

On Minimality of Convolutional Ring Encoders

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Abstract—Convolutional codes are considered with code sequences modeled as semi-infinite Laurent series. It is well known that a convolutional code \mathcal{C} over a finite group \mathcal{G} has a minimal trellis representation that can be derived from code sequences. It is also well known that, for the case that \mathcal{G} is a finite field, any polynomial encoder of \mathcal{C} can be algebraically manipulated to yield a minimal polynomial encoder whose controller canonical realization is a minimal trellis. In this paper we seek to extend this result to the finite ring case $\mathcal{G} = \mathbb{Z}_{p^r}$ by introducing a so-called “ p -encoder”. We show how to manipulate a polynomial encoding scheme of a noncatastrophic convolutional code over \mathbb{Z}_{p^r} to produce a particular type of p -encoder (“minimal p -encoder”) whose controller canonical realization is a minimal trellis with nonlinear features. The minimum number of trellis states is then expressed as p^γ , where γ is the sum of the row degrees of the minimal p -encoder. In particular, we show that any convolutional code over \mathbb{Z}_{p^r} admits a delay-free p -encoder which implies the novel result that delay-freeness is not a property of the code but of the encoder, just as in the field case. We conjecture that a similar result holds with respect to catastrophicity, i.e., any catastrophic convolutional code over \mathbb{Z}_{p^r} admits a noncatastrophic p -encoder.

Index Terms—Convolutional codes over rings, minimal polynomial encoder, minimal trellis.

I. INTRODUCTION

HERE exists a considerable body of literature on convolutional codes over finite groups. In this paper, we are interested in trellis representations that use a minimum number of states. Since decoders, such as the Viterbi decoder, are based on trellis representations, minimality is a desirable property that leads to low complexity decoding. In [1, Section VI-D] a minimal encoder construction is presented in terms of code sequences of the code, involving so-called “granule representatives”, see also [2]. This is a powerful method that applies to convolutional codes over any finite group \mathcal{G} . It is well known that, for the case that \mathcal{G} is a field, any polynomial encoder of a convolutional code can be algebraically manipulated to yield a so-called “canonical polynomial encoder” (left prime and row reduced) whose controller canonical realization yields a minimal trellis representation of the code. This is a fundamental result that is useful in practice because codes are usually

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specified in terms of encoders rather than code sequences. In this paper we seek to extend this result to the finite ring case $\mathcal{G} = \mathbb{Z}_{p^r}$, where r is a positive integer and p is a prime integer. The open problem that we solve is also mentioned in the 2007 paper [3]. We first tailor the concept of encoder to the \mathbb{Z}_{p^r} case, making use of the specific algebraic finite chain structure of \mathbb{Z}_{p^r} . This leads to concepts of “ p -encoder” and “minimal p -encoder”. We then show how to construct a minimal p -encoder from a polynomial encoding scheme of the code. The minimal p -encoder translates immediately into a minimal trellis realization. Thus our results allow for easy construction of a minimal trellis representation from a polynomial encoding scheme and parallel the field case.

Convolutional codes over rings were introduced in [4], [5] where they are motivated for use with phase modulation. In particular, convolutional codes over the ring \mathbb{Z}_M are useful for M -ary phase modulation (with M a positive integer). By the Chinese Remainder Theorem, results on codes over \mathbb{Z}_{p^r} can be extended to codes over \mathbb{Z}_M , see also [6]–[9].

Most of the literature on convolutional codes over rings adopts an approach in which code sequences are semi-infinite Laurent series [1], [2], [9]–[14]. In order to make a connection with this literature, we adopt this approach in our definition of a convolutional code: a linear convolutional code \mathcal{C} of length n over \mathbb{Z}_{p^r} is defined as a subset of $(\mathbb{Z}_{p^r}^n)^{\mathbb{Z}}$ for which there exists a polynomial matrix $G(z) \in \mathbb{Z}_{p^r}^{k \times n}[z]$, such that

$$\mathcal{C} = \{\mathbf{c} \in (\mathbb{Z}_{p^r}^n)^{\mathbb{Z}} \mid \exists \mathbf{u} \in (\mathbb{Z}_{p^r}^k)^{\mathbb{Z}} : \mathbf{c} = \mathbf{u}G(z) \text{ and } \text{supp } \mathbf{u} \subset [N, \infty) \text{ for some integer } N\}. \quad (1)$$

Here $\text{supp } \mathbf{u}$ denotes the support of \mathbf{u} , i.e., the set of time-instants $t \in \mathbb{Z}$ for which $u(t)$ is nonzero. Further, z denotes the right shift operator $zu(t) = u(t-1)$. Clearly, (1) implies that \mathcal{C} is linear and shift-invariant with respect to both z and z^{-1} . If the matrix $G(z)$ has full row rank then $G(z)$ is called an *encoder* of \mathcal{C} .

For the field case any linear convolutional code admits a left prime polynomial encoder, i.e., an encoder that has a polynomial right inverse. Such an encoder $G(z)$ gives rise to the following two properties:

1) *delay-free property*: for any $N \in \mathbb{Z}$

$$\text{supp } \mathbf{c} \subset [N, \infty) \implies \text{supp } \mathbf{u} \subset [N, \infty)$$

2) *noncatastrophic property*:

$$\text{supp } \mathbf{c} \text{ is finite} \implies \text{supp } \mathbf{u} \text{ is finite}$$

where $\mathbf{c} = \mathbf{u}G(z)$. Clearly, in the field case, “delay-free”-ness and “catastrophicity” are encoder properties, not code properties. For the ring case, however, there are codes that do not admit

a noncatastrophic encoder. For example (see [1], [10], [15]) the convolutional code over \mathbb{Z}_4 with encoder $G(z) = [1+z \ 1+3z]$ does not admit a noncatastrophic encoder. Similarly, the rotationally invariant convolutional code over \mathbb{Z}_4 with encoder $G(z) = [3+3z+3z^2 \ 3+z+z^2]$ does not admit a noncatastrophic encoder. The reader is referred to [5] for motivation and characterization of rotationally invariant codes over rings. Further, there are codes that do not admit a delay-free encoder. For example (see [2], [5], [15]) the convolutional code over \mathbb{Z}_4 with encoder $G(z) = [2 \ 2+z]$ does not admit a delay-free encoder. Note that some codes over \mathbb{Z}_{p^r} do not even admit an encoder, for example over \mathbb{Z}_4 the code given by (1) with

$$G(z) = \begin{bmatrix} 1+z & z & z^2 \\ 2 & 2 & 2 \end{bmatrix}.$$

For the ring case, the literature (see, e.g., [15, subsection V-C]) has declared the properties of “delay-free” and “catastrophic” to be properties of the code rather than the encoding procedure. By resorting to a particular type of polynomial encoder, named “ p -encoder”, we show in Section III that delay-freeness is not a property of the code but of the encoding procedure, just as in the field case, see also [16]. We conjecture that the same is true for catastrophicity. To support this argument, in Section IV we examine specific catastrophic convolutional codes over \mathbb{Z}_{p^r} and show that a noncatastrophic p -encoder exists for these examples.

A more recent approach [17] (see also [3], [18]) to convolutional codes focuses on so-called “finite support convolutional codes” in which the input sequence \mathbf{u} corresponds to a polynomial. Thus the natural time axis is \mathbb{Z}_+ and both input sequences and code sequences have finite support. Finite support convolutional codes are, by definition, noncatastrophic (Property 2 above) and can be interpreted as submodules of $\mathbb{Z}_{p^r}^n[z]$. For $n = 1$ connections can be made with polynomial block codes. For more details the reader is referred to our paper [19].

II. PRELIMINARIES

A set that plays a fundamental role throughout the paper is the set of “digits”, denoted by $\mathcal{A}_p = \{0, 1, \dots, p-1\} \subset \mathbb{Z}_p$. Recall that any element $a \in \mathbb{Z}_p$ can be written uniquely as $a = \theta_0 + \theta_1 p + \dots + \theta_{r-1} p^{r-1}$, where $\theta_\ell \in \mathcal{A}_p$ for $\ell = 0, \dots, r-1$ (p -adic expansion). This fundamental property of the ring \mathbb{Z}_{p^r} essentially expresses a type of linear independence among the elements $1, p, p^2, \dots, p^{r-1}$. It leads to specific notions of “ p -linear independence” and “ p -generator sequence” for modules in $\mathbb{Z}_{p^r}^n$, as developed in the 1996 paper [20]. For example, for the simplest case $n = 1$, the elements $1, p, p^2, \dots, p^{r-1}$ are called “ p -linearly independent” in [20] and the module $\mathbb{Z}_{p^r} = \text{span}\{1\}$ is written as $\mathbb{Z}_{p^r} = p\text{-span}\{1, p, p^2, \dots, p^{r-1}\}$. The module \mathbb{Z}_{p^r} is said to have “ p -dimension” r .

In this section we recall the main concepts from [21] on modules in $\mathbb{Z}_{p^r}^n[z]$, that are needed in the sequel. We present the notions of p -basis and p -dimension of a submodule of $\mathbb{Z}_{p^r}^n[z]$, which are extensions from [20]’s notions for submodules of $\mathbb{Z}_{p^r}^n$. From [21] we also recall the concept of a reduced p -basis in $\mathbb{Z}_{p^r}^n[z]$ that plays a crucial role in the next section.

Definition 1: [21] Let $\{v_1(z), \dots, v_m(z)\} \subset \mathbb{Z}_{p^r}^n[z]$. A **p -linear combination** of $v_1(z), \dots, v_m(z)$ is a vector $\sum_{j=1}^m a_j(z)v_j(z)$, where $a_j(z) \in \mathbb{Z}_{p^r}[z]$ is a polynomial with coefficients in \mathcal{A}_p for $j = 1, \dots, m$. Furthermore, the set of all p -linear combinations of $v_1(z), \dots, v_m(z)$ is denoted by **p -span** $(v_1(z), \dots, v_m(z))$, whereas the set of all linear combinations of $v_1(z), \dots, v_m(z)$ with coefficients in $\mathbb{Z}_{p^r}[z]$ is denoted by $\text{span}(v_1(z), \dots, v_m(z))$.

Definition 2: [21] A sequence $(v_1(z), \dots, v_m(z))$ of vectors in $\mathbb{Z}_{p^r}^n[z]$ is said to be a **p -generator sequence** if $p v_m(z) = 0$ and $p v_i(z)$ is a p -linear combination of $v_{i+1}(z), \dots, v_m(z)$ for $i = 1, \dots, m-1$.

The next lemma is a straightforward result that is used in Section III.

Lemma 1: Let $(v_1(z), \dots, v_m(z))$ be a p -generator sequence in $\mathbb{Z}_{p^r}^n[z]$. Then $(v_1(0), \dots, v_m(0))$ is a p -generator sequence in $\mathbb{Z}_{p^r}^n$.

Theorem 1: [21] Let $v_1(z), \dots, v_m(z) \in \mathbb{Z}_{p^r}^n[z]$. If $(v_1(z), \dots, v_m(z))$ is a p -generator sequence then

$$p\text{-span}(v_1(z), \dots, v_m(z)) = \text{span}(v_1(z), \dots, v_m(z)).$$

In particular, $p\text{-span}(v_1(z), \dots, v_m(z))$ is a submodule of $\mathbb{Z}_{p^r}^n[z]$.

Definition 3: [21] The vectors $v_1(z), \dots, v_m(z) \in \mathbb{Z}_{p^r}^n[z]$ are said to be **p -linearly independent** if the only p -linear combination of $v_1(z), \dots, v_m(z)$ that equals zero is the trivial one.

Definition 4: Let M be a submodule of $\mathbb{Z}_{p^r}^n[z]$, written as a p -span of a p -generator sequence $(v_1(z), v_2(z), \dots, v_m(z))$. Then $(v_1(z), v_2(z), \dots, v_m(z))$ is called a **p -basis** for M if the vectors $v_1(z), \dots, v_m(z)$ are p -linearly independent in $\mathbb{Z}_{p^r}^n[z]$.

Lemma 2: [21] Let M be a submodule of $\mathbb{Z}_{p^r}^n[z]$ and let $(v_1(z), v_2(z), \dots, v_m(z))$ be a p -basis for M . Then each vector of M is written in a unique way as a p -linear combination of $v_1(z), \dots, v_m(z)$.

All submodules of $\mathbb{Z}_{p^r}^n[z]$ can be written as the p -span of a p -generator sequence. In fact, if $M = \text{span}(g_1(z), \dots, g_k(z))$ then M is the p -span of the p -generator sequence

$$(g_1(z), p g_1(z), \dots, p^{r-1} g_1(z), \dots, g_k(z), \dots, p^{r-1} g_k(z)).$$

Next, we recall a particular p -basis for a submodule of $\mathbb{Z}_{p^r}^n[z]$, called “reduced p -basis”. We first recall the concept of “degree” of a vector in $\mathbb{Z}_{p^r}^n[z]$, which is the same as in the field case.

Definition 5: Let $v(z)$ be a nonzero vector in $\mathbb{Z}_{p^r}^n[z]$, written as $v(z) = v_0 + v_1 z + \dots + v_d z^d$, with $v_i \in \mathbb{Z}_{p^r}$, $i = 0, \dots, d$, and $v_d \neq 0$. Then $v(z)$ is said to have **degree** d , denoted by $\text{deg } v(z) = d$. Furthermore, v_d is called the **leading coefficient vector** of $v(z)$, denoted by v^{lc} .

In the sequel, we denote the **leading row coefficient matrix** of a polynomial matrix $V(z)$ by V^{lrc} . A matrix $V(z)$ is called **row-reduced** if V^{lrc} has full row rank.

Lemma 3: [21] Let M be a submodule of $\mathbb{Z}_{p^r}^n[z]$, written as a p -span of a p -generator sequence $(v_1(z), \dots, v_m(z))$

with $v_1^{lc}, \dots, v_m^{lc}$ p -linearly independent in $\mathbb{Z}_{p^r}^n$. Then $(v_1(z), \dots, v_m(z))$ is a p -basis for M .

Definition 6: [21] Let M be a submodule of $\mathbb{Z}_{p^r}^n[z]$, written as a p -span of a p -generator sequence $(v_1(z), \dots, v_m(z))$. Then $(v_1(z), \dots, v_m(z))$ is called a **reduced p -basis** for M if the vectors $v_1^{lc}, \dots, v_m^{lc}$ are p -linearly independent in $\mathbb{Z}_{p^r}^n$.

A reduced p -basis in $\mathbb{Z}_{p^r}^n[z]$ generalizes the concept of row reduced basis from the field case. Moreover, it also leads to the predictable degree property and gives rise to several invariants of M , see [21]. In particular, the number of vectors in a reduced p -basis as well as the degrees of these vectors (called **p -degrees**), are invariants of M . Consequently, their sum is also an invariant of M .

Every submodule M of $\mathbb{Z}_{p^r}^n[z]$ has a reduced p -basis. A constructive proof is given by Algorithm 3.11 in [21] that takes as its input a set of spanning vectors and produces a reduced p -basis of M . It is easy to see that if the input is already a p -basis of m vectors, then the algorithm produces a reduced p -basis of again m vectors. Since m is an invariant of the module, it follows that all p -bases of M have the same number of elements. As a result, the next definition is well-defined and not in conflict with the slightly different definition of [21].

Definition 7: The number of elements of a p -basis of a submodule M of $\mathbb{Z}_{p^r}^n[z]$ is called the **p -dimension** of M , denoted as $p\text{-dim } M$.

In recent work [22], [23] it is shown that computational packages for computing minimal Gröbner bases can be used to construct a reduced p -basis for a module M of $\mathbb{Z}_{p^r}^n[z]$. More specifically, it should first be noted that a minimal Gröbner basis of M is not necessarily a basis of M . It is shown in [23] that an ordered minimal Gröbner basis $\{g_1, g_2, \dots, g_\ell\}$ of M gives rise to a reduced p -basis of M of the form

$$(g_1, pg_1, \dots, p^{\beta_1-1}g_1, g_2, pg_2, \dots, p^{\beta_2-1}g_2, \dots, \dots, g_\ell, pg_\ell, \dots, p^{\beta_\ell-1}g_\ell)$$

where $\beta_1 + \dots + \beta_\ell = p\text{-dim } M$.

III. MINIMAL TRELLIS CONSTRUCTION FROM A p -ENCODER

Formally, we define a *trellis section* as a three-tuple $X = (\mathbb{Z}_{p^r}^n, S, K)$, where S is the *trellis state set* and K is the *set of branches* which is a subset of $S \times \mathbb{Z}_{p^r}^n \times S$, see also [1], [2]. A *trellis* is a sequence $\mathcal{X} = \{X_t\}_{t \in \mathbb{Z}}$ of trellis sections $X_t = (\mathbb{Z}_{p^r}^n, S, K_t)$. A *path* through the trellis is a sequence $(\dots, b_{t-1}, b_t, b_{t+1}, \dots)$ of branches $b_t = (s_t, c_t, s_{t+1}) \in K_t$ such that b_{t+1} starts in the trellis state where b_t ends for $t \in \mathbb{Z}$. The set of all trellis paths that start at the zero state is denoted by $\pi(\mathcal{X})$. The mapping $\lambda : \pi(\mathcal{X}) \mapsto (\mathbb{Z}_{p^r}^n)^{\mathbb{Z}}$ assigns to every path $(\dots, b_{t-1}, b_t, b_{t+1}, \dots)$ its label sequence $(\dots, c_{t-1}, c_t, c_{t+1}, \dots)$. A trellis \mathcal{X} is called a *trellis representation* for a convolutional code \mathcal{C} if $\mathcal{C} = \lambda(\pi(\mathcal{X}))$.

A trellis representation \mathcal{X} for a convolutional code \mathcal{C} is called *minimal* if the size of its trellis state set S is minimal among all trellis representations of \mathcal{C} . It is well known how to construct a minimal trellis representation in terms of the code sequences of \mathcal{C} . In fact, the theory of canonical trellis representations from the field case carries through to the ring case, see [1], [2], [24].

Since it plays a crucial role in the proof of our main result, we recall the definition of canonical trellis in Appendix A.

Let us recall the well-known controller canonical form. Let \mathcal{R} be a ring. A matrix $E(z) \in \mathcal{R}^{\kappa \times n}[z]$ is realized in controller canonical form [25] (see also [26, Sec. 5]) as

$$E(z) = B(z^{-1}I - A)^{-1}C + D \quad (2)$$

as follows. Denoting the i 'th row of $E(z)$ by $e_i(z) = \sum_{\ell=0}^{\delta_i} e_{i,\ell} z^\ell$, where $e_{i,\ell} \in \mathcal{R}^{1 \times n}$ and $e_{i,\delta_i} \neq 0$, the matrices A, B, C and D in (2) are given by

$$A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_\kappa \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_\kappa \end{bmatrix}$$

$$C = \begin{bmatrix} C_1 \\ \vdots \\ C_\kappa \end{bmatrix}, \quad D = \begin{bmatrix} e_{1,0} \\ \vdots \\ e_{\kappa,0} \end{bmatrix}$$

where A_i is a $\delta_i \times \delta_i$ matrix, B_i is a $1 \times \delta_i$ matrix and C_i is a $\delta_i \times 1$ matrix, given by

$$A_i = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}, \quad B_i = [1 \quad 0 \quad \dots \quad 0]$$

$$C_i = \begin{bmatrix} e_{i,1} \\ \vdots \\ e_{i,\delta_i} \end{bmatrix} \quad \text{for } i = 1, \dots, \kappa. \quad (3)$$

Whenever $\delta_i = 0$, the i th block in A as well as C is absent and a zero row occurs in B . Denoting the sum of the δ_i 's by δ , it is clear that A is a $\delta \times \delta$ nilpotent matrix. The above controller canonical realization can be visualized as a feedforward shift-register with δ registers.

In the case that \mathcal{R} is a field with q elements it is well known [2], [27] how to obtain a minimal trellis representation for \mathcal{C} from a polynomial encoder. For this, the rows of the polynomial encoder should first be algebraically manipulated (using Smith form and row reduction operations) to yield a left prime and row reduced encoder $G(z)$. Then $G(z)$ is called *canonical* in the literature, see [2, Appendix II]. A minimal trellis representation of \mathcal{C} is then provided by the controller canonical realization $G(z) = B(z^{-1}I - A)^{-1}C + D$ as in (3). Although this result is known, in Appendix B we give a proof by showing that there exists an isomorphism between the trellis state set of the controller canonical realization and the trellis state set of the canonical trellis (as defined in Appendix A) of \mathcal{C} . The set is thus minimal and has q^ν elements, where q is the number of elements of the field and ν is the sum of the row degrees of $G(z)$. The invariant ν is commonly referred to as the “degree” of the code \mathcal{C} (but called the “overall constraint length” in the early literature). The row degrees are called the “Forney indices” of the code [28].

Below we consider convolutional codes over \mathbb{Z}_{p^r} that admit a noncatastrophic encoder, for simplicity, we call such codes noncatastrophic. We show that such codes admit a particular type of polynomial encoder (later called “minimal p -encoder”), whose

controller canonical realization provides a minimal trellis representation, just as in the field case. We are then also able to express the minimal number of trellis states in terms of the sum of the row degrees of a minimal p -encoder.

Let us now first introduce the notion of “ p -encoder”. Recall that $\mathcal{A}_p = \{0, 1, \dots, p - 1\} \subset \mathbb{Z}_{p^r}$.

Definition 8: Let \mathcal{C} be a convolutional code of length n over \mathbb{Z}_{p^r} . Let $E(z) \in \mathbb{Z}_{p^r}^{\kappa \times n}[z]$ be a polynomial matrix whose rows are a p -linearly independent p -generator sequence. Then $E(z)$ is said to be a **p -encoder** for \mathcal{C} if

$$\mathcal{C} = \{\mathbf{c} \in (\mathbb{Z}_{p^r}^n)^{\mathbb{Z}} \mid \exists \mathbf{u} \in (\mathcal{A}_p^\kappa)^{\mathbb{Z}} : \mathbf{c} = \mathbf{u}E(z) \text{ and } \text{supp } \mathbf{u} \subset [N, \infty) \text{ for some integer } N\}.$$

The integer κ is called the **p -dimension** of \mathcal{C} . Furthermore, $E(z)$ is said to be a **delay-free** p -encoder if for any $N \in \mathbb{Z}$ and any $\mathbf{c} \in \mathcal{C}$, written as $\mathbf{c} = \mathbf{u}E(z)$ with $\mathbf{u} \in (\mathcal{A}_p^\kappa)^{\mathbb{Z}}$ we have

$$\text{supp } \mathbf{c} \subset [N, \infty) \implies \text{supp } \mathbf{u} \subset [N, \infty).$$

Also, $E(z)$ is said to be a **noncatastrophic** p -encoder if for any $\mathbf{c} \in \mathcal{C}$, written as $\mathbf{c} = \mathbf{u}E(z)$ with $\mathbf{u} \in (\mathcal{A}_p^\kappa)^{\mathbb{Z}}$ we have

$$\text{supp } \mathbf{c} \text{ is finite} \implies \text{supp } \mathbf{u} \text{ is finite.}$$

Finally, a convolutional code \mathcal{C} that admits a noncatastrophic p -encoder is called **noncatastrophic**.

Thus a difference between a p -encoder $E(z)$ and the encoding matrix $G(z)$ of (1), is that the inputs of $E(z)$ take their values in \mathcal{A}_p rather than in \mathbb{Z}_{p^r} . Note that the idea of using a p -adic expansion for the input sequence is already present in the 1993 paper [1]. It was not until 1996 that the crucial notion of p -generator sequence appeared in [20], but only for constant vectors—it was extended to polynomial vectors in [21]. In our definition the rows of a p -encoder are required to be a p -generator sequence consisting of polynomial vectors.

Recall that a convolutional code over \mathbb{Z}_{p^r} is given by (1):

$$\mathcal{C} = \{\mathbf{c} \in (\mathbb{Z}_{p^r}^n)^{\mathbb{Z}} \mid \exists \mathbf{u} \in (\mathbb{Z}_{p^r}^k)^{\mathbb{Z}} : \mathbf{c} = \mathbf{u}G(z) \text{ and } \text{supp } \mathbf{u} \subset [N, \infty) \text{ for some integer } N\}.$$

Also recall that there exist convolutional codes over \mathbb{Z}_{p^r} that do not admit a $G(z)$ of full row rank, i.e., an encoder. An important observation is that *any* convolutional code over \mathbb{Z}_{p^r} admits a p -encoder, even a p -encoder $E(z)$, such that the rows of E^{lrc} are p -linearly independent in $\mathbb{Z}_{p^r}^n$. Indeed, any reduced p -basis of the polynomial module M spanned by the rows of $G(z)$, produces the rows of such a p -encoder $E(z)$. This shows that the concept of p -encoder is more natural than the concept of encoder as it is tailored to the algebraic structure of \mathbb{Z}_{p^r} . We note that $E(z)$ can be computed from a minimal Gröbner basis of M , using recent ideas from [22] and [23]. The next lemma is straightforward.

Lemma 4: Let $E(z) \in \mathbb{Z}_{p^r}^{\kappa \times n}[z]$ be a p -encoder for a convolutional code \mathcal{C} of length n . Then $E(z)$ is delay-free (Definition 8) if and only if the rows of $E(0)$ are p -linearly independent in $\mathbb{Z}_{p^r}^n$.

Theorem 2: Let \mathcal{C} be a convolutional code of length n over \mathbb{Z}_{p^r} . Then \mathcal{C} admits a delay-free p -encoder $E(z) \in \mathbb{Z}_{p^r}^{\kappa \times n}[z]$ for some integer κ , such that the rows of E^{lrc} are p -linearly independent in $\mathbb{Z}_{p^r}^n$.

Proof: As noted above, \mathcal{C} admits a p -encoder $E(z)$, such that the rows of E^{lrc} are p -linearly independent in $\mathbb{Z}_{p^r}^n$, i.e., they constitute a reduced p -basis. Without loss of generality we may assume that the row degrees of $E(z)$ are nonincreasing. Let L be the smallest nonnegative integer such that the last $\kappa - L$ rows of $E(z)$ are a delay-free p -encoder.

Now assume that $L > 0$ (otherwise we are done). If $L = \kappa$ it means that the last row $e_\kappa(z)$ of $E(z)$ can be written as

$$e_\kappa(z) = z^\ell \bar{e}_\kappa(z),$$

where $\ell > 0$ and $\bar{e}_\kappa(z) \in \mathbb{Z}_{p^r}^n[z]$ with $\bar{e}_\kappa(0) \neq 0$. Note that $\deg \bar{e}_\kappa(z) < \deg e_\kappa(z)$. Clearly, $(e_1(z), \dots, e_{\kappa-1}(z), \bar{e}_\kappa(z))$ is a p -encoder of \mathcal{C} , whose rows are still a reduced p -basis.

If $L < \kappa$, then, by construction, there exist $\alpha_j \in \mathcal{A}_p$ for $j = L + 1, \dots, \kappa$, such that

$$e_L(0) + \sum_{j>L} \alpha_j e_j(0) = 0$$

(use the fact that $(e_1(0), \dots, e_\kappa(0))$ is a p -generator sequence by Lemma 1). Replacing $e_L(z)$ by $\tilde{e}_L(z) := e_L(z) + \sum_{j>L} \alpha_j e_j(z)$ obviously gives a p -basis $(e_1(z), \dots, e_{L-1}(z), \tilde{e}_L(z), e_{L+1}(z), \dots, e_\kappa(z))$ of the module spanned by $e_1(z), \dots, e_L(z), \dots, e_\kappa(z)$ and, consequently, a p -encoder of \mathcal{C} . Moreover, by the p -predictable degree property (Theorem 3.8 of [21]), $\deg \tilde{e}_L(z) = \deg e_L(z)$, which means that $(e_1(z), \dots, \tilde{e}_L(z), \dots, e_\kappa(z))$ is still a reduced p -basis. Since $\tilde{e}_L(0) = 0$, we can write $\tilde{e}_L(z) = z^{\tilde{\ell}} \bar{e}_L(z)$, with $\bar{e}_L(0) \neq 0$ and $\tilde{\ell} > 0$. Note that $p\tilde{e}_L(z)$ is a p -linear combination $p\tilde{e}_L(z) = \sum_{j>L} \beta_j(z) e_j(z)$ with $\beta_j(z) \in \mathcal{A}_p[z]$. Because of the p -linear independence of $e_{L+1}(0), \dots, e_\kappa(0)$, we must have that the coefficients $\beta_j(z)$ are of the form $\beta_j(z) = z^{\ell_j} \tilde{\beta}_j(z)$ with $\ell_j \geq \tilde{\ell}$ for $L + 1 \leq j \leq \kappa$. Consequently, the sequence $(e_1(z), \dots, e_{L-1}(z), \bar{e}_L(z), e_{L+1}(z), \dots, e_\kappa(z))$ is a p -encoder of \mathcal{C} , which is still a reduced p -basis with $\deg \bar{e}_L(z) < \deg e_L(z)$. If $(e_1(z), \dots, \bar{e}_L(z), e_{L+1}(z), \dots, e_\kappa(z))$ is not a delay-free p -encoder, then re-order the vectors so that their degrees are nonincreasing and repeat this procedure until a delay-free p -encoder for \mathcal{C} is obtained. Since the sum of the row degrees of p -bases obtained at each step of the procedure is lower than in the previous step, a delay-free p -encoder is obtained after finitely many iterations. \square

The next example is a simple example that illustrates the above theorem.

Example 1: Over \mathbb{Z}_4 : consider the $(2, 1)$ convolutional code \mathcal{C} of [2, p. 1668] given by the polynomial encoder

$$G(z) = [2 \quad 2 + z].$$

A delay-free p -encoder for \mathcal{C} is given by

$$E(z) = \begin{bmatrix} 2 & 2 + z \\ 0 & 2 \end{bmatrix}.$$

Theorem 3: Let \mathcal{C} be a noncatastrophic convolutional code of length n over \mathbb{Z}_{p^r} . Then \mathcal{C} admits a delay-free noncatastrophic p -encoder $E(z) \in \mathbb{Z}_{p^r}^{\kappa \times n}[z]$ for some integer κ , such that the rows of E^{lrc} are p -linearly independent in $\mathbb{Z}_{p^r}^n$.

Proof: By definition there exists a noncatastrophic p -encoder $E_1(z)$ for \mathcal{C} . Apply Algorithm 3.11 of [21] to the rows of $E_1(z)$. This gives us a reduced p -basis $e_1(z), \dots, e_\kappa(z)$ for the module spanned by the rows of $E_1(z)$. Define $E_2(z)$ as the $\kappa \times n$ polynomial matrix with $e_1(z), \dots, e_\kappa(z)$ as rows. By construction the rows of E_2^{lrc} are p -linearly independent in $\mathbb{Z}_{p^r}^n$. It is easy to see that $E_2(z)$ is still noncatastrophic. If $E_2(z)$ is not delay-free apply the procedure of the proof of Theorem 2 to $E_2(z)$ to obtain a delay-free p -encoder $E(z)$, such that the rows of E^{lrc} are p -linearly independent in $\mathbb{Z}_{p^r}^n$. It is easy to see that $E(z)$ is still noncatastrophic. \square

Definition 9: Let \mathcal{C} be a noncatastrophic convolutional code of length n over \mathbb{Z}_{p^r} . Let $E(z) \in \mathbb{Z}_{p^r}^{\kappa \times n}[z]$ be a delay-free noncatastrophic p -encoder for \mathcal{C} , such that the rows of E^{lrc} are p -linearly independent in $\mathbb{Z}_{p^r}^n$. Then $E(z)$ is called a **minimal p -encoder** of \mathcal{C} . Furthermore, the **p -indices** of \mathcal{C} are defined as the row degrees of $E(z)$ and the **p -degree** of \mathcal{C} is defined as the sum of the p -indices of \mathcal{C} .

Thus, in the terminology of Section II, the rows of a minimal p -encoder are a reduced p -basis. If the code \mathcal{C} has a canonical encoder $G(z)$, then both $G^{\text{lrc}} \bmod p$ and $G(0) \bmod p$ have full row rank in $\mathbb{Z}_p^{\kappa \times n}$, so that a minimal p -encoder is trivially constructed as

$$E(z) = \begin{bmatrix} G(z) \\ pG(z) \\ \vdots \\ p^{r-1}G(z) \end{bmatrix}. \quad (4)$$

An important observation is that all noncatastrophic codes admit a minimal p -encoder $E(z)$ but not all such codes admit an encoder $G(z)$ that is row reduced and/or delay-free.

Definition 10: Let \mathcal{C} be a convolutional code of length n with p -encoder $E(z) \in \mathbb{Z}_{p^r}^{\kappa \times n}[z]$. Denote the sum of the row degrees of $E(z)$ by γ and let

$$(A, B, C, D) \in \mathbb{Z}_{p^r}^{\gamma \times \gamma} \times \mathbb{Z}_{p^r}^{\kappa \times \gamma} \times \mathbb{Z}_{p^r}^{\gamma \times n} \times \mathbb{Z}_{p^r}^{\kappa \times n}$$

be a controller canonical realization of $E(z)$. Then the **controller canonical trellis** corresponding to $E(z)$ is defined as $\mathcal{X} = \{X_t\}_{t \in \mathbb{Z}}$, where $X_t = (\mathbb{Z}_{p^r}^n, \mathcal{A}_p^\gamma, K_t)$ with

$$K_t = \{(s(t), s(t)C + u(t)D, s(t)A + u(t)B) \text{ such that } s(t) \in \mathcal{A}_p^\gamma, u(t) \in \mathcal{A}_p^\kappa\}.$$

Note that the states take their values in the nonlinear set \mathcal{A}_p^γ , which is not closed with respect to addition or scalar multiplication. Similarly, the inputs take their values in the nonlinear set \mathcal{A}_p^κ . The next theorem presents our main result.

Theorem 4: Let \mathcal{C} be a noncatastrophic convolutional code of length n with minimal p -encoder $E(z) \in \mathbb{Z}_{p^r}^{\kappa \times n}[z]$. Denote

the p -degree of \mathcal{C} by γ . Then the controller canonical trellis corresponding to $E(z)$ is a minimal trellis representation for \mathcal{C} . In particular, the minimum number of trellis states equals p^γ .

Proof: See Appendix B. \square

In the field case $r = 1$ the above theorem coincides with the classical result, i.e., the minimum number of trellis states equals p^γ , where γ is the degree of the code.

For convolutional codes that admit a canonical encoder, we have the following corollary, which follows immediately from applying Theorem 4 to the minimal p -encoder given by (4). Note that the result coincides with results in [14, Sec. 7.4], where a canonical encoder is called “minimal-basic”.

Corollary 1: Let \mathcal{C} be a (n, k) convolutional code that has a canonical encoder $G(z) \in \mathbb{Z}_{p^r}^{\kappa \times n}[z]$. Then the rk p -indices of \mathcal{C} are the k row degrees of $G(z)$, each occurring r times. The minimum number of trellis states equals q^ν , where ν is the sum of the row degrees of $G(z)$ and where $q = p^r$.

The next example illustrates our theory for the more interesting case where the code does not admit a canonical encoder.

Example 2: Over \mathbb{Z}_4 : consider the $(3, 2)$ convolutional code \mathcal{C} given by the polynomial encoder

$$G(z) = \begin{bmatrix} g_1(z) \\ g_2(z) \end{bmatrix}$$

where

$$g_1(z) = [z^2 + 1 \quad 1 \quad 0]$$

and

$$g_2(z) = [2z \quad 2 \quad 1].$$

Clearly, $G(z)$ is a left prime encoder whose controller canonical trellis has $4^3 = 64$ trellis states. Note that G^{lrc} does not have full row rank and therefore $G(z)$ is not canonical. Denote by $\text{im } G(z)$ the polynomial module spanned by the rows of $G(z)$. A p -basis for the module $\text{im } G(z)$ is provided by the rows of the matrix

$$\begin{bmatrix} g_1(z) \\ 2g_1(z) \\ g_2(z) \\ 2g_2(z) \end{bmatrix} = \begin{bmatrix} z^2 + 1 & 1 & 0 \\ 2z^2 + 2 & 2 & 0 \\ 2z & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

which has leading row coefficient matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The row reduction algorithm of [21, Algorithm 3.11] is particularly simple in this case: by adding z times the third row to the second row, we obtain the matrix $E(z)$, given by

$$E(z) = \begin{bmatrix} z^2 + 1 & 1 & 0 \\ 2 & 2z + 2 & z \\ 2z & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

whose rows are a reduced p -basis for the module $\text{im } G(z)$. Indeed, the rows of its leading row coefficient matrix, given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

are p -linearly independent. As a result, the p -indices of C are $2, 1, 1, 0$ and the p -degree of C equals 4 . The controller canonical trellis corresponding to $E(z)$ is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 0 \end{bmatrix}; \quad D = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

This trellis is minimal with $2^4 = 16$ trellis states.

Example 3: Over \mathbb{Z}_4 : consider the $(2, 1)$ convolutional code C of Example 1, given by the polynomial encoder $G(z) = [2 \ 2+z]$ (note that $G(z)$ is not delay-free). The delay-free p -encoder

$$E(z) = \begin{bmatrix} 2 & 2+z \\ 0 & 2 \end{bmatrix}.$$

of Example 1 is clearly minimal, so that its corresponding trellis is minimal with two states which concurs with [1].

IV. CONCLUSION

An important class of polynomial encoders for convolutional codes over a field are the canonical ones. Their feedforward shift register implementations are minimal trellis representations of the code. The trellis state space is linear. However, for convolutional codes over the finite ring \mathbb{Z}_{p^r} , the literature has generalized this result only for restricted cases. In this paper we introduce the concept of p -encoder and define *minimal p -encoder* for the class of noncatastrophic convolutional codes. We show how to obtain a minimal p -encoder from a polynomial encoding scheme of the code. We show that the feedforward shift register implementation of such a minimal p -encoder is a minimal trellis representation of the code. Its trellis state space is nonlinear. We also express the minimal number of states in terms of the row degrees of the minimal p -encoder. In our view a minimal p -encoder is the ring analogon of the ‘‘canonical polynomial encoder’’ from the field case. We also present the novel concepts of p -indices and p -degree of a code as analogons of the field notions of ‘‘Forney indices’’ and ‘‘degree’’, respectively. Our approach allows us to view ‘‘delay-freeness’’ as a property of the p -encoder. Thus we arrive at the novel result that delay-freeness is a property of the encoding scheme (just as in the field case) rather than a property of the code, as in the literature so far (see e.g., [15, Sec. V-C]). We conjecture that a similar

phenomenon occurs with respect to catastrophicity, i.e., ‘‘non-catastrophic’’ is a property of the p -encoder, not the code. This would imply that minimal p -encoders can be obtained for *all* convolutional codes over \mathbb{Z}_{p^r} , including the catastrophic codes. This is of particular importance for rotationally invariant catastrophic codes, see e.g., [5]. It is a topic of future research to investigate this conjecture which is likely to involve a generalization of a type of ‘‘normal form’’ for polynomial matrices over \mathbb{Z}_{p^r} . To support our conjecture, let us examine the rotationally invariant catastrophic code C_1 over \mathbb{Z}_4 given by the encoder $G_1(z) = [3+3z+3z^2 \ 3+z+z^2]$. A noncatastrophic minimal p -encoder for C_1 is given by

$$E_1(z) = \begin{bmatrix} 3+3z+3z^2 & 3+z+z^2 \\ 2 & 2 \end{bmatrix}$$

yielding a minimal trellis representation of C_1 with four states. Similarly the catastrophic code C_2 over \mathbb{Z}_4 with encoder $G_2(z) = [1+z \ 1+3z]$ has a noncatastrophic minimal p -encoder

$$E_2(z) = \begin{bmatrix} 1+z & 1+3z \\ 2 & 2 \end{bmatrix}$$

yielding a minimal trellis representation of C_2 with two states.

APPENDIX A

In this Appendix, we recall the construction of a minimal trellis for a convolutional code C as a so-called *two-sided realization* of C , see [1], [2], [10], [11], [14], and [24]. Consider two code sequences $\mathbf{c} \in C$ and $\tilde{\mathbf{c}} \in C$. Conform [24], the *concatenation* at time $t \in \mathbb{Z}$ of \mathbf{c} and $\tilde{\mathbf{c}}$, denoted by $\mathbf{c} \wedge_t \tilde{\mathbf{c}}$, is defined as

$$\mathbf{c} \wedge_t \tilde{\mathbf{c}}(t') := \begin{cases} c(t') & \text{for } t' < t \\ \tilde{c}(t') & \text{for } t' \geq t \end{cases}.$$

The code sequences \mathbf{c} and $\tilde{\mathbf{c}}$ are called *equivalent*, denoted by $\mathbf{c} \simeq \tilde{\mathbf{c}}$, if

$$\mathbf{c} \wedge_0 \tilde{\mathbf{c}} \in C.$$

Definition 11: Let C be a linear convolutional code of length n over a finite ring \mathcal{R} . The *canonical trellis* of C is defined as $\mathcal{X} = \{X_t\}_{t \in \mathbb{Z}}$, where $X_t = (\tilde{\mathcal{R}}^n, S, K_t)$ with $S := C \bmod \simeq$ and

$$K_t := \{(s(t), c(t), s(t+1)) | s(t) = z^{-t}\mathbf{c} \bmod \simeq \text{ and } s(t+1) = z^{-t-1}\mathbf{c} \bmod \simeq\}.$$

It has been shown in [24] that the above trellis is minimal. Intuitively this is explained from the fact that, by construction, states cannot be merged.

APPENDIX B

In this Appendix, we prove Theorem 4 via a bijective mapping from the controller canonical trellis state set to the trellis state set of the canonical trellis that is defined in Appendix A.

We first provide the proof for the field case. In our proof of Theorem 4, which is the ring case, we are then able to highlight the parts that are different from the proof for the field case.

Theorem 5: Let \mathcal{C} be a (n, k) convolutional code of degree ν over a finite field \mathcal{R} with canonical encoder $G(z) \in \mathcal{R}^{k \times n}[z]$. Then the controller canonical trellis corresponding to $G(z)$ is a minimal trellis representation for \mathcal{C} . In particular, the minimum number of trellis states equals q^ν , where q is the size of the field \mathcal{R} .

Proof: Denote the memory of \mathcal{C} by ν_* , i.e., ν_* is the maximal Forney index of \mathcal{C} . Consider the mapping $\Theta : \mathcal{R}^\nu \mapsto C \bmod \simeq$, given by

$$\Theta(s) := [\mathbf{c}]_{\simeq}$$

where $\mathbf{c} \in \mathcal{C}$ passes through state s at time 0. The mapping Θ is well-defined since for any s there exists such a code sequence and any two code sequences that pass through state s at time 0 are obviously equivalent.

Since the trellis state set $C \bmod \simeq$ of the canonical trellis of Appendix A is minimal, it suffices to prove that Θ is an isomorphism, as follows. Surjectivity follows immediately from the fact that all code sequences pass through some state at time 0. Furthermore, the mapping Θ is linear since $\Theta(s_1 + s_2) = [\mathbf{c}_1 + \mathbf{c}_2]_{\simeq}$. It remains to prove that Θ is injective.

For this, let $s \in \mathcal{R}^\nu$ be such that $\Theta(s) = 0$. Define $u(-\nu_*), \dots, u(-2), u(-1)$ as elements of \mathcal{R}^k for which

$$[u(-\nu_*) \ \cdots \ u(-2) \ u(-1)] \begin{bmatrix} BA^{\nu_*-1} \\ \vdots \\ BA \\ B \end{bmatrix} = s.$$

Define $\mathbf{u} := (\dots, 0, 0, u(-\nu_*), \dots, u(-2), u(-1), 0, 0, \dots)$ and let $\mathbf{c} := G(z)\mathbf{u}$ be the corresponding code sequence. Then clearly \mathbf{c} passes through s . From $\Theta(s) = 0$ it now follows that the sequence $\mathbf{c} \wedge_0 0$ is a code sequence. Denote its state at time 0 by s' and its input sequence by \mathbf{u}' . Then clearly

$$[u'(-\nu_*) \ \cdots \ u'(-2) \ u'(-1)] \begin{bmatrix} BA^{\nu_*-1} \\ \vdots \\ BA \\ B \end{bmatrix} = s'.$$

We now prove that $s = s'$, as follows. First, it is clear that

$$\begin{aligned} & [c(-\nu_*) \ \cdots \ c(-2) \ c(-1)] \\ &= [u(-\nu_*) \ \cdots \ u(-2) \ u(-1)] \\ &\quad \times \begin{bmatrix} D & BC & BAC & \cdots \\ 0 & D & BC & \cdots \\ 0 & 0 & D & \cdots \\ & & & \ddots \end{bmatrix}. \end{aligned} \quad (5)$$

Furthermore, from the fact that the encoder is delay-free (Property 1 in Section I) it follows that $D = G(0)$ has full row rank and that $u'(\ell) = 0$ for $\ell < -\nu_*$. As a result

$$\begin{aligned} & [c(-\nu_*) \ \cdots \ c(-2) \ c(-1)] \\ &= [u'(-\nu_*) \ \cdots \ u'(-2) \ u'(-1)] \end{aligned}$$

$$\times \begin{bmatrix} D & BC & BAC & \cdots \\ 0 & D & BC & \cdots \\ 0 & 0 & D & \cdots \\ & & & \ddots \end{bmatrix}. \quad (6)$$

Since D has full row rank, the matrix in the above equation also has full row rank. Since the right-hand sides of (5) and (6) are equal, it then follows that $u(\ell) = u'(\ell)$ for $-\nu_* \leq \ell \leq -1$. As a result $s = s'$.

We now prove that $s = 0$. By the above, $\mathbf{c} \wedge_0 0$ is a code sequence that passes through s at time 0. Its input sequence \mathbf{u}' is of the form

$$(\dots, 0, 0, u'(-\nu_*), \dots, u'(M), 0, 0, \dots)$$

where $M \geq 0$. Here we used the fact that the encoder is non-catastrophic (Property 2 in Section I). By construction the state of $\mathbf{c} \wedge_0 0$ at time $M + \nu_* + 1$ then equals zero. We now use the row reducedness of $G(z)$ to conclude that $s = 0$, as follows. Denote the state at time $M + \nu_*$ by \bar{s} . Now recall the formula (3) for the controller canonical form. Since $\bar{s}A = 0$, the nonzero components of \bar{s} must be last components in a $(1 \times \nu_i)$ -block in \bar{s} . Also, $c(M + \nu_*) = 0$, so that $\bar{s}C = 0$. By construction, the last rows of the $(\nu_i \times n)$ -blocks of C are rows from G^{lrc} and are therefore linearly independent. As a result, $\bar{s} = 0$. Repeating this argument again and again, we conclude that $u'(0) = \dots = u'(M) = 0$ and all states for time ≥ 0 are zero, so that, in particular $s = 0$, which proves the theorem. Obviously, the size of the trellis state set S equals q^ν . \square

We now turn to the ring case to prove the analogon of the above theorem. As compared to the field case, the proof requires some care because the trellis state set \mathcal{A}_p^γ is not linear.

Proof of Theorem 4: Define ν_* as the maximal p -index of \mathcal{C} . Consider the mapping $\Theta : \mathcal{A}_p^\gamma \mapsto C \bmod \simeq$, given by

$$\Theta(s) := [\mathbf{c}]_{\simeq}$$

where $\mathbf{c} \in \mathcal{C}$ passes through state s at time 0. Then Θ can be shown to be well-defined and surjective, as in the proof of Theorem 5. Note that Θ is not necessarily a linear mapping. As a result, injectivity can no longer be proven by showing that $\Theta(s) = 0$ only for $s = 0$, as in the proof of Theorem 5. Thus, to show that Θ is injective, let s and $\tilde{s} \in \mathcal{A}_p^\gamma$ be such that $\Theta(s) = \Theta(\tilde{s})$. Let \mathbf{c} be the code sequence that passes through s at time 0, as defined in the proof of Theorem 5. Let $\tilde{\mathbf{c}}$ be the analogous code sequence that passes through \tilde{s} at time 0. Note that both \mathbf{c} and $\tilde{\mathbf{c}}$ have finite support. From $\Theta(s) = \Theta(\tilde{s})$ it now follows that the sequence $\mathbf{c} \wedge_0 \tilde{\mathbf{c}}$ is a code sequence. Denote its state at time 0 by s' and its input sequence by $\mathbf{u}' \in (\mathcal{A}_p^\kappa)^\mathbb{Z}$. Since $E(z)$ is a delay-free p -encoder, the rows of $E(0)$ are a p -basis (use also Lemma 1). By Lemma 2.8 of [21] (see also [20]), it now follows from the fact that inputs only take values in \mathcal{A}_p that $s = s'$. The reasoning is as in the proof of Theorem 5.

We now prove that $s = \tilde{s}$. By the above, $\mathbf{c} \wedge_0 \tilde{\mathbf{c}}$ is a code sequence that passes through s at time 0. As in the proof of Theorem 5, it follows that its state equals zero at time $M + \nu_* + 1$ for some $M \geq 0$. Since $E(z)$ is a minimal p -encoder, the rows of E^{lrc} are p -linearly independent. It now follows from the fact that states only take values in \mathcal{A}_p that the state at time $M + \nu_*$

must also be zero. The reasoning is as in the proof of Theorem 5. Repeating this argument again and again, we conclude that all states for time $\geq \nu_*$ are zero. As a result, $u'(0) = u'(1) = \dots = u'(\nu_* - 1) = 0$, so that

$$s[C \ AC \ \dots \ A^{\nu_*-1}C] = \tilde{s}[C \ AC \ \dots \ A^{\nu_*-1}C].$$

We now prove that the above equation implies that $s = \tilde{s}$. By Theorem 3.10 of [21], the rows of E^{hrc} are not only p -linearly independent but also a p -generator sequence. By Lemma 2.8 of [21] any p -linear combination of these rows is then unique. By construction, this property is inherited by the rows of $[C \ AC \ \dots \ A^{\nu_*-1}C]$. Since both s and \tilde{s} take their values in \mathcal{A}_p , it therefore follows that $s = \tilde{s}$, which proves the theorem. Obviously, the size of the trellis state set S equals p^γ . \square

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