is S-stable and therefore finite dimensional by Lemma 4. Hence, a finite dimensional behavior is implementable by regular full interconnection from $\ker L$. According to [9, Theorem 16] we conclude that the controllable part of $\ker L$ is rectifiable. But since $\ker L$ is controllable (being L an MLA of R), it coincides with its controllable part and then $\ker L$ is itself rectifiable, or in other words, L is ℓZP.

Let $\ker L$ be rectifiable. Then, there exists another rectifiable behavior $\ker \overline{K}$ such that $\ker L \cap_{\mathrm{reg}} \ker \overline{K} = 0$, see [29, Corollary 5.2], [18, Lemma 3.5]. In terms of the associated modules this is equivalent to $\mathrm{RM}(L) \oplus \mathrm{RM}(\overline{K}) = \mathbb{R}^q[\underline{s}, \underline{s}^{-1}]$ where q is the number of columns of L or $\overline{K}$. As $\mathrm{RM}(L) \cap \mathrm{RM}(E) = \{0\}$, $\mathrm{RM}(E) \subset \mathrm{RM}(\overline{K})$ and therefore $\overline{A}\,\overline{K} = E$ for some L-polynomial matrix $\overline{A}$. Clearly $\ker \begin{bmatrix} L \\ \overline{K} \end{bmatrix} = \ker \begin{bmatrix} L \\ E \\ \overline{K} \end{bmatrix}$. Besides, $\mathrm{RM}(L) \cap \mathrm{RM}(\overline{K}) = \{0\}$ together with the fact that L has full row rank (being an MLA of R) implies that an MLA of $\begin{bmatrix} L \\ \begin{bmatrix} \overline{A} \\ I \end{bmatrix} \overline{K} \end{bmatrix} = \begin{bmatrix} L \\ E \\ \overline{K} \end{bmatrix}$ is of the form $[0 \quad X]$ with X an MLA of $\begin{bmatrix} \overline{A} \\ I \end{bmatrix} \overline{K} = \begin{bmatrix} E \\ \overline{K} \end{bmatrix}$. In addition, as $\overline{K}$ and E have full row rank (being the former ℓZP and the later an MLA), we have that $[I \quad -\overline{A}]$ is an MLA of $\begin{bmatrix} E \\ \overline{K} \end{bmatrix}$ and consequently

$$[0 \quad I \quad -\overline{A}] \text{ is an MLA of } \begin{bmatrix} L \\ E \\ \overline{K} \end{bmatrix},$$

which implies in particular that

$$\begin{bmatrix} 0 & I & -\overline{A} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ is an MLA of } \begin{bmatrix} L \\ E \\ \overline{K} \\ 0 \end{bmatrix},$$

It is now easy to verify that $\begin{bmatrix} \overline{A} & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} \overline{K} \\ 0 \end{bmatrix}$ satisfy the conditions of 4 in Theorem 12 and therefore $\mathfrak{B}_{(w,c)}$ is S-stabilizable by regular partial interconnection. This concludes the proof. □

When dealing with *partial* control problems, one normally seeks for reducing them to equivalent problems in the context of *full* control, as happens for instance, for the problem of implementation by regular partial interconnection, see [5,6,16,19,22]. This is so because full interconnection problems can be handled more easily and in many cases there exist computational effective solutions. However, for the problem under consideration a characterization in terms of stabilization by regular full control seems to be impossible. Nevertheless, by imposing a condition on the hidden behavior $\mathfrak{B}_{(0,c)}$, we can obtain a characterization in the context of full control. The following results treat this issue.

Theorem 14 *Let $\mathfrak{B}_{(w,c)}$ be a behavior and S a stability cone. Assume that $\mathfrak{B}_{(0,c)}$ is S-stabilizable by regular full interconnection. Then, the following are equivalent:*

1. *$\mathfrak{B}_{(w,c)}$ is S-stabilizable by regular partial interconnection.*
2. *$\mathfrak{B}_c$ is S-stabilizable by regular full interconnection.*

Proof (1) $\Rightarrow$ (2) Assume $\overline{\mathcal{K}}$ stabilizes $\mathfrak{B}_{(w,c)} = \ker[R \;\; M]$ by regular partial interconnection and let E and L be an MLA of M and R, respectively. Define $\mathcal{K} := \overline{\mathcal{K}} + \mathfrak{B}_{(0,c)}$. As $\ker M = \mathfrak{B}_{(0,c)} \subset \mathcal{K}$, $\mathcal{K} = \ker KM$, for some L-polynomial matrix K. According to Lemma 9, $\mathcal{K}$ also stabilizes $\mathfrak{B}_{(w,c)}$ by regular partial interconnection, i.e., $\pi_w(\mathfrak{B}_{(w,c)} \cap \mathcal{K})$ is S-stable, and $\mathrm{RM}[R \quad M] \cap \mathrm{RM}[0 \quad KM] = \{0\}$, which implies that $\mathrm{RM}(LM) \cap \mathrm{RM}(KM) = \{0\}$. By assumption there exists a behavior $\widehat{\mathcal{K}}$ such that $\mathfrak{B}_{(0,c)} \cap_{\mathrm{reg}} \widehat{\mathcal{K}}$ is S-stable. Take

$$\mathcal{C} := \mathcal{K} \cap \widehat{\mathcal{K}} = \ker \begin{bmatrix} KM \\ \widehat{K} \end{bmatrix}$$

where $\widehat{K}$ is a representation of $\widehat{\mathcal{K}}$. We claim that $\mathcal{C}$ stabilizes $\mathfrak{B}_c$ by regular full interconnection.

Denote $\widetilde{\mathfrak{B}}_{(w,c)} := \mathfrak{B}_{(w,c)} \cap \mathcal{C} = \ker \begin{bmatrix} R & M \\ 0 & KM \\ 0 & \widehat{K} \end{bmatrix}$. Applying the equivalence of 1, 2 and 4 of Lemma 11 to $\widetilde{\mathfrak{B}}_{(w,c)}$, together with the fact that $\pi_c(\widetilde{\mathfrak{B}}_{(w,c)}) = \mathfrak{B}_c \cap \mathcal{C}$ and that $\widetilde{\mathfrak{B}}_{(0,c)} = \mathfrak{B}_{(0,c)} \cap \mathcal{C} = \ker \begin{bmatrix} M \\ KM \\ \widehat{K} \end{bmatrix}$ is S-stable (since it is contained in $\mathfrak{B}_{(0,c)} \cap_{\mathrm{reg}} \widehat{K}$), we obtain that

$$\begin{aligned}\pi_w(\widetilde{\mathfrak{B}}_{(w,c)}) &\Leftrightarrow \begin{bmatrix} M(\underline{\sigma}, \underline{\sigma}^{-1}) \\ KM(\underline{\sigma}, \underline{\sigma}^{-1}) \\ \widehat{K}(\underline{\sigma}, \underline{\sigma}^{-1}) \end{bmatrix} (\mathfrak{B}_c \cap \mathcal{C}) \text{ is } S\text{-stable} \\ &\Leftrightarrow \mathfrak{B}_c \cap \mathcal{C} \text{ is } S\text{-stable.}\end{aligned} \tag{5}$$

Clearly $\pi_w(\widetilde{\mathfrak{B}}_{(w,c)})$ is S-stable as it is contained in $\pi_w(\mathfrak{B}_{(w,c)} \cap \mathcal{K})$ which is S-stable, and therefore $\mathfrak{B}_c \cap \mathcal{C}$ is S-stable. We are thus reduced to proving that the interconnection of $\mathfrak{B}_c = \ker LM$ and $\mathcal{C} = \ker \begin{bmatrix} KM \\ \widehat{K} \end{bmatrix}$ is regular, i.e., $\mathrm{RM}(LM) \cap (\mathrm{RM}(\widehat{K}) + \mathrm{RM}(KM)) = \{0\}$. Note that

$$\mathrm{RM}(\widehat{K}) \cap (\mathrm{RM}(LM) + \mathrm{RM}(KM)) = \{0\} \quad \text{since } \mathrm{RM}(\widehat{K}) \cap \mathrm{RM}(M) = \{0\}.$$

Thus,

$$\mathrm{RM}(LM) \cap (\mathrm{RM}(\widehat{K}) + \mathrm{RM}(KM)) = \mathrm{RM}(LM) \cap \mathrm{RM}(KM),$$

which in particular implies that

$$\mathfrak{B}_c \cap_{\mathrm{reg}} \mathcal{C} \quad \text{if and only if } \mathfrak{B}_{(w,c)} \cap_{\mathrm{reg}} \mathcal{K}.$$

By assumption $\mathfrak{B}_{(w,c)} \cap_{\mathrm{reg}} \mathcal{K}$ and so $\mathfrak{B}_c \cap_{\mathrm{reg}} \mathcal{C}$.

(2) $\Rightarrow$ (1) Let $\mathcal{C}$ be such that $\mathfrak{B}_c \cap_{\mathrm{reg}} \mathcal{C}$ is S-stable. Then, $M(\underline{\sigma}, \underline{\sigma}^{-1})(\mathfrak{B}_c) \cap_{\mathrm{reg}} \mathcal{C})$ is also S-stable and therefore part 2 of Theorem 12 is satisfied. □

Note that the assumption that $\mathfrak{B}_{(0,c)}$ is S-stabilizable by regular full interconnection from $\mathfrak{B}_c$ is a relaxation of the condition that $\mathfrak{B}_{(0,c)}$ is S-stable, or in other words, that c is S-detectable from w in $\mathfrak{B}_{(w,c)}$. This leads to the following corollary.

Corollary 15 *Let $\mathfrak{B}_{(w,c)}$ be a behavior and S a stability cone. If c is S-detectable from w, then $\mathfrak{B}_{(w,c)}$ is S-stabilizable by regular partial interconnection if and only if $\mathfrak{B}_c$ is S-stabilizable by regular full interconnection.*

This result, can be compared with [1, Theorem 6] on the stabilization of 1D behaviors by regular partial interconnection, with the difference that now the conditions are given on the behavior of the control variable c rather than on the behavior of the system variable w. This is not unexpected since, as shown in [16,19], contrary to what happens in the 1D case, the partial implementation conditions for nD behaviors are equivalent to full implementation conditions on the behaviors of c rather than of w.

However, the following theorem shows that if $\mathfrak{B}_{(0,c)}$ is controllable, a necessary condition for stabilization of $\mathfrak{B}_{(w,c)}$ by regular partial control is the stabilization of $\mathfrak{B}_w$ by regular full interconnection. In turn, as shown in [9], this amounts to saying that

$$\mathfrak{B} = \mathfrak{B}^c \oplus \mathfrak{B}^s, \tag{6}$$

where $\mathfrak{B}^c$ (the controllable part of $\mathfrak{B}$) is rectifiable and $\mathfrak{B}^s$ is an S-stable behavior.

Theorem 16 *Let $\mathfrak{B}_{(w,c)}$ be a behavior and S a stability cone. Assume that $\mathfrak{B}_{(0,c)}$ is controllable. If $\mathfrak{B}_{(w,c)}$ is S-stabilizable by regular partial interconnection, then the following two equivalent conditions hold:*

1. *$\mathfrak{B}_w$ is S-stabilizable by regular full interconnection.*
2. *There exists an S-stable behavior $\mathfrak{B}^s$ such that*

$$\mathfrak{B}_w = \mathfrak{B}_w^c \oplus \mathfrak{B}^s, \tag{7}$$

where $\mathfrak{B}_w^c$ (the controllable part of $\mathfrak{B}_w$) is rectifiable.

Proof The equivalence of between (1) and (2) has been shown in [9]. Assume now that $\mathfrak{B}_{(w,c)}$ is S-stabilizable by regular partial interconnection. Write $\mathfrak{B}_{(w,c)} = \ker[R \quad M]$ and let $\widetilde{\mathfrak{C}}$ be a controller behavior such that $\pi_w(\mathfrak{B}_{(w,c)} \cap_{\text{reg}} \widetilde{\mathfrak{C}})$) is S-stable. Define $\mathfrak{C} := \widetilde{\mathfrak{C}} + \mathfrak{B}_{(0,c)}$. We know from parts 1 and 2 of Lemma 9 that $\pi_w(\mathfrak{B}_{(w,c)} \cap_{\text{reg}} \mathfrak{C})$ is S-stable. Let $\mathfrak{C}^c$ be the controllable part of $\mathfrak{C}$. Obviously, $\mathfrak{B}_{(w,c)} \cap \mathfrak{C}^c \subset \mathfrak{B}_{(w,c)} \cap \mathfrak{C}$ implies $\pi_w(\mathfrak{B}_{(w,c)} \cap \mathfrak{C}^c) \subset \pi_w(\mathfrak{B}_{(w,c)} \cap \mathfrak{C})$ and therefore $\pi_w(\mathfrak{B}_{(w,c)} \cap \mathfrak{C}^c)$ is S-stable as $\pi_w(\mathfrak{B}_{(w,c)} \cap \mathfrak{C})$ is S-stable. Moreover, the intersection of $\mathfrak{B}_{(w,c)}$ and $\mathfrak{C}^c$ is regular by [9, Prop. 11].

By [11, Corollary 4] (or [28]) we have that $\mathrm{RM}(\mathfrak{C}^c)$ is a free $\mathbb{R}[\underline{s}, \underline{s}^{-1}]$-module since $\mathfrak{C}^c$ is controllable, i.e., there exists a full row rank representation for $\mathfrak{C}^c$. Further, $\mathfrak{B}_{(0,c)} \subset \mathfrak{C}$ implies $\mathfrak{B}^c_{(0,c)} \subset \mathfrak{C}^c$, where $\mathfrak{B}^c_{(0,c)}$ is the controllable part of $\mathfrak{B}_{(0,c)}$. By assumption, $\mathfrak{B}^c_{(0,c)} = \mathfrak{B}_{(0,c)} = \ker M$ and therefore $\ker M \subset \mathfrak{C}^c$. From this, together with the fact that $\mathfrak{C}^c$ has a full row rank representation, it follows that there exists an L-polynomial matrix K such that $\mathfrak{C}^c = \ker KM$ with KM full row rank.

Note that $\begin{bmatrix} I & 0 \\ K & -I \end{bmatrix} \cdot \begin{bmatrix} R & M \\ 0 & KM \end{bmatrix} = \begin{bmatrix} R & M \\ KR & 0 \end{bmatrix}$ and let E be an MLA of M. Then,

$$\begin{aligned} \pi_w(\mathfrak{B}_{(w,c)} \cap \mathfrak{C}^c) &= \pi_w(\ker \begin{bmatrix} R & M \\ 0 & KM \end{bmatrix}) = \pi_w(\ker \begin{bmatrix} R & M \\ KR & 0 \end{bmatrix}) \\ &= \ker \begin{bmatrix} ER \\ KR \end{bmatrix} = \mathfrak{B}_w \cap \ker KR. \end{aligned} \tag{8}$$

Therefore, $\mathfrak{B}_w \cap \ker KR$ is S-stable as $\pi_w(\mathfrak{B}_{(w,c)} \cap \mathfrak{C}^c)$ is S-stable. It is left to prove that the interconnection between $\mathfrak{B}_w$ and $\ker KR$ is regular. According to part 3 of Lemma 9, it is enough to show that the interconnection between $\mathfrak{B}_{(w,c)} = \ker[R \quad M]$ and $\ker[KR \quad 0]$ is regular. This amounts to showing that $v[R \quad M] = z[KR \quad 0]$ for some row vectors v and z, implies $v[R \quad M] = 0 = z[KR \quad 0]$. Suppose that $v[R \quad M] = z[KR \quad 0]$. Note that $z[KR \quad 0] = z[[0 \quad -KM] + [KR \quad KM]]$ and then $v[R \ M] - z[KR \quad KM] = (v - zK)[R \quad M] = z[0 \quad -KM]$. By the assumption that the interconnection of $\mathfrak{B}_{(w,c)}$ and $\mathfrak{C}^c$ is regular one has that $(v - zK)[R \quad M] = z[0 \quad -KM] = 0$ and since KM has full row rank one obtains that $z = 0$ and therefore $v[R \quad M] = z[KR \quad 0] = 0$ which proves that the interconnection between $\mathfrak{B}_w$ and $\ker KR$ is regular. This concludes the proof that $\mathfrak{B}_w$ is S-stabilizable by regular full interconnection. □

4 Conclusions

In this paper, we have studied the problem of stabilization of two dimensional systems within the behavioral approach to system interconnection and control. In particular, we considered the case of control by partial interconnection. This type of control is more difficult to analyze than control by full interconnection, since it assumes that only some of the system variables are available for control.

This question has already been solved in [1] for the case of 1D systems, by translating the stabilization by partial interconnection into a problem of stabilization by full control. However, the extension of the corresponding results to the 2D case is non-trivial, mainly due to the fact that the translation of partial control problems into full control problems, is not as easy as in the 1D case.

Within these limitations, several characterizations of necessary and sufficient conditions for the problem solvability were established. Moreover, we have shown that for several relevant particular cases it is possible to derive constructive solutions.

In our opinion, the paper provides a suitable framework and tools in order to address some extensions of this work, e.g., one might consider different underlying notions of stability. Also note that Theorems 12, 13 and 16 hold only for the 2D case (as opposed to the nD case when $n > 2$) and therefore further research need to be done for the extensions of these results to the general case.

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