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Stability of Switched Systems With Partial State Reset

Isabel Brás, Ana Catarina Carapito, and Paula Rocha

***Abstract*—In this note, we consider switched systems and switched systems with state reset. In particular we focus on the case of partial reset, i.e., where only some state components may undergo the action of a reset. First we consider switched systems with pre-specified (partial) reset and investigate under which conditions such systems are stable. In a second stage we consider the problem of stabilization by (partial) reset, which consists in finding a suitable (partial) reset for a given switched system that makes this system stable under arbitrary switching.**

***Index Terms*—Lyapunov functions, partial reset, quadratic stability, switched systems.**

I. Introduction

Switched linear systems can be constructed from a family of linear time invariant systems together with a switching law. The switching law determines which of the linear system within the family is active at each time instant, hence defining how the time invariant systems commute among themselves. This type of systems may appear either as a direct result of the mathematical modeling of a phenomenon or as the consequence of certain control techniques using switching schemes, see, for instance, [1], [2]. In these schemes, a bank of controllers (multi-controller) is considered and the control procedure is performed by commutation among the controllers of the bank. In this context, finding conditions that guarantee that the obtained switched system is stable is a crucial issue, [1]. For a good survey of the state of the art in the area of switched systems we refer to [3], [4].

When dealing with switched systems, two different approaches are possible: either to consider that the state evolution is continuous, i.e., the state components are not subject to "jumps" during switching, or to allow (or even force) state discontinuities at the switching instants, in which case one says that there is a state reset.

A common approach when dealing with switched systems is not to allow jumps in the state during the switching instants. In such case, even if each individual time invariant system is stable the correspondent switched system may be unstable, [5]. The stability of switched systems with continuous state trajectories has been widely investigated, see, for instance, [6], [1], [7] and [8]. On the other hand, the case where the state components suffer a reset has been considered in [2], [9], [10].

In this technical note we focus on the case of partial reset, i.e., where only some state components may suffer the action of a reset. This situation occurs, for instance, in switching control, where it is possible to reset the state of the controllers, but may be impossible to change

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the state of the controlled process. Our aim is the study of stability from two perspectives. First we consider switched systems with pre-specified (partial) reset and investigate under which conditions such systems are stable. In a second stage we consider the problem of stabilization by (partial) reset, which consists in finding a suitable (partial) reset for a given switched system that makes this system stable under arbitrary switching. In particular, we prove that the existence of a set of quadratic Lyapunov functions associated to matrices with a common Schur complement is a sufficient condition for stabilization by partial reset. Although more modern approaches have been developed in recent years, namely using multiple Lyapunov functions or parameter dependent Lyapunov function [5], [11], the issue of quadratic stability (based on the existence of common quadratic functions) is still relevant and keeps attracting the attention of several researchers, see for instance [12]–[14].

II. Preliminaries

Let $\mathcal{P}$ be a finite index set, $\Sigma_{\mathcal{P}} = \{\Sigma_p : p \in \mathcal{P}\}$ a family of linear time invariant systems and (A_p, B_p, C_p, D_p) a state representation of Σ_p, for $p \in \mathcal{P}$. Additionally, define a *switching law* or a *switching signal* as a piecewise constant function of time, $\sigma : [t_0, +\infty[\; \to \mathcal{P}$, such that $\sigma(t) = i_{k-1}$, for $t \in [t_{k-1}, t_k[_{k \in \mathbb{N}}$; the time instants $t_k, k \in \mathbb{N}, t_0 < t_1 < \cdots < t_{k-1} < t_k \ldots$, are called *switching instants*. The set of all switching signals is represented by $\mathcal{S}_{\mathcal{P}}$. A triple $\mathbb{S} = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}})$ is said to be a *switched system with switching bank* $\Sigma_{\mathcal{P}}$. Each switching signal $\sigma(\cdot) \in \mathcal{S}_{\mathcal{P}}$ produces a linear time varying system Σ_σ defined by

$$\Sigma_\sigma := \begin{cases} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{cases} \tag{1}$$

for all $t \geq t_0$, where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input and $y(t) \in \mathbb{R}^p$ is the output. The system Σ_σ is said to be a *σ-switched system*. Note that in this definition it is assumed that there are no state discontinuities in the switching instants.

A more general notion of switched system can be considered by allowing state discontinuities during the switching instants. Here, we consider that these discontinuities are determined by a family of resets

$$\mathcal{R} = \left\{(q, p, R_{(q,p)}) : (q,p) \in \mathcal{P} \times \mathcal{P}, q \neq p\right\}$$

where $R_{(q,p)}$ are invertible real matrices that act on the state of $\mathbb{S}$ during the switching instants. These matrices will be called *reset matrices*. More precisely, we define a *switched system with state reset* as a quadruple $\mathbb{S}_R = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R})$ in the sense that each switching signal $\sigma(\cdot) \in \mathcal{S}_{\mathcal{P}}$ produces a linear time varying system defined as in (1), such that at each switching time instant $t_k, k \in \mathbb{N}$, with $t_0 < t_1 < \cdots < t_{k-1} < t_k < \cdots$

$$x(t_k) = R_{(i_{k-1}, i_k)}x(t_k^-), \text{ if } \sigma(t) = i_{k-1} \text{ for } t \in [t_{k-1}, t_k[, \tag{2}$$

where $x(t_k^-) := \lim_{t \to t_k^-} x(t)$ and, for each k, $(i_{k-1}, i_k, R_{(i_{k-1}, i_k)}) \in \mathcal{R}$. Notice that, the matrices $R_{(i_{k-1}, i_k)}$ are determined by which systems (within the bank) are active before and after the switching. Clearly if, for all $(q,p) \in \mathcal{P} \times \mathcal{P}$, $R_{(q,p)} = I_n$, where I_n denotes the identity matrix of order n, the correspondent switched system is a system without state reset.

Our definition of switched system with state reset allows, in principle, to reset all the state components. However, as already mentioned, in some situations the reset of some of those components may be forbidden or inconvenient. This leads to the analysis of switched systems with state reset with a certain block structure, that is, switched systems where the reset matrices in $\mathcal{R}$ are of the form

$$R_{(q,p)} = \begin{bmatrix} I_{n-z} & 0 \\ R_{21}^{(q,p)} & R_{22}^{(q,p)} \end{bmatrix}. \tag{3}$$

In opposition to the case in which all of the state components can be subject to reset (total reset), we say that this type of reset is a *partial reset of order* z.

It should be noticed that there are other ways to define state resets, for instance by setting some state components to zero [15] or by considering not necessarily invertible reset matrices [2], [16]. State jumps are also treated in a different context in the field of impulsive systems.

III. Stability of Switched Systems With Reset

In this section we analyze the stability of switched systems with pre-specified state reset.

Definition 1: A switched system (with or without reset) is stable if there exist $\gamma, \lambda \in \mathbb{R}^+$ such that, for every switching signal σ, for every $t_0 \in \mathbb{R}$ and every $x_0 \in \mathbb{R}^n$, the solution $x(t)$ of $\dot{x}(t) = A_{\sigma(t)}x(t)$, with $x(t_0) = x_0$, satisfies $\|x(t)\| \leq \gamma e^{-\lambda(t-t_0)}\|x_0\|$ for $t \geq t_0$.

A sufficient condition for the stability of switched systems without state reset is stated in terms of quadratic Lyapunov functions. A function $V(x) = x^T P x$, where P is a square symmetric positive definite matrix, is said to be a common quadratic Lyapunov function (CQLF) for the switched system $\mathbb{S} = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}})$ if $A_p^T P + P A_p < 0$, for all $p \in \mathcal{P}$, i.e., if $V(x)$ is a Lyapunov function for all the individual systems Σ_p. With some abuse of language, we shall call the matrix P a CQLF for Σ_p and A_p, $p \in \mathcal{P}$. The existence of a CQLF is a well known suficient condition for stability, see for instance [2], [8], [17].

For switched systems $\mathbb{S}_R = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}}, \mathcal{R})$ with a pre-specified state reset $\mathcal{R} = \left\{(q, p, R_{(q,p)}) : (q,p) \in \mathcal{P} \times \mathcal{P}, q \neq p\right\}$ it was proved in [2] that stability is assured by the existence of a set $\mathcal{L} = \{P_p : p \in \mathcal{P}\}$ of QLFs for $\mathbb{S}$ (*i.e.,* a set of QLFs such that P_p is a QLF for A_p) that satisfies the condition $R_{(q,p)}^T P_p R_{(q,p)} \leq P_q$, for all $p, q \in \mathcal{P}$, where $R_{(q,q)} = I_n$, by convention. When this condition is satisfied we say that $\mathcal{L}$ is *$\mathcal{R}$-contractive*.

In the sequel, we derive a necessary condition for the $\mathcal{R}$-contractivity of a set of QLFs, in the case where the reset is a partial reset of order z as defined in the previous section. More concretely, we shall show that a necessary condition for the $\mathcal{R}$-contractivity of a set of QLFs is the existence of a common Schur complement (CSC) of order $n - z$ for that set. We begin by reviewing the concept of Schur complement of a block in a matrix, [18]. Let

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{4}$$

be a partitioned matrix of order n such that the block P_{22} is an invertible matrix of order $z \in \{1, 2, \ldots, n-1\}$. The *Schur complement of P_{22} in P* is the matrix $P_{11} - P_{12}P_{22}^{-1}P_{21}$. We denote this matrix by $\mathcal{C}_{n-z}(P)$ and call it *Schur complement of order $n - z$ of P*; by convention, $\mathcal{C}_n(P) := P$.

Definition 2: Let $\{P_p : p \in \mathcal{P}\}$ be a set of positive definite matrices of order n and $z \in \{0, 1, \ldots, n-1\}$. The set $\{P_p : p \in \mathcal{P}\}$ is said to have a common Schur complement of order $n - z$ if there exists a matrix C of order $n - z$ such that $\mathcal{C}_{n-z}(P_p) = C$. Briefly, we say that the set $\{P_p : p \in \mathcal{P}\}$ has $(n - z)$-CSC.

Remark 1: If $\{P_p : p \in \mathcal{P}\}$ is a set of QLFs with n-CSC, then $\mathcal{C}_n(P_p) = C = P_p$. So, the problem of existence of a set of QLFs with n-CSC reduces to the problem of existence of a CQLF.

In order to obtain the aforementioned necessary condition for the $\mathcal{R}$-contractivity, we need the following two properties of the Schur complements (for the proofs, see the Appendix).

Lemma 1: Let

$$G = \begin{bmatrix} G_{11} & G_{12} \\ 0 & G_{22} \end{bmatrix}, H = \begin{bmatrix} H_{11} & 0 \\ H_{21} & H_{22} \end{bmatrix} \text{ and } P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

be square matrices of order n such that G_{22}, H_{22} and P_{22} are invertible matrices of order z. Then

$$\mathcal{C}_{n-z}(GPH) = G_{11}\mathcal{C}_{n-z}(P)H_{11}.$$

Lemma 2: If $P \geq Q > 0$ then $\mathcal{C}_{n-z}(P) \geq \mathcal{C}_{n-z}(Q) > 0$.

We are now ready to state a necessary condition for the $\mathcal{R}$-contractivity of a set of QLFs when $\mathcal{R}$ is a family of partial resets.

*Theorem 1: Let $\mathbb{S}_R = (\mathcal{P}, \Sigma_\mathcal{P}, \mathcal{S}_\mathcal{P}, \mathcal{R})$ be a switched system with partial state reset of order z. If $\mathcal{L}$ is a $\mathcal{R}$-contractive set of QLFs for $\mathbb{S}_R$, then $\mathcal{L}$ is a QLFs set with $(n-z)$-*CSC.

Proof: If $\mathcal{L} = \{P_p: p \in \mathcal{P}\}$ is a $\mathcal{R}$-contractive set of QLFs for $\mathbb{S}_R$ then, for all $p, q \in \mathcal{P}$, $R_{(q,p)}^T P_p R_{(q,p)} \leq P_q$. Since $R_{(q,p)}^T P_p R_{(q,p)}$ and P_q are positive definite then, by Lemma 2, $\mathcal{C}_{(n-z)}\left(R_{(q,p)}^T P_p R_{(q,p)}\right) \leq \mathcal{C}_{n-z}(P_q), \forall p, q \in \mathcal{P}$. Moreover, by Lemma 1, $\mathcal{C}_{n-z}\left(R_{(q,p)}^T P_p R_{(q,p)}\right) = \mathcal{C}_{n-z}(P_p), \forall p, q \in \mathcal{P}$. Thus, $\mathcal{C}_{n-z}(P_p) = \mathcal{C}_{n-z}(P_q), \forall p, q \in \mathcal{P}$. ■

Notice that, the reciprocal statement of Theorem 1 does not hold. For example

$$P_1 = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \text{ and } P_2 = \operatorname{diag}\left(\frac{7}{4}, 4\right)$$

have 1-CSC equal to 7/4. If $R_{(2,1)} = I_2$ is one of the reset matrices of $\mathcal{R}$, we have $R_{(2,1)}^T P_1 R_{(2,1)} - P_2$, which is not negative semidefinite. However, as we shall see next, given a set $\mathcal{L}$ of QLFs with $(n-z)$-CSC it is possible to construct an $\mathcal{R}$ such that $\mathcal{L}$ is $\mathcal{R}$-contractive.

IV. Stabilization by Total and Partial Reset

Whereas in the previous section we have studied the stability of a switched system with a pre-specified state reset, in this section, we analyze the possibility of defining suitable state resets so that the resulting switched system is stable.

Definition 3: A switched system $\mathbb{S} = (\mathcal{P}, \Sigma_\mathcal{P}, \mathcal{S}_\mathcal{P})$ is said to be stabilizable by state reset if there exists a family of resets $\mathcal{R}$ such that the switched system with state reset $\mathbb{S}_R = (\mathcal{P}, \Sigma_\mathcal{P}, \mathcal{S}_\mathcal{P}, \mathcal{R})$ is stable. In particular, if each reset matrix of $\mathcal{R}$ is an invertible matrix of the form

$$\begin{bmatrix} I_{n-z} & 0 \\ R_{21}^{(q,p)} & R_{22}^{(q,p)} \end{bmatrix}, (q,p) \in \mathcal{P} \times \mathcal{P}$$

the system $\mathbb{S}$ is called stabilizable by partial state reset of order z.

First we consider the case where all state variables are free for reset, and show that it is always possible to assure stability of the switched system with a proper choice of reset matrices. For this purpose we need the following lemma, which is a consequence of results presented in [2].

Lemma 3: Let $\{A_p: p \in \mathcal{P}\} \subset \mathbb{R}^{n\times n}$ be a set of stable matrices and $P \in \mathbb{R}^{n\times n}$ a symmetric and positive definite matrix. Then there exists a set of invertible matrices $\{S_p: p \in \mathcal{P}\}$ such that $\overline{A}_p = S_p A_p S_p^{-1}$ share P as a CQLF.

Based on Lemma 3, it is possible to define a total reset that ensures the stabilization of an arbitrary switched system.

Theorem 2: Let $\mathbb{S} = (\mathcal{P}, \Sigma_\mathcal{P}, \mathcal{S}_\mathcal{P})$ be a switched system and S_p be invertible matrices such that $S_p A_p S_p^{-1}$ share a CQLF. If $\mathcal{R} = \{(q, p, S_p^{-1} S_q): (q,p) \in \mathcal{P} \times \mathcal{P}, q \neq p\}$ then, $\mathbb{S}_R = (\mathcal{P}, \Sigma_\mathcal{P}, \mathcal{S}_\mathcal{P}, \mathcal{R})$ is a stable system.

Proof: Let $\{S_p: p \in \mathcal{P}\}$ be a set of invertible matrices S_p such that all matrices $S_p A_p S_p^{-1}$ share a CQLF P. Then, for each p, $P_p := S_p^T P S_p$ is a QLF for A_p. Since $R_{(q,p)} = S_p^{-1} S_q$, it is easy to conclude that $R_{(q,p)}^T P_p R_{(q,p)} = S_q^T P S_q = P_q$ for all $p, q \in \mathcal{P}$. Therefore, $\left\{S_p^T P S_p: p \in \mathcal{P}\right\}$ is a $\mathcal{R}$-contractive set of QLFs for $\mathbb{S}_R$. Hence, $\mathbb{S}_R$ is stable. ■

As could be expected, contrary to what happens in the total reset case, stability is not always achievable with partial resets.

Example 1: A switched system $\mathbb{S} = (\mathcal{P}, \Sigma_\mathcal{P}, \mathcal{S}_\mathcal{P})$ where the switching bank is associated to the stable matrices $A_p = \operatorname{diag}(A_{11}^p, -I_z)$ cannot be stabilized with partial resets if the switched system $\mathbb{S}^1$ associated with the matrices A_{11}^p is unstable, because, in this case, the application of partial resets of the form (3) does not change the dynamics of $\mathbb{S}^1$.

Next we investigate when a switched system is *stabilizable by a partial reset* of a certain order. This is done by establishing a relation between the $\mathcal{R}$-contractivity property and the existence of a family of QLFs with common Schur complement of a suitable order. For this purpose we need the following result, which can be regarded as the counterpart of Lemma 3 for the partial reset case.

Lemma 4: A set of matrices $\{A_p: p \in \mathcal{P}\}$ has a QLFs set with $(n-z)$-CSC if and only if there exist invertible matrices

$$S_p = \begin{bmatrix} I_{n-z} & 0 \\ S_{21}^p & S_{22}^p \end{bmatrix}$$

such that the set $\left\{S_p^{-1} A_p S_p: p \in \mathcal{P}\right\}$ has a CQLF.

Proof:

($\Rightarrow$) Let P_p be QLFs for the matrices $A_p, p \in \mathcal{P}$, written in form

$$P_p = \begin{bmatrix} P_{11}^p & P_{12}^p \\ P_{12}^{p\,T} & P_{22}^p \end{bmatrix}, \text{ where } P_{11}^p \in \mathbb{R}^{(n-z)\times(n-z)}.$$

Since P_{22}^p is positive definite, there exist invertible matrices V_p such that $(P_{22}^p)^{-1} = V_p V_p^T$. Taking

$$S_p := \begin{bmatrix} I_{n-z} & 0 \\ (P_{12}^p V_p)^T & V_p^{-1} \end{bmatrix}$$

it easy to verify that $P_p = S_p^T \operatorname{diag}(\mathcal{C}_{n-z}(P_p), I_z) S_p$. Therefore, $\operatorname{diag}(\mathcal{C}_{n-z}(P_p), I_z)$ is a CQLF for the matrices $S_p^{-1} A_p S_p$, for all $p \in \mathcal{P}$.

($\Leftarrow$) Reciprocally, if there are invertible matrices S_p such that the matrices $S_p^{-1} A_p S_p$ share a CQLF P, then $\left\{S_p^{-T} P S_p^{-1}: p \in \mathcal{P}\right\}$ is a set of QLFs for $\{A_p: p \in \mathcal{P}\}$. Moreover, by Lemma 1, $\mathcal{C}_{n-z}\left(S_p^{-T} P S_p^{-1}\right) = \mathcal{C}_{n-z}(P)$.

■

The next theorem gives the relation between the contractivity and the common complement Schur property.

Theorem 3: Let $\mathbb{S} = (\mathcal{P}, \Sigma_\mathcal{P}, \mathcal{S}_\mathcal{P})$ be a switched system and $\mathcal{L} = \{P_p: p \in \mathcal{P}\}$ a set of symmetric and positive definite matrices of order n. The following statements are equivalent:

1) *There exists a family $\mathcal{R}$ of partial resets of order z such that $\mathcal{L}$ is a $\mathcal{R}$-contractive set of QLFs for $\mathbb{S}_R = (\mathcal{P}, \Sigma_\mathcal{P}, \mathcal{S}_\mathcal{P}, \mathcal{R})$.*
2) *$\mathcal{L}$ is a set of QLFs for $\mathbb{S}_R$ with $(n-z)$-*CSC.

Proof:

($1. \Rightarrow 2.$) This is a direct consequence of Theorem 1.

($2. \Rightarrow 1.$) Now suppose that $\mathcal{L} = \{P_p: p \in \mathcal{P}\}$ is a set of QLFs with $(n-z)$-CSC of the switched system $\mathbb{S}$. According to the proof the Lemma 4, $P_p = S_p^T \operatorname{diag}(\mathcal{C}_{n-z}(P_p), I_z) S_p$, where

$$S_p := \begin{bmatrix} I_{n-z} & 0 \\ (P_{12}^p V_p)^T & V_p^{-1} \end{bmatrix} \text{ and } P_{22}^p = V_p^{-T} V_p^{-1}$$

for some invertible matrix $V_p, p \in \mathcal{P}$. Moreover, according to the same lemma, the matrices $S_p^{-1} A_p S_p$ share a CQLF. Finally, by similar arguments as in the proof of Theorem 2, we conclude that $\mathcal{L}$ is a $\mathcal{R}$-contractive set of QLFs for $\mathbb{S}_R = (\mathcal{P}, \Sigma_\mathcal{P}, \mathcal{S}_\mathcal{P}, \mathcal{R})$, where $\mathcal{R} = \left\{(q, p, S_p^{-1} S_q) : (q,p) \in \mathcal{P} \times \mathcal{P}, q \neq p\right\}$.

■

As a consequence of Theorem 3, the following corollary gives a sufficient condition for the stabilizability of a switched system by means of a partial state reset of order z.

*Corollary 1: If the switched system $\mathbb{S}$ has a set of QLFs with $(n-z)$-*CSC *then $\mathbb{S}$ is stabilizable by partial reset of order z.*

Example 2: The switched system $\mathbb{S} = (\mathcal{P}, \Sigma_{\mathcal{P}}, \mathcal{S}_{\mathcal{P}})$ with switching bank $\Sigma_{\mathcal{P}}$ associated to the matrices

$$A_1 = \begin{bmatrix} -0.05 & 2 \\ -1 & -0.05 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} -0.05 & 1 \\ -2 & -0.05 \end{bmatrix}$$

is unstable, [19]. However, $\mathbb{S}$ is stabilizable by partial reset of order 1. Indeed, $P_1 = \operatorname{diag}(1, 2)$ and $P_2 = \operatorname{diag}(1, 1/2)$ are Lyapunov functions for A_1 and A_2, respectively, with 1-CSC.

Notice that a switched system is always stabilizable by partial reset of order $n-1$. This is due to the fact that any set of stable matrices A_p has a set of QLFs with 1-CSC. Indeed, if P_p are QLFs of A_p, respectively, then $(1/C_p)P_p$, where $C_p = \mathcal{C}_1(P_p)$, are also QLFs for the matrices A_p with 1-CSC, equal to 1. This was the case in the previous example.

Remark 2: In theory, if a set of QLFs with common Schur complement exists, this set can be determined by means of LMI methods, similar to what is done to the case of CQLF, [20]. However, such methods are not effective for large banks of systems, [3]. Moreover, the use of those numerical optimization methods does not produce an answer to the problem of existence of a set of QLFs with common Schur complement (as a general problem). That is, from the application of such methods, the identification/characterization of the classes of systems that allow the existence of sets of QFLs with common Schur complement cannot be achieved, as happens for CQLFs, [3].

V. Existence of CSC: The Block Diagonal Case

In the sequel, we tackle the problem of identifying switched systems that have sets of QLFs with a common Schur complement of a certain order z. Notice that, once z is fixed, the QLFs must be considered as partitioned into a block structure of suitable dimensions, as in (4). For the sake of simplicity, we study the existence of sets of QLFs $\{P_p : p \in \mathcal{P}\}$ with common Schur complement, where the matrices P_p have a block diagonal structure, i.e., $P_p = \operatorname{diag}(P_{11}^p, P_{22}^p)$. In this case, the Schur complement is P_{11}^p. This restriction is in the spirit of most of the algebraic approaches to the CQLF problem, where only diagonal CQLFs are considered [12], [14], [21].

Our first result is the following necessary condition.

Theorem 4: Let $\{A_p : p \in \mathcal{P}\}$ be a set of stable matrices of order n that are partitioned into 2×2 blocks:

$$A_p = \begin{bmatrix} A_{11}^p & A_{12}^p \\ A_{21}^p & A_{22}^p \end{bmatrix}, \textit{where } A_{22}^p \textit{is a square matrix of order } z$$

for $p \in \mathcal{P}$. If there exists a set of block diagonal QLFs for the matrices $\{A_p : p \in \mathcal{P}\}$ with $(n-z)$-CSC equal to P then the blocks A_{11}^p share P as a CQLF and the blocks A_{22}^p are stable.

Proof: Suppose that $P_p = \operatorname{diag}(P, P_{22}^p)$ are block diagonal QLFs for matrices A_p i.e., $-A_p^T P_p - P_p A_p$ are positive definite matrices. So, their blocks (1,1) and (2,2), $Q_1^p = -{A_{11}^p}^T P - P A_{11}^p$ and $Q_2^p = -{A_{22}^p}^T P_{22}^p - P_{22}^p A_{22}^p$, respectively, are also positive definite matrices. This means that, P is a CQLF for the matrices A_{11}^p and $A_{22}^p, p \in \mathcal{P}$, are stable. ■

The previous theorem becomes a necessary and sufficient condition in the case where the A_p matrices are lower or upper block triangular. Here, we only consider the upper block triangular case, since the lower case may be treated in a similar way.

Corollary 2: Let $\mathcal{A} = \{A_p : p \in \mathcal{P}\}$ be a set of stable matrices of order n, where

$$A_p = \begin{bmatrix} A_{11}^p & A_{12}^p \\ 0 & A_{22}^p \end{bmatrix}, \text{ where } A_{11}^p \in \mathbb{R}^{(n-z)\times(n-z)}, p \in \mathcal{P}.$$

Then, $\mathcal{A}$ has a set of block diagonal QLFs with $(n-z)$-CSC equal to P if and only if the (1,1) blocks of A_p share P as a CQLF.

Proof:

($\Rightarrow$) By Theorem 4.

($\Leftarrow$) Suppose that the (1,1) blocks of A_p share P as a CQLF and P_{22}^p are QLFs of the matrices A_{22}^p. Take $P_p = \operatorname{diag}(P, \alpha_p P_{22}^p)$, where $\alpha_p \in \mathbb{R}^+$, as candidates block diagonal Lyapunov QLFs for A_p, with $(n-z)$-CSC. Considering the block structure of the matrices A_p and P_p, we have

$$M_p := -A_p^T P_p - P_p A_p = \begin{bmatrix} Q_1^p & -PA_{12}^p \\ -(A_{12}^p)^T P & \alpha_p Q_2^p \end{bmatrix}$$

where $Q_1^p := -(A_{11}^p)^T P - P A_{11}^p > 0$ and $Q_2^p := -(A_{22}^p)^T P_{22}^p - P_{22}^p A_{22}^p > 0$. In order that M_p is positive definite, the following inequality must be satisfied, see [18, pp. 181, 472], $\alpha_p \lambda_{\min}[Q_2^p] > \lambda_{\max}\left[PA_{12}^p \left(Q_p^1\right)^{-1} (A_{12}^p)^T P\right]$. If $A_{12}^p = 0$, then α_p may be taken arbitrarily in $\mathbb{R}^+$ and if not, we must choose α_p such that

$$\alpha_p > \frac{\lambda_{\max}\left[PA_{12}^p \left(Q_p^1\right)^{-1} (A_{12}^p)^T P\right]}{\lambda_{\min}[Q_2^p]}.$$

■

Notice that the previous corollary can be regarded as a relaxation of the result according to which for switched systems with block-triangular structure the existence of CQLFs for the diagonal blocks in the same position is a necessary and sufficient condition for the existence of a CQLF of the overall system, [21]. However, unlike the result in [21], the existence of a set of QLFs with $(n-z)$-CSC does not hold for simultaneously block triangularizable switched systems. The reason for this is that similarity transformations do not in general preserve the common Schur complement property. Nevertheless, in our case, the block-triangularizing similarity transformation need not be the same. In fact, it is not difficult to show that:

Theorem 5: Let T_p be complex invertible matrices of the form

$$T_p = \begin{bmatrix} T_{11} & 0 \\ T_{21}^p & T_{22}^p \end{bmatrix}, \text{ where } T_{11} \in \mathbb{C}^{(n-z)\times(n-z)}, p \in \mathcal{P}.$$

*The set $\{A_p : p \in \mathcal{P}\}$ has a set of QLFs with $(n-z)$-*CSC *if and only if $\{T_p^{-1} A_p T_p : p \in \mathcal{P}\}$ has a set of complex QLFs with $(n-z)$-*CSC.

Based on Corollary 2 and on the previous theorem we state the following result.

Corollary 3: Let $\{A_p : p \in \mathcal{P}\}$ be a set of stable matrices of order n. If there exist invertible matrices

$$T_p = \begin{bmatrix} T_{11} & 0 \\ T_{21}^p & T_{22}^p \end{bmatrix}, \text{ where } T_{11} \in \mathbb{C}^{(n-z)\times(n-z)}$$

such that

$$T_p^{-1} A_p T_p = \begin{bmatrix} \widetilde{A}_{11}^p & \widetilde{A}_{12}^p \\ 0 & \widetilde{A}_{22}^p \end{bmatrix}$$

*for $p \in \mathcal{P}$, where the set of blocks $\{\widetilde{A}_{11}^p : p \in \mathcal{P}\}$ has CQLF, then $\mathcal{A}$ has a set of QLFs with $(n-z)$-*CSC.

Example 3: The matrices

$$A_1 = \begin{bmatrix} -1 & -1 & -2 & 0 \\ 1 & -2 & -3 & 0 \\ 4 & 1 & -2 & 0 \\ 21 & 2 & -13 & -2 \end{bmatrix}$$

$$\text{and} \quad A_2 = \begin{bmatrix} -1 & -1 & -2 & 0 \\ 3 & 3 & -2 & -1 \\ -2 & 0 & -1 & 0 \\ 13 & 18 & -14 & -6 \end{bmatrix}$$

are such that

$$T_1^{-1}A_1T_1 = \left[\begin{array}{ccc|c} -1 & -1 & -2 & 0 \\ 1 & -2 & -3 & 0 \\ 4 & 1 & -2 & 0 \\ \hline 0 & 0 & 0 & -2 \end{array}\right]$$

and

$$T_2^{-1}A_1T_2 = \left[\begin{array}{ccc|c} -1 & -1 & -2 & 0 \\ 1 & -2 & -3 & 1 \\ -2 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right]$$

where

$$T_1 = \left[\begin{array}{ccc|c} & I_3 & & 0_{3\times 1} \\ \hline 2 & 3 & 4 & 5 \end{array}\right] \text{ and } T_2 = \left[\begin{array}{ccc|c} & I_3 & & 0_{3\times 1} \\ \hline 2 & 5 & 1 & -1 \end{array}\right].$$

From Theorem 5, we may conclude the non-existence of QLFs with 3-CSC with block-diagonal structure, since the (1,1) blocks of size 3 do not have a CQLF. In fact, their difference has rank 1 and its product has negative real eigenvalues, [22].

VI. Conclusion

In this technical note we have analyzed stability and stabilization problems for switched linear systems under arbitrary switching. The main contributions of the technical note are made in Sections III and IV. Among those contributions we emphasize the sufficient condition for stabilization by partial reset of the state (Theorem 3 and Corollary 1). We have established that, if a switched system has a set of QLFs with $(n-z)$-CSC then, the system is stabilizable by partial rest of order z. This sufficient condition is somehow a generalization of the well-known sufficient condition for stability of a switched system (the existence of a CQLF). In fact, if $z=0$ (no reset is done), Corollary 1 becomes that sufficient condition. The existence problem of a set of QLFs with $(n-z)$-CSC for a switched system seems not to be an easy one. The difficulty of this problem is certainly related to its connection to the existence problem of a CQLF. Notice that there are no simple algebraic conditions that characterize systems (matrices) that have a CQLF.

The block triangular case that we have studied in Section V is somehow the counterpart of the well-known sufficient condition of the existence of CQLF: the simultaneous triangularization.

Finally, it should be mentioned that, in practice, the construction of sets of QLFs with a common Schur complement of a pre-fixed order, whenever they exist, is also an important issue, since partial resets used in the stabilization procedure are determined by those sets of QLFs. As an LMI problem, this construction may be tackled numerically using optimization techniques, [20].

Appendix

Proof of Lemma 1: Having in account the structure of matrices G, H and P, after trivial computations, we obtain

$$GPH = \begin{bmatrix} G_{11}P_{11}H_{11}+K & G_{11}P_{12}H_{22}+G_{12}P_{22}H_{22} \\ G_{22}P_{21}H_{11}+G_{22}P_{22}H_{21} & G_{22}P_{22}H_{22} \end{bmatrix}$$

with $K := G_{12}P_{21}H_{11} + G_{11}P_{12}H_{21} + G_{12}P_{22}H_{21}$. Since G_{22}, H_{22} and P_{22} are invertible matrices, it is easy to conclude that $\mathcal{C}_{n-z}(GPH) = G_{11}P_{11}H_{11} - G_{11}P_{12}P_{22}^{-1}P_{21}H_{11}$, which is equivalent to $\mathcal{C}_{n-z}(GPH) = G_{11}\mathcal{C}_{n-z}(P)H_{11}$. ■

Proof of Lemma 2: Let us suppose that P and Q are such that $P \geq Q > 0$. Then $Q^{-1} \geq P^{-1}$, see [18, p. 471]. Since the blocks (1,1) of Q^{-1} and P^{-1} are equal to $(\mathcal{C}_{n-z}(Q))^{-1}$ and $(\mathcal{C}_{n-z}(P))^{-1}$, respectively, see [18, p. 472], we have $(\mathcal{C}_{n-z}(Q))^{-1} - (\mathcal{C}_{n-z}(P))^{-1} \geq 0$. Since, $(\mathcal{C}_{n-z}(Q))^{-1}$ and $(\mathcal{C}_{n-z}(P))^{-1}$ are positive definite, then, [18, p. 472], $\mathcal{C}_{n-z}(P) \geq \mathcal{C}_{n-z}(Q)$. ■

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