STRESS INTENSITY FACTORS BY NUMERICAL EVALUATION IN CRACKED STRUCTURES

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Abstract. This paper reviews post processing techniques to estimate stress intensity factors (SIF) using the stress and displacement fields calculated by numerical methods. The stress intensity factor is an important parameter for estimating the residual life in cracked structures. Several techniques, presenting various levels of precision, particularly when dealing with complex boundaries, can be used for calculating SIF. In this paper the following techniques are reviewed: compounding method, displacement extrapolation method, force method, J-integral, singularity subtraction technique and virtual crack closure method (VCCT) in its classical and modified form. Two benchmark problems are considered to demonstrate the advantages and disadvantages of the analyzed SIF calculation techniques. The modified virtual crack closure technique proved to be an accurate method for determining SIFs with low computational effort and with good accuracy for common problems.

Keywords. Stress intensity factor calculation, compounding method, displacement extrapolation method, force method, J-integral method, singularity subtraction technique, virtual crack closure method.

1 INTRODUCTION

Stress Intensity Factor (SIF) is an essential Linear Elastic Fracture Mechanics (LEFM) parameters for structural integrity assessment of structures containing cracks and singular stress fields. SIF gives a measure of the intensity of the stress field in the crack tip region. This parameter gives the possibility to analyze the possible crack growth or the possible catastrophic failure if a given load is applied to the structure. The stress intensity factors can be calculated using stress and strain analysis or parameters that measure the energy released by crack growth. The estimation of stress intensity factors can be done by analytical or numerical techniques. Normally, the analytical ones are more complex to calculate; however they have some advantages, because an analytical solution can be applied for a range of crack lengths. The numerical techniques require the calculation of stress or strain field for each crack length and therefore for each value of SIF.

For complex structures, it is difficult to perform an analysis taking into account all boundary effects near the crack tip, so the numerical calculation of SIF has some advantages for these structures. The evolution of computers (hardware and software) permits the use of more complex numerical techniques and to obtain solutions with smaller calculation time. Hence, the numerical techniques for estimating stress intensity factors are nowadays more popular than the analytical techniques.

In this paper a semi-analytical technique, the compounding method was also reviewed. This method allows estimating SIFs in complex structures, by decomposing them in simple structures for which SIF solutions may be found in the scientific literature or may be calculated using analytical or numerical techniques.

Five numerical methods for SIFs are reviewed in this study. These methods are post-processing techniques, since they require the stress field in crack tip vicinity. Two of these techniques are based on the extrapolation of a variable to the crack tip – displacement extrapolation and force method, one is based on the known solution of SIF and assumptions of the stress in a calibrated point – Singularity subtraction method, and the other two are based on energetic assumptions – J-integral and virtual crack closure technique (VCCT).

The stress/strain field for a structure can be calculated using several techniques. The most common and available in several commercial packages is the finite element method (FEM) – [1]; however the boundary element method (BEM) – [2], is also widely used. Actually, new techniques based on meshless methods as the extended finite element model (XFEM), are emerging and have several advantages compared with the traditional methods in problems of fracture mechanics, [3].

In this study, the stress/strain fields were calculated using both finite element~~s~~ and dual boundary element method (DBEM). The DBEM is an extension to the BEM, developed by Portela, [4], with the capacity to add internal boundaries to the principal boundary, as internal cracks.

Two examples were chosen to check and compare the different SIF~~s~~ calculation techniques. The first example was a finite plate with two collinear cracks and loaded remotely in tension. This situation can be found in multi-site damage condition; where the stress at the crack tip can be influenced by the other cracks.

The other example was a reinforced finite panel, with two stiffeners and with a central crack. This type of structure is largely found in situations where stiffened panels with low weight are required, for example in aircrafts, ships and trains.

2 STRESS INTENSITY FACTORS TECHNIQUES

The concept of stress intensity factor (K) is a result of the bi-dimensional analysis of the stress field at the crack tip. This analysis was carried out by Williams in 1957 - [5], taking into account Westergaard's work, [6]. Using a coordinates system centred in the crack tip and according to William's analysis, the near crack tip components of the stress field are proportional to $K/r^{1/2}$, where K is the stress intensity factor. The crack opening may correspond to one of the three basic cases, the opening mode, the sliding mode and the tearing mode, or to any of their combination; thus, there are three basic stress intensity factor values denoted with the subscripts I, II and III. For each mode, the stress field in the crack tip region may be calculated using the expressions:

$$\begin{aligned} \lim_{r\to 0} \sigma_{ij}^{(I)} &= \frac{K_I}{\sqrt{2\pi r}} f_{ij}^{(I)}(\theta) \\ \lim_{r\to 0} \sigma_{ij}^{(II)} &= \frac{K_{II}}{\sqrt{2\pi r}} f_{ij}^{(II)}(\theta) \\ \lim_{r\to 0} \sigma_{ij}^{(III)} &= \frac{K_{III}}{\sqrt{2\pi r}} f_{ij}^{(III)}(\theta) \end{aligned} \tag{1}$$

where r and θ are the polar coordinates in the system of axes having the origin at the crack tip. According to Williams' analysis, [5], the components of the stress field can be written as series expansions; from this expansion, the stress intensity factors can be calculated from the stress field in the crack tip.

2.1 Compounding method

The compounding method, proposed by Cartwright and Rooke in 1974, [7], is used for stress intensity factor determination in complex structures, starting from available solutions for simpler problems. This method consists of decomposing a cracked structure with N boundaries into N ancillary configurations, each one containing one boundary and for which stress intensity factor solutions are available, [8].

The stress intensity factor for a crack tip can be expressed as a function of the N ancillary stress intensity factor values:

$$K_{1N} = K_0 + \sum_{n=1}^{N} \left(K_n - K_0 \right) \tag{2}$$

where K_0 is the stress intensity factor for the same body without the boundaries and K_n is the SIF for the body with the boundary n. When boundaries interact one to another, stresses at the location of these boundaries will be different, leading to an increase or decrease of stress intensity factor values. Due to this effect, another term K_e, corresponding to the boundary-to-boundary interaction is added in equation (2):

$$K_{1N} = K_0 + \sum_{n=1}^{N}\left(K_n - K_0\right) + K_e \tag{3}$$

If a crack crosses a boundary, a modification in equations (2) and (3) is required in order to account for the effect of the stress originated by this boundary at the crack tip. This effect was not taken into account in the preceding considerations, but must be considered because of the change in stress and displacement fields at the crack tip due to the boundary. In these circumstances, the structure with a crack of length $2a$, containing a boundary crossed by the crack, is converted into an equivalent structure without the boundary but with an equivalent crack of length $2a'$. The stress intensity factor calculation takes into account the other boundaries interacting with the crack of length $2a'$. Taking into account that $K_{0n} = Q_0K_n$, where K_n is the stress intensity factor for configuration n containing a crack of length $2a$, the stress intensity factor can be re-written in non-dimensional form, dividing all terms by $\sigma\sqrt{\pi a}$, obtaining:

$$Q_{1N} = Q_0\left[1 + \sum_{n=1}^{N}\left(Q_n - 1\right)\right] \tag{4}$$

2.2 Displacement extrapolation

The displacement extrapolation method was developed in order to obtain crack tip singular stresses and stress intensity factors using only nodal displacements of elements around the crack tip, [9]. The near crack tip displacement field may be expressed as a series in function of the stress intensity factors, distance to the crack tip and the angle with the propagation direction. For θ = 180° (along the crack line) and using the first term of this series, the displacements are given by:

$$\begin{aligned} u_x &= \frac{K_{II}}{2G}\sqrt{\frac{r}{2\pi}}\left(1+\kappa\right) \\ u_y &= \frac{K_I}{4G}\sqrt{\frac{r}{2\pi}}\left(1+\kappa\right) \\ u_z &= \frac{2K_{III}}{G}\sqrt{\frac{r}{2\pi}} \end{aligned} \tag{5}$$

where $\kappa = 3 - 4\upsilon$ in plane stress and $\kappa = \frac{3-\upsilon}{1+\upsilon}$ in plane strain. From these equations, a relationship between displacements and the apparent stress intensity factor K_0 is obtained. Using a linear extrapolation to $r = 0$, the stress intensity factor at the crack tip can be estimated with a high accuracy. This technique can be more accurate using the quarter node point or collapsed elements; however, for these situations modifications are required in the equations above, [10].

2.3 Force method

The force method is an alternative to the displacement method, using nodal reactions obtained in a finite element model. The first paper dealing with this method was published in 1977 by Raju and Newman, [11]. Curiously, it then disappeared from the literature, until it was mentioned again in [12]; recently, Morais [13] published an application of the method to isotropic center-cracked infinite plates and orthotropic beam specimens showing good results.

Using the first term of Williams series expansion of the stress it is possible to estimate the SIF value using extrapolation. The stress along the line defined by crack tip direction (*θ =0°)* are:

$$\begin{aligned} \sigma_{xx} &= \frac{K_I}{\sqrt{2\pi r}} \\ \sigma_{yy} &= \frac{K_I}{\sqrt{2\pi r}} \\ \tau_{xy} &= \frac{K_{II}}{\sqrt{2\pi r}} \end{aligned} \tag{6}$$

Analyzing the forces along a distance r, the following expressions for the forces transmitted along this line in the x and y direction, are obtained:

$$F_y = \int_0^{x_c} \sigma_{yy} dy = K_I \sqrt{\frac{2x_c}{\pi}}$$
$$F_x = \int_0^{x_c} \tau_{xy} dy = K_{II} \sqrt{\frac{2x_c}{\pi}} \qquad (7)$$

The F_x and Fy values may be obtained as post-processing results of a FE~~M~~ model as a function of the coordinate xc. This coordinate is the distance from the crack tip to the intermediate location between the node under consideration and the next node. Therefore the Mode I and Mode II stress intensity factor values as a function of distance r are given by:

$$K_I' = \sqrt{\frac{\pi}{2x_c}} \sum_{i=1}^{n} F_{y,i}$$
$$K_{II}' = \sqrt{\frac{\pi}{2x_c}} \sum_{i=1}^{n} F_{x,i} \qquad (8)$$

Calculating K_0 for several values of x_c, a linear extrapolation to $r = 0$ can be carried out, giving the value of K. Since stress values very close to the crack tip cannot be accurately determined, it is usually found that K_0 values in that region do not follow the K_0 *vs.* r trend of other points, and are neglected.

2.4 Singularity subtraction technique

The Singularity Subtraction Technique (SST) uses a singular solution of the stress field in order to calculate the stress intensity factor, [14]. From William's series expansion, giving the stress fields, [5], and the equations from statics, stress tensor components in any location close to the crack tip are known, and may be used for calculating stresses in any direction.

If a point P close to the crack tip where the stress components are known is selected, the stress σ_P calculated in P, at a distance ε from the crack tip, is related with the new t_I and t_{II} components that are obtained using the next equations:

$$K_I = t_I \sqrt{2\pi r}$$
$$K_{II} = t_{II} \sqrt{2\pi r} \qquad (9)$$

For the K determination the value of r is independent of ε and it is calibrated from known stress intensity factor solutions.

2.5 J-integral

The J-integral is a contour integral characterizing the strain energy release rate for an elastic non-linear material. The stress field is related to the strain energy density as:

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \qquad (10)$$

From the definition of potential energy along a contour, work theorem and the previous equation, Rice [15] defined an integral independent of the integration contour Γ around the crack tip as:

$$J = \oint_{\Gamma} \left(w dy - \mathbf{T} \frac{\partial \mathbf{u}}{\partial x} ds \right) \qquad (11)$$

where~~,~~ w is strain energy density per unit volume, $\mathbf{T}$ is the traction vector ($\mathbf{T} = \sigma\mathbf{n}$), $\mathbf{u}$ is the displacement vector and y is the direction perpendicular to the crack line. For linear or non-linear elastic materials, the strain energy release rate is equal to the strain energy release rate along a contour at crack tip vicinity (J = G); this parameter is related to the stress intensity factor as $G = K^2/E$ in plane stress or $G = K^2/(E(1-\nu^2))$ in plane strain.

2.6 Virtual crack closure technique

The Virtual Crack Closure Technique (VCCT) is based on energy release rate when the crack grows with an infinitesimal increment. It is based on the calculation of the strain energy release rate, using the energy variation when an extension of crack length is imposed:

$$G = \frac{\partial U}{\partial a} \approx \frac{U_{a+\Delta a} - U_a}{\Delta a} \qquad (12)$$

This technique was proposed by Rybicki and Kanninen in 1977, [16]; however it requires two finite elements analysis in order to calculate the strain energy release rate for a specific crack length. A review of VCCT can be ~~is~~ found in the report referred in [17]. In this paper, a modified version is proposed where only one model is needed to calculate the energy release rate.

The modified VCCT is based on the same assumptions as VCCT in two steps, but in addition it is assumed that the conditions at the crack tip are not significantly altered when the crack extends by an increment Δa, from a crack length $a+\Delta a$ to a length $a+2\Delta a$. This implies that the displacements of a region close to the crack tip, when the tip is at specific node, are approximately the same as the displacements at the same location when the tip is at the previous node.

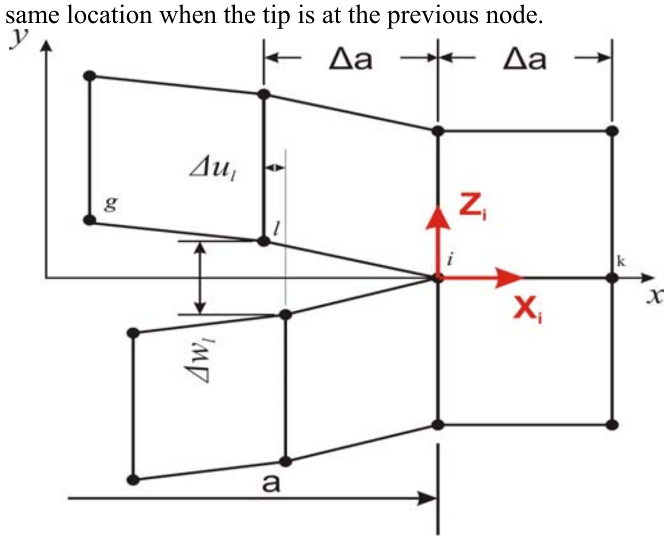

Fig. 1 – Modified virtual crack closure technique notation.

The energy variation ΔE necessary to close the crack along a distance Δa is:

$$\Delta E = \frac{1}{2}\left(X_i \cdot \Delta u_l + Z_i \cdot \Delta w_l\right) \tag{13}$$

where X_i and Z_i are the nodal forces at point i and Δu_l and Δw_l are the node l displacements. Therefore the information required for the calculation of the energy variation is obtained from a single finite elements analysis.

After obtaining the energy variation, the energy release rate is calculated as:

$$G = \frac{\Delta E}{\Delta A} = \frac{\Delta E}{\Delta a \cdot b} \tag{14}$$

where ΔA is the surface area created by a crack propagation of Δa; in the case of plates with a thickness b, this area is $\Delta a \cdot b$. The calculation of strain energy release rates for each mode is made using the displacements and nodal forces corresponding to the strain energy of that mode.
Thus, for the case of Fig. 1, the energy release rate is:

$$\begin{aligned} G_I &= -\frac{1}{2\Delta a} Z_i \Delta w_l = -\frac{1}{2\Delta a} Z_i \left(w_l - w_{l^*}\right) \\ G_{II} &= -\frac{1}{2\Delta a} X_i \Delta u_l = -\frac{1}{2\Delta a} X_i \left(u_l - u_{l^*}\right) \end{aligned} \tag{15}$$

If the finite element model was built using other types of elements (solid elements or plate elements with 8 nodes), the strain energy release rate equation must be modified in order to take into account the effects of the other reaction forces.

3 EXAMPLES

Two examples of stress intensity factor determination using the techniques described above are presented in order to compare the methods.

3.1 Finite plate with two collinear cracks

The first example considered in order to illustrate the practical application of the stress intensity factors determination is the case of a finite width plate having two collinear cracks of unequal length and subjected to remote uniform stress, (Fig. 2). For this structure, the stress intensity factors were calculated using the compounding technique, the finite element method with different techniques and the dual boundary element method. Several a_1/b and a_2/b ratios were considered.

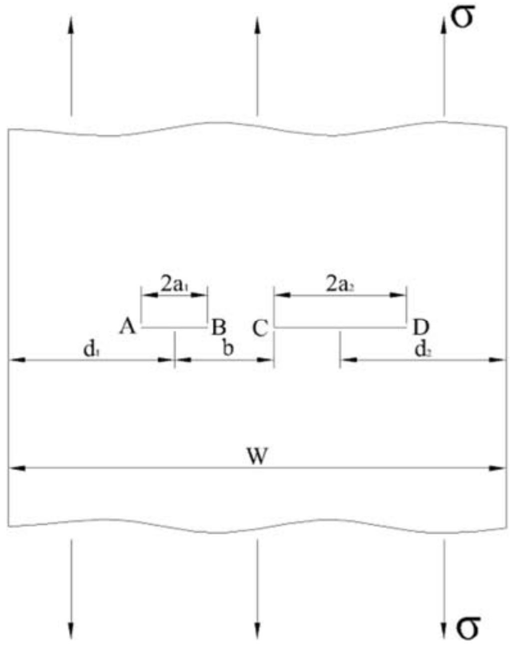

Fig. 2 –The finite width plate with two collinear cracks.

The values d_1 and d_2 were considered as follows: $d_1 = 2b+a_1$ and $d_2 = 2b+a_2$. The width W, depends on the values of a_1 and a_2, and is $W = d_1+b+a_2+d_2$ or $W = 5b+a_1+2a_2$. The compounding method is applied by dividing the structure in three ancillary configurations. For these configurations analytical solutions found in literature were used, [18], [19]. The results for crack tips C and D are presented in Fig. 3.

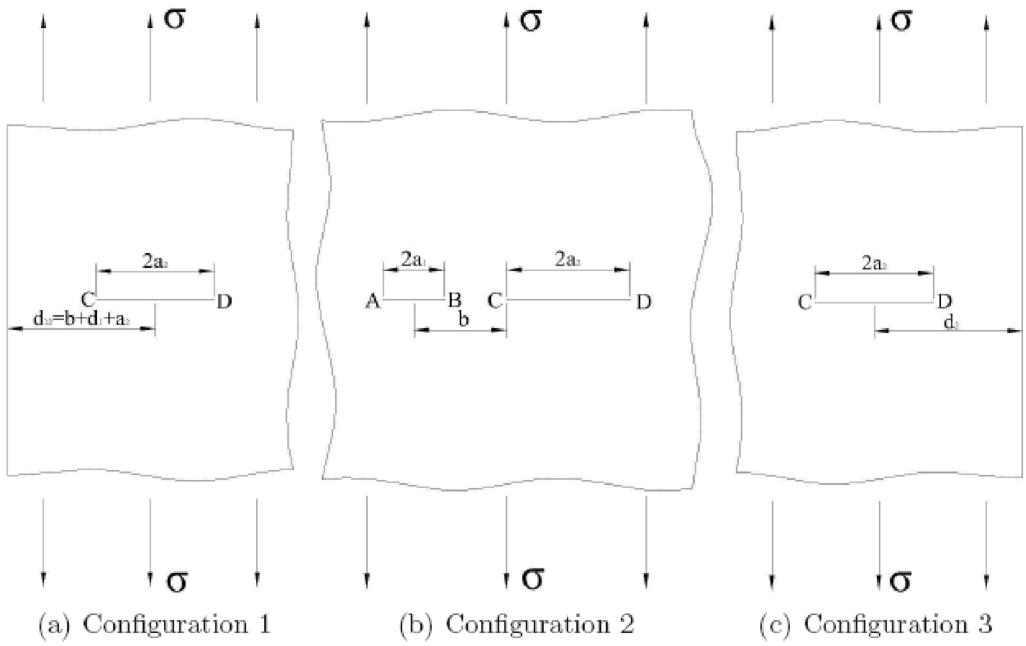

Fig. 3 –Ancillary configurations.

A finite element model of the plate was designed for stress intensity factor calculations using the J-integral and displacements extrapolation techniques. The model was created with a batch for ANSYS with capacity to define the geometry and mesh automatically and calculate the SIF for a crack tip.

Using the dual boundary element software CRACKER, [4], the stress intensity factor values for three values of the ratio a_2/b (0.4, 2 and 5) were calculated using two post processing techniques: the singularity subtraction technique and J-Integral.

Fig. 4 shows the stress intensity factors calculated using the compounding method for the crack tips C and D with different ratios of a_1/b and a_2/b. Fig. 5 shows a comparison of all techniques used for this example, where it is verified that the they follow the same trend; however, the compounding method yields the lowest values. The J-integral have convergence problems when the crack tips are close to a boundary (as the crack tips A and D when they approach the plate boundary).

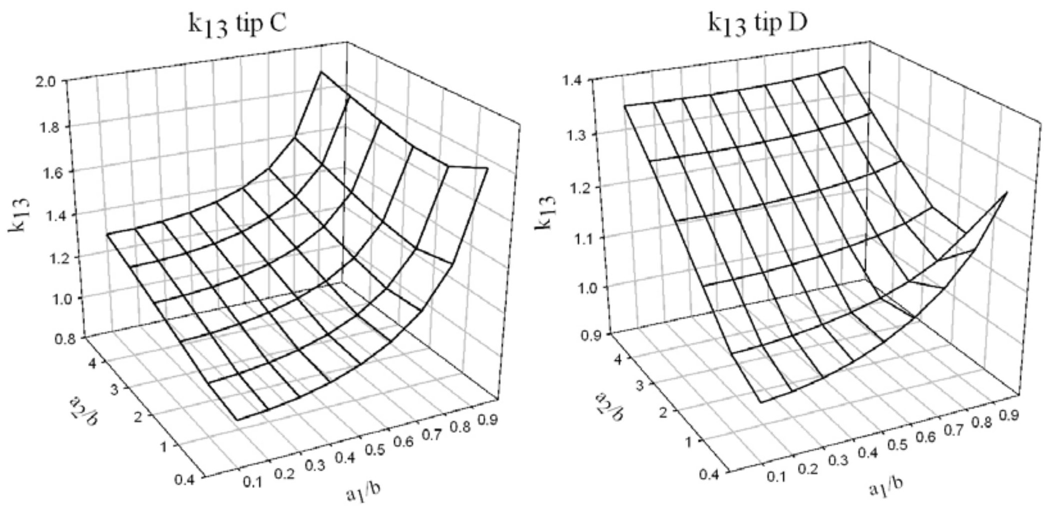

Fig. 4 – Stress intensity factors calculated using the compounding method for the crack tips C and D.

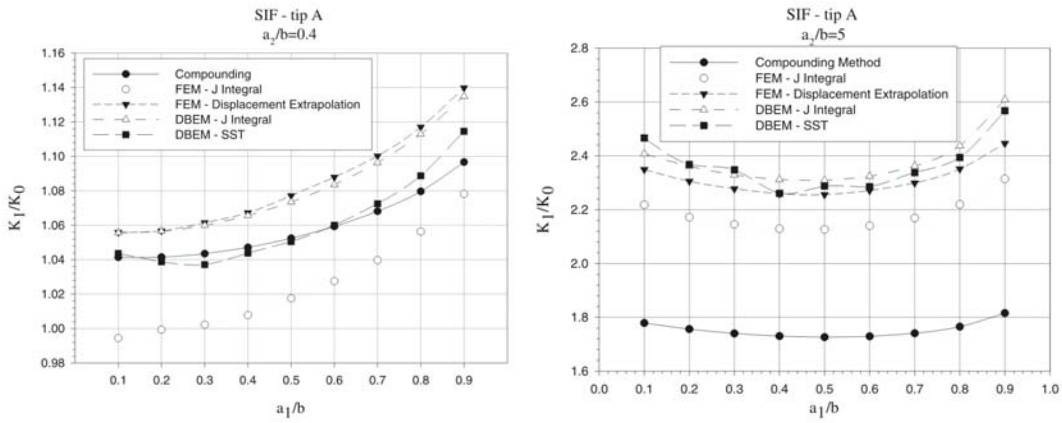

Fig. 5 – Comparison of the stress intensity factors obtained with different techniques (crack tip A, $a_2/b = 0.4$ and $a_2/b = 5$).

3.2 Reinforced plate with central crack

The second example chosen for this comparative analysis is a plate with reinforcements (stiffeners). The stress intensity factors were calculated for the case of a central crack, perpendicular to the stiffeners (Fig. 6), for crack lengths up to the distance between the stiffeners.

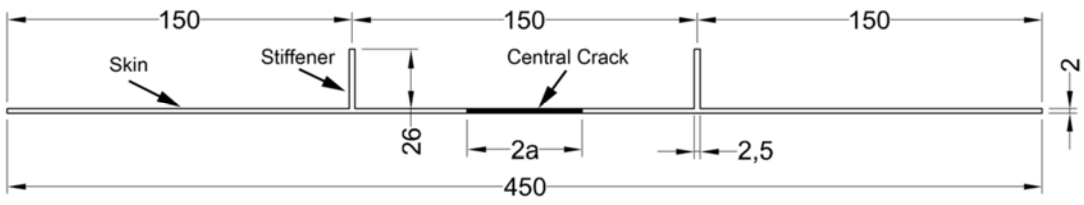

Fig. 6 – Cross section through the stiffened plate with central crack.

The compounding method with three ancillary configurations was used for the analysis of this structure: a finite plate with a central crack, an infinite plate with reinforcement at the left of the crack, and an infinite plate with reinforcement at the right side. The solutions for the infinite plate with reinforcement were found in [19].

A finite element model for this geometry was designed using shell elements with 6 degrees of freedom per node and reduced integration. Eight different crack lengths were considered for the designed model. The stress intensity factors were calculated using the Force method, the J-Integral technique and the modified VCCT. The model was processed using the FEM code ABAQUS, with 11517 nodes and 3644 S8R elements of the ABAQUS library (shell elements with 6 degrees of freedom per node and reduced integration). Fig. 7 presents a detail of the mesh and the deformed shape of the structure with a contour map of σ_y stress field.

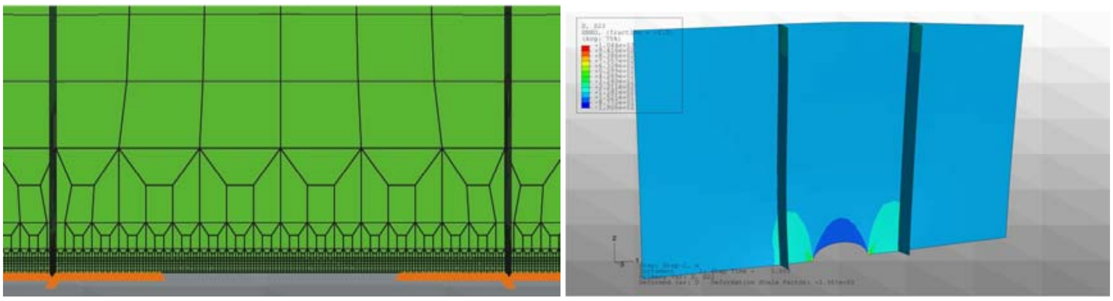

Fig. 7 – Mesh detail in the crack area and deformed shape with the contour map of σ_y stress.

Fig. 8 presents a comparison of the stress intensity factor values obtained with all the considered procedures-for different crack lengths in the stiffened plate.

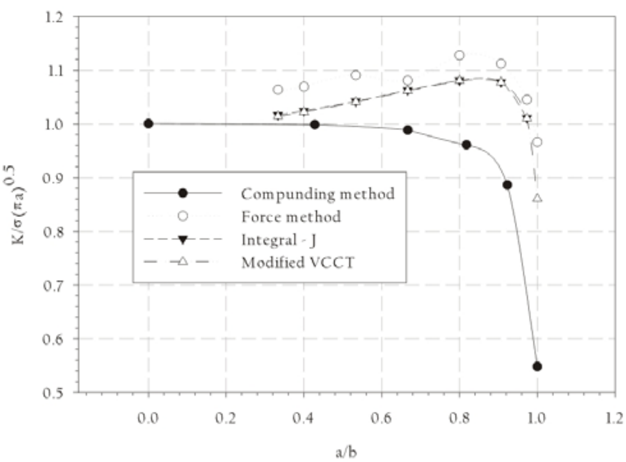

Fig. 8 – Comparison of the SIFs results obtained with different methods.

One can notice that the compounding method leads to the lowest values of stress intensity factor, while the force method presents the–highest values. VCCT and J-integral yield very similar solutions (although the J-integral data point for $2a$ = 150*mm*, is not plotted here because it was affected by the stiffener boundary and the integral didn't converge to a constant value). The J-integral and modified VCCT, yielded very close results. This is a remarkable conclusion because the modified VCCT is the simplest technique in terms of computation compared with the others and can be applied boundaries exist in the crack tip vicinity.

4 CONCLUSIONS

A comparison of numerical techniques for calculation of stress intensity factors for cracked plates subjected to in-plane loading was carried out.

The compounding technique was found to give reasonable results for relatively simple geometries; it becomes too time consuming for geometries of great complexity or when interaction between their boundaries exists.

The force method gives accurate results; however it requires preparation of a suitable finite element mesh in the crack front.

The modified virtual crack closure technique, proved to be an accurate tool for calculating SIFs in post-processing of a finite element analysis, with low computational effort and good accuracy both in simple cases as well as in complex structures.

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