Dual Radio Networks: Capacity and Connectivity

Rui A. Costa[1] João Barros[1]

***Abstract*— Motivated by the proliferation of dual radio devices, we consider a wireless network model in which all devices have short-range transmission capability, but a subset of the nodes has a secondary long-range wireless interface. For the resulting class of random graph models, we present analytical bounds for both the connectivity and the max-flow min-cut capacity. Perhaps the most striking conclusion to be drawn from our results is that the capacity of this class grows quadratically with the fraction of dual radio devices, thus indicating that a small percentage of such devices is sufficient to improve significantly the capacity of the network.**

I. Introduction

As wireless interfaces become standard commodities and communication devices with multiple radio interfaces appear in various products, it is only natural to ask whether the aforementioned devices can lead to substantial performance gains in wireless communication networks. Promising examples include [1], where multiple radios are used to provide better performance and greater functionality for users, and [2], where it is shown that using radio hierarchies can reduce power consumption. In addition, [3] presents a link-layer protocol that works with multiple IEEE 802.11 radios and improves TCP throughput and latency. This growing interest in wireless systems with multiple radios (for example, a Bluetooth interface and an IEEE 802.11 wi-fi card) motivates us to study the impact of dual radio devices on the connectivity and capacity of wireless networks.

For classical single-radio networks, random geometric graphs provide a widely accepted model, whose connectivity is well understood. In [4] Penrose shows a relation between connectivity and minimum degree in terms of the value of the radio range. Gupta and Kumar derive in [5], the critical radio range for which the probability that the network is connected goes to one as the number of nodes goes to infinity. Ganesh and Xue [6] studied the connectivity and diameter of a class of networks similar to random geometric graphs, with the new feature of adding random shortcuts to the network, thus creating a so called *small-world* network.

The capacity of networks (described by general weighted graphs) supporting multiple communicating parties is largely unknown, although progress has recently been reported in several relevant instances of this problem. In the case where the network has one or more independent sources of information but only one sink, it is known that routing offers an optimal solution for transporting messages [7] — in this case the transmitted information behaves like *water in pipes* and the capacity can be obtained by classical network flow methods. Specifically, the capacity of the network follows from the well-known Ford-Fulkerson *max-flow min-cut* theorem [8], which asserts that the maximal amount of a flow (provided by the network) is equal to the capacity of a minimal cut, i.e. a nontrivial partition of the graph node set V into two parts such that the sum of the capacities of the edges connecting the two parts (the cut capacity) is minimum.

The case of general multicast networks, in which a single source broadcasts a number of messages to a set of sinks, is considered in [9], where it is shown that applying coding operations at intermediate nodes (i.e. *network coding*) is necessary to achieve the max-flow/min-cut bound of the network. In other words, if k messages are to be sent then the minimum cut between the source and each sink must be of size at least k. A converse proof for this problem, known as the *network information flow problem*, was provided by [10], whereas linear network codes were proposed and discussed in [11] and [12]. Max-flow min-cut capacity bounds for Erdös-Rényi

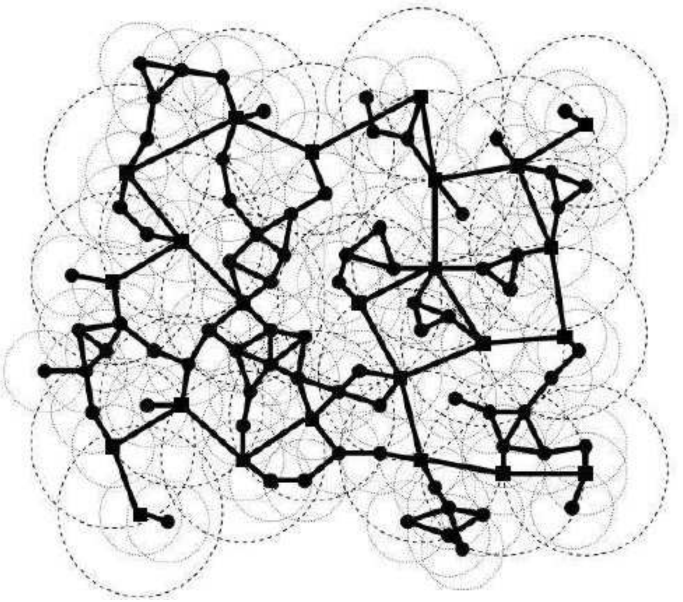

Fig. 1. Illustration of Dual Radio Networks. The square nodes represent the devices with two wireless technologies, and the circular nodes represent the nodes with only one wireless technology. The small and large circumferences represent the coverage area of the short-range and long-range wireless interfaces, respectively.

[1]The authors are with Instituto de Telecomunicações and the Department of Computer Science, Universidade do Porto, Portugal. URL: http://www.dcc.fc.up.pt/~barros/. This work was partly supported by the Fundação para a Ciência e Tecnologia (Portuguese Foundation for Science and Technology) under grant POSC/EIA/62199/2004.

graphs and random geometric graphs were presented in [13]. For small-world networks, capacity bounds were presented in [14], [15].

Our main contributions are as follows:

- *Network Model:* We introduce a simple random graph model, the *Dual Radio Network* (DRN), where nodes with low-range radios are represented by a primary random geometric graph and the set of dual radio nodes with their additional long-range wireless links form a secondary random geometric graph (see Fig. 1).
- *Connectivity Bounds:* For this class of networks, we provide upper and lower bounds for the probability that an instance of a Dual Radio Network is connected;
- *Capacity Bounds:* Using a set of probabilistic tools, we derive upper and lower bounds for the max-flow min-cut capacity of this class of random networks.

The rest of the paper is organized as follows. Section II offers a formal problem statem. Our main results with respect to connectivity and capacity are stated and proved in Sections III and IV, respectively. The paper concludes with Section V.

II. Problem Statement

In this section, we give a rigorous definition for the class of networks under consideration in the rest of the paper.

Definition 1: A *Dual Radio Network* (DRN) is a graph $G(n,p,r_s,r_L) = (V,E)$ constructed by the following procedure. Assign n nodes uniformly at random in the set T, where T is the torus obtained by identifying the opposite sides of the box $[0,1]^2$, and define V as the set of these n nodes. For a parameter r_S, each pair of nodes (a,b), with $a,b \in V$, is connected if their euclidian distance verifies $d(a,b) \leq r_S$, and let E_S be the set of edges created in this step. Now, for a parameter p, define the set V_L by the following: for node i, $i \in V_L$ with probability p, and repeat this procedure $\forall i \in V$. For a parameter r_L, each pair of nodes (a,b), $a,b \in V_L$ is connected if their euclidian distance verifies $d(a,b) \leq r_L$, and let E_L be the set of edges created in this step. Finally, the set of edges of a DRN is defined by $E = E_S \cup E_L$.

Fig. 1 provides an illustration of Dual Radio Networks. In the definition above, notice that, for two nodes $a,b \in V$ such that $r_S < d(a,b) \leq r_L$, they are connected only if both are elements of the set V_L. In terms of the wireless systems that this class of networks pretends to model, this is a realistic feature, since devices with the higher-level wireless technology can only communicate, using this technology, with devices that have the higher-level wireless technology as well.

In the rest of the paper, we study this class of networks in terms of connectivity and capacity. We say that a network is *connected* if for each pair of nodes there exist a path connecting them. In the spirit of the max-flow min-cut theorem of Ford and Fulkerson [8], we will refer to the global minimum cut of a graph as the max-flow min-cut capacity (or simply the *capacity*) of the graph.

III. Results on the connectivity of a Dual Radio Network

In this section, we study the connectivity of the class of networks introduced in Section II, providing an upper and a lower bound on the probability of an instance of a Dual Radio Network being connected.

Lemma 1: For $r_S \leq 1/\sqrt{\pi}$ and $r_L \leq 1/\sqrt{\pi}$, the probability that there is no isolated node in $G(n,p,r_S,r_L)$ verifies:

$$\mathcal{P}\{\text{no isolated node}\} \leq 1 - \left(1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2)\right)^{n-1}.$$

Proof: First, we calculate the probability that a node $\mathbf{Y}$ is connected to node $\mathbf{X}$, given the position of $\mathbf{X}$. This probability is given by $\mathcal{P}(\mathbf{X} \leftrightarrow \mathbf{Y}|\mathbf{X}) = \mathcal{P}(\{d(\mathbf{X},\mathbf{Y}) \leq r_S\} \cup (\{\mathbf{X} \in V_L\} \cap \{\mathbf{Y} \in V_L\} \cap \{d(\mathbf{X},\mathbf{Y}) \leq r_L\}) | \mathbf{X})$. Using the notation $\mathcal{P}(A|\mathbf{X}) = \mathcal{P}_{\mathbf{X}}(A)$ and $d(\mathbf{X},\mathbf{Y}) = D$, we have the following:

$$\begin{aligned}\mathcal{P}_{\mathbf{X}}(\mathbf{X} \leftrightarrow \mathbf{Y}) &\overset{(a)}{=} \mathcal{P}_{\mathbf{X}}(D \leq r_S) + \mathcal{P}_{\mathbf{X}}(\{\mathbf{X} \in V_L\} \cap \{\mathbf{Y} \in V_L\} \cap \{D \leq r_L\}) \\ &\quad - \mathcal{P}_{\mathbf{X}}(\{D \leq r_S\} \cap \{\mathbf{X} \in V_L\} \cap \{\mathbf{Y} \in V_L\} \cap \{D \leq r_L\}) \\ &\overset{(b)}{=} \mathcal{P}_{\mathbf{X}}(D \leq r_S) + \mathcal{P}_{\mathbf{X}}(\{\mathbf{X} \in V_L\} \cap \{\mathbf{Y} \in V_L\} \cap \{D \leq r_L\}) \\ &\quad - \mathcal{P}_{\mathbf{X}}(\{D \leq r_S\} \cap \{\mathbf{X} \in V_L\} \cap \{\mathbf{Y} \in V_L\})\end{aligned}$$

where (a) follows from the fact that for any two events A and B, $\mathcal{P}(A \cup B) = \mathcal{P}(A) + \mathcal{P}(B) - \mathcal{P}(A \cap B)$, and (b) is justified by noticing that $D \leq r_S \Rightarrow D \leq r_L$, thus $\{D \leq r_S\} \cap \{D \leq r_L\} = \{D \leq r_S\}$.

The events $\{D \leq r_L\}$ and $\{\mathbf{X} \in V_L\}$ are independent, and the same is true for the events $\{D \leq r_L\}$ and $\{\mathbf{Y} \in V_L\}$. Because the set of nodes V_L is formed by nodes selected at random and in an independent fashion, we have that the events $\{\mathbf{X} \in V_L\}$ and $\{\mathbf{Y} \in V_L\}$ are independent. Therefore,

$$\begin{aligned}&\mathcal{P}_{\mathbf{X}}(\{\mathbf{X} \in V_L\} \cap \{\mathbf{Y} \in V_L\} \cap \{D \leq r_L\}) = \\ &\quad = \mathcal{P}_{\mathbf{X}}(\mathbf{X} \in V_L) \cdot \mathcal{P}_{\mathbf{X}}(\mathbf{Y} \in V_L) \cdot \mathcal{P}_{\mathbf{X}}(D \leq r_L).\end{aligned}$$

Using analogous arguments, we have that

$$\begin{aligned}&\mathcal{P}_{\mathbf{X}}(\{\mathbf{X} \in V_L\} \cap \{\mathbf{Y} \in V_L\} \cap \{D \leq r_S\}) = \\ &\quad = \mathcal{P}_{\mathbf{X}}(\mathbf{X} \in V_L) \cdot \mathcal{P}_{\mathbf{X}}(\mathbf{Y} \in V_L) \cdot \mathcal{P}_{\mathbf{X}}(D \leq r_S).\end{aligned}$$

Noticing that the events $\{\mathbf{X} \in V_L\}$ and $\{\mathbf{Y} \in V_L\}$ are independent of the position of $\mathbf{X}$, we have that $\mathcal{P}_{\mathbf{X}}(\mathbf{X} \leftrightarrow \mathbf{Y}) = \mathcal{P}_{\mathbf{X}}(D \leq r_S) + \mathcal{P}(\mathbf{X} \in V_L) \cdot \mathcal{P}(\mathbf{Y} \in V_L) \cdot (\mathcal{P}_{\mathbf{X}}(D \leq r_L) - \mathcal{P}_{\mathbf{X}}(D \leq r_S))$.

Because the set where the nodes are placed is a torus, we have that $\mathcal{P}_{\mathbf{X}}(D \leq \rho) = \pi\rho^2$, with $\rho \leq 1/\sqrt{\pi}$. Noticing that $\mathcal{P}(\mathbf{X} \in V_L) = \mathcal{P}(\mathbf{Y} \in V_L) = p$, we have that:

$$\mathcal{P}_{\mathbf{X}}(\mathbf{X} \leftrightarrow \mathbf{Y}) = \pi r_S^2 + \pi p^2 (r_L^2 - r_S^2).$$

Now, to compute the probability that a node at a position $\mathbf{X}$ is isolated, we argue that the events $\{\mathbf{X} \leftrightarrow \mathbf{Y}_1\}, \{\mathbf{X} \leftrightarrow \mathbf{Y}_2\}, \ldots, \{\mathbf{X} \leftrightarrow \mathbf{Y}_{n-1}\}$, conditioning on the fact that the position of node $\mathbf{X}$ is given (say $\mathbf{X} = (x_1,x_2) = \mathbf{x}$), are mutually independent. Without loss of generality, we may write:

$$\begin{aligned}&\mathcal{P}(\mathbf{X} \leftrightarrow \mathbf{Y}_1 | \mathbf{X} \leftrightarrow \mathbf{Y}_2, \ldots, \mathbf{X} \leftrightarrow \mathbf{Y}_{n-1}, \mathbf{X} = \mathbf{x}) \\ &= \mathcal{P}(\mathbf{Y}_1 \leftrightarrow \mathbf{x} | \mathbf{Y}_2 \leftrightarrow \mathbf{x}, \ldots, \mathbf{Y}_{n-1} \leftrightarrow \mathbf{x}),\end{aligned}$$

where we exploited the fact that the position of $\mathbf{X}$ is fixed. Now, notice that none of the events $\{\mathbf{Y}_2 \leftrightarrow \mathbf{x}\}, \ldots, \{\mathbf{Y}_{n-1} \leftrightarrow \mathbf{x}\}$ afects the event $\{\mathbf{Y}_1 \leftrightarrow \mathbf{x}\}$, because we do not have information about the existence of connection between $\mathbf{Y}_1$ and any of the $\mathbf{Y}_i$. Therefore, $\mathcal{P}(\mathbf{X} \leftrightarrow \mathbf{Y}_1 | \mathbf{X} \leftrightarrow \mathbf{Y}_2, \ldots, \mathbf{X} \leftrightarrow \mathbf{Y}_{n-1}, \mathbf{X} = \mathbf{x}) = \mathcal{P}(\mathbf{X} \leftrightarrow \mathbf{Y}_1 | \mathbf{X} = \mathbf{x})$. Since we can use similar arguments for different subsets of the collection $\{\{\mathbf{X} \leftrightarrow \mathbf{Y}_1\}, \{\mathbf{X} \leftrightarrow \mathbf{Y}_2\}, \ldots, \{\mathbf{X} \leftrightarrow \mathbf{Y}_{n-1}\}\}$, we have that the events $\{\mathbf{X} \leftrightarrow \mathbf{Y}_1\}, \{\mathbf{X} \leftrightarrow \mathbf{Y}_2\}, \ldots, \{\mathbf{X} \leftrightarrow \mathbf{Y}_{n-1}\}$ are mutually independent, conditioned on the fact that the position of node $\mathbf{X}$ is given.

Thus, the probability that a node at a position $\mathbf{X}$ is isolated is given by:

$$\mathcal{P}_{\mathbf{X}}(\{\mathbf{X} \text{ is isolated}\}) = \left(1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2)\right)^{n-1}.$$

Therefore, the probability of a node being isolated is given by:

$$\mathcal{P}(\{\text{a node is isolated}\}) = \int_0^1 \int_0^1 \mathcal{P}_{\mathbf{X}}(\{\mathbf{X} \text{ is isolated}\}) d\mathbf{x} = \left(1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2)\right)^{n-1} \quad (1)$$

Now, let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ represent the nodes of the graph. We have that:

$$\begin{aligned}\mathcal{P}(\{\text{no isolated node}\}) &= \mathcal{P}(\{\mathbf{X}_1 \text{ is not isolated}\} \cap \ldots \\ &\qquad \cdots \cap \{\mathbf{X}_n \text{ is not isolated}\}) \\ &= 1 - \mathcal{P}(\{\mathbf{X}_1 \text{ is isolated}\} \cup \ldots \\ &\qquad \cdots \cup \{\mathbf{X}_n \text{ is isolated}\}).\end{aligned}$$

Now, notice that

$$\mathcal{P}(\{\mathbf{X}_1 \text{ is isolated}\} \cup \cdots \cup \{\mathbf{X}_n \text{ is isolated}\}) \geq \mathcal{P}(\mathbf{X}_1 \text{ is isolated}).$$

Therefore, by (1), we have that

$$\mathcal{P}(\{\mathbf{X}_1 \text{ is isolated}\} \cup \ldots \cup \{\mathbf{X}_n \text{ is isolated}\}) \geq \left(1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2)\right)^{n-1}$$

and the result follows. ■

After determining the probability of having an isolated node in a DRN, we calculate a bound on the probability that a DRN is disconnected.

Lemma 2: For $r_S \leq 1/\sqrt{\pi}$ and $r_L \leq 1/\sqrt{\pi}$, the probability that $G(n, p, r_S, r_L)$ is disconnected, $P_d(n, p, r_S, r_L)$, verifies

$$P_d(n, p, r_S, r_L) \leq \frac{1 - \left(1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2)\right)^n}{\pi r_S^2 + \pi p^2 (r_L^2 - r_S^2)} - 1.$$

Proof: For $k > 1$, select a node from $G(k, p, r_S, r_L)$, say node k. To $G(k, p, r_S, r_L)$ be disconnected, or node k is isolated, or the subgraph obtained by removing node k and all its edges (which can be viewed as $G(k-1, p, r_S, r_L)$) is disconnected. Thus, we have that $\{G(k, p, r_S, r_L) \text{ is disconnected}\} = \{G(k-1, p, r_S, r_L) \text{ is disconnected}\} \cup \{\text{node } k \text{ is isolated}\}$. Therefore

$$\begin{aligned}P_d(k, p, r_S, r_L) &\leq \mathcal{P}(\text{node } k \text{ is isolated in } G(k, p, r_S, r_L)) \\ &\quad + P_d(k-1, p, r_S, r_L).\end{aligned}$$

After recursion, we have that:

$$\begin{aligned}P_d(n, p, r_S, r_L) &\leq \mathcal{P}(\textit{a node is isolated in } G(2, p, r_S, r_L)) \\ &\quad + \sum_{k=3}^{n} \mathcal{P}(\textit{node } k \textit{ is isolated in } G(k, p, r_S, r_L)) \\ &\leq 1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2) \\ &\quad + \sum_{k=3}^{n} \left(1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2)\right)^{k-1} \\ &\leq 1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2) \\ &\quad + \sum_{k=2}^{n-1} \left(1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2)\right)^{k} \\ &\leq \sum_{k=1}^{n-1} \left(1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2)\right)^{k}.\end{aligned}$$

Because $\sum_{k=1}^{n} a^k = \frac{a - a^{n+1}}{1-a}$, we have that $P_d(n, p, r_S, r_L) \leq \frac{1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2) - \left(1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2)\right)^n}{\pi r_S^2 + \pi p^2 (r_L^2 - r_S^2)}$ and the result follows. ■

Using the previous two lemmas, we are able to state our main result in terms of connectivity.

Theorem 1: For $r_S \leq 1/\sqrt{\pi}$ and $r_L \leq 1/\sqrt{\pi}$, the probability that $G(n, p, r_S, r_L)$ is connected, $P_c(n, p, r_S, r_L)$, verifies

$$P_c(n, p, r_S, r_L) \geq \max\{2 - \frac{1 - \left(1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2)\right)^n}{\pi r_S^2 + \pi p^2 (r_L^2 - r_S^2)}, 0\}$$

and

$$P_c(n, p, r_S, r_L) \leq 1 - \left(1 - \pi r_S^2 - \pi p^2 (r_L^2 - r_S^2)\right)^{n-1}.$$

Proof: It is easy to see that

$$P_c(n, p, r_S, r_L) \leq \mathcal{P}\left(\{\textit{no isolated node in } G(n, p, r_S, r_L)\}\right).$$

Thus, using *Lemma 1*, we have the upper bound for $P_c(n, p, r_S, r_L)$. Noticing that $P_c(n, p, r_S, r_L) = 1 - P_d(n, p, r_S, r_L)$, using *Lemma 2* and taking the maximum between the lower bound obtained and zero (because a probability is always lower bounded by zero), the result follows. ■

IV. Capacity Results for Dual Radio Networks

We consider a multiple-source multiple-terminal transmission on a DRN with n nodes, denoting by $s_1, \ldots, s_\alpha$ the set of the α sources and by $t_1, \ldots, t_\beta$ the set of the β terminals. Let i and j be two nodes of a DRN. C_{ij} is the capacity of the edge (i, j), defined by $C_{ij} = 1$, if $d(i, j) \leq r_S$ or $i \in V_L \wedge j \in V_L \wedge d(i, j) \leq r_L$, and $C_{ij} = 0$ otherwise. This means that $C_{ij} = 1$ if nodes i and j are connected, and $C_{ij} = 0$ otherwise. Notice that $E[C_{ij}] = \mathcal{P}(i \leftrightarrow j)$ and, as we have seen in Section III, $\mathcal{P}(i \leftrightarrow j) = \pi r_S^2 + \pi p^2 (r_L^2 - r_S^2)$, with $r_S \leq 1/\sqrt{\pi}$ and $r_L \leq 1/\sqrt{\pi}$, which we assume in the following. Let $\mu = \pi r_S^2 + \pi p^2 (r_L^2 - r_S^2)$. The techniques used for proving the following results are similar to those used in [13]. In the following, we consider unitary weight for all the edges in an instance of a DRN, for simplicity.

First, we determine an upper bound on the probability that the capacity of a cut does not takes a value much greater than its expected value.

Lemma 3: Let G be a random instance of a DRN, and consider a single-source single-terminal transmission (i.e. $\alpha = \beta = 1$). Let N be the number of relay nodes, i.e. $N = n - 2$. Let C_k be the capacity of a cut in G in which one of the partitions consists of k nodes and the source. For $\epsilon > 0$ and $N \geq 2$,

$$\mathcal{P}(C_k \leq (1-\epsilon)E\left[C_k\right]) \leq e^{-(N+1+k(N-k))\mu^2\epsilon^2/N^2}.$$

Proof: We have that

$$\mathcal{P}(C_k \leq (1-\epsilon)E(C_k)) = \mathcal{P}(-C_k - E(-C_k) \geq \epsilon E(C_k)). \quad (2)$$

To compute the desired upper bound, we shall use the Hoeffding's inequality [16], which states that, for $X_1, X_2, \ldots, X_m$ independent random variables with $\mathcal{P}(X_i \in [a_i, b_i]) = 1, \forall i \in \{1, 2, \ldots, m\}$, if we define $S = X_1 + X_2 + \cdots + X_m$, then

$$\mathcal{P}(S - E(S) \geq mt) \leq \exp\left(-\frac{2m^2t^2}{\sum_{i=1}^{m}(b_i - a_i)^2}\right).$$

More precisely, we shall use this inequality for $m = 1$. First, notice that C_k is upper bounded by the value of a similar cut in the complete graph, i.e.

$$C_k \leq (k+1)(N-k+1) = N + 1 + k(N-k).$$

Therefore, we have that $C_k \in [0, N+1+k(N-k)]$. Thus, applying Hoeffding's inequality in (2), we have that

$$\mathcal{P}(C_k \leq (1-\epsilon)E(C_k)) \leq \exp\left(-\frac{2\epsilon^2(E(C_k))^2}{(N+1+k(N-k))^2}\right). \quad (3)$$

Now, notice that C_k is the sum of $N+1+k(N-k)$ random variables of the form C_{ij}, with $C_{ij} = 1,$ if $i \leftrightarrow j$ and $C_{ij} = 0,$ if $i \not\leftrightarrow j$, i.e. i is not connected to j. Therefore, for each of these random variables, we have that $E(C_{ij}) = \mathcal{P}(i \leftrightarrow j) = \mu$. Thus

$$E(C_k) = (N+1+k(N-k))\mu.$$

Now, notice that $N+1+k(N-k) \leq 2N^2$, for $N \geq 2$, thus $\frac{1}{N+1+k(N-k)} \geq \frac{1}{2N^2}$, for $N \geq 2$. Therefore

$$\exp\left(-\frac{2\epsilon^2(E(C_k))^2}{(N+1+k(N-k))^2}\right) \leq \exp\left(-\frac{(N+1+k(N-k))\mu^2\epsilon^2}{N^2}\right).$$

Thus, by (3), the result follows. ■

Corollary 1: Let C_k and N be as defined in *Lemma 3* and let A_k be the event $\{C_k < (1-\epsilon)E\left[C_k\right]\}$. Then

$$\mathcal{P}(\cup_k A_k) \leq 2e^{-\mu^2\epsilon^2/N} \cdot \left[1 + e^{-\mu^2\epsilon^2/2N}\right]^N.$$

Proof: By *Lemma 3*, we have that $\mathcal{P}(A_k) \leq e^{-(N+1+k(N-k)\mu^2)\epsilon^2/N^2}$, which also gives

$$\mathcal{P}(A_k) \leq e^{-(N+k(N-k))\mu^2\epsilon^2/N^2}.$$

Notice that, for each $k \in \{0, ..., N\}$, there are $\binom{N}{k}$ cuts in which one of the partitions consists on k nodes and the source. Therefore,

$$\begin{aligned}\mathcal{P}(\cup_k A_k) &\leq \sum_{k=0}^{N}\binom{N}{k}\mathcal{P}(A_k) \\ &\leq \sum_{k=0}^{N}\binom{N}{k}e^{-(N+k(N-k))\mu^2\epsilon^2/N^2}\end{aligned}$$

Let $\beta = e^{-\mu^2\epsilon^2/N}$. Then:

$$\begin{aligned}\mathcal{P}(\cup_k A_k) &\leq \beta\sum_{k=0}^{N}\binom{N}{k}\beta^{N\frac{k}{N}(1-\frac{k}{N})} \\ &= \beta\left(\sum_{k=0}^{\lfloor N/2\rfloor}\binom{N}{k}\beta^{N\frac{k}{N}(1-\frac{k}{N})} + \sum_{k=\lfloor N/2\rfloor+1}^{N}\binom{N}{k}\beta^{N\frac{k}{N}(1-\frac{k}{N})}\right).\end{aligned}$$

Notice that, when $\frac{k}{N} \in [0, 1/2]$,

$$\frac{k}{N}(1 - \frac{k}{N}) \geq \frac{k}{2N},$$

and when $\frac{k}{N} \in [1/2, 1]$,

$$\frac{k}{N}(1 - \frac{k}{N}) \geq \frac{N-k}{2N}.$$

Therefore:

$$\begin{aligned}\mathcal{P}(\cup_k A_k) &\leq \beta\left(\sum_{k=0}^{\lfloor N/2\rfloor}\binom{N}{k}\beta^{N\frac{k}{2N}} + \sum_{k=\lfloor N/2\rfloor+1}^{N}\binom{N}{k}\beta^{N\frac{1}{2}(1-\frac{k}{N})}\right) \\ &\leq \beta\left(\sum_{k=0}^{N}\binom{N}{k}\left(\beta^{\frac{1}{2}}\right)^k + \sum_{k=0}^{N}\binom{N}{k}\left(\beta^{\frac{1}{2}}\right)^{N-k}\right) \\ &\overset{(a)}{\leq} 2\beta(1+\sqrt{\beta})^N \\ &\overset{(b)}{\leq} 2e^{-\mu^2\epsilon^2/N} \cdot \left[1 + e^{-\mu^2\epsilon^2/2N}\right]^N\end{aligned}$$

where (a) follows from the fact that $(x+y)^m = \sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k}$, thus

$$\sum_{k=0}^{N}\left(\sqrt{\beta}\right)^k = (1+\sqrt{\beta})^N = \sum_{k=0}^{N}\left(\sqrt{\beta}\right)^{N-k},$$

and (b) follows from substituting β by $e^{-\mu^2\epsilon^2/N}$. ■

Now, using *Corollary 1*, we obtain the first result related to the capacity of a DRN, which is valid for the single-source single-terminal transmission problem.

Corollary 2: Let $C_{\min}(s_1 \to t_1)$ be the global minimum cut in an instance of DRN. Then

$$\mathcal{P}(C_{\min}(s_1 \to t_1) \leq (1-\epsilon)(N+1)\mu) \leq 2e^{-\mu^2\epsilon^2/N} \cdot \left[1 + e^{-\mu^2\epsilon^2/2N}\right]^N.$$

Proof: Let $\tilde{A}_k$ be the event $\{C_k < (1-\epsilon)E\left[C_0\right]\}$ and let A_k be the event $\{C_k < (1-\epsilon)E\left[C_k\right]\}$. We have that $E\left[C_k\right] = (N+1+k(N-k))\mu$. Therefore, $E\left[C_k\right] \geq E\left[C_0\right], \forall k \in$

$0, ..., N$. Thus $\tilde{A}_k \subseteq A_k$, which implies that $\cup_k \tilde{A}_k \subseteq \cup_k A_k$. Therefore,

$$\begin{aligned}\mathcal{P}(C_{\min}(s_1 \to t_1) \le (1-\epsilon)E[C_0]) &= \mathcal{P}(\cup_k \tilde{A}_k) \\ &\le \mathcal{P}(\cup_k A_k).\end{aligned}$$

Using *Corollary 1* and noticing that $E[C_0] = (N+1)\mu$, the result follows. ■

Now, we are ready to state our main result in terms of capacity of a DRN:

Theorem 2: Let $C_{\min}(\alpha,\beta)$ be the global minimum cut for a transmission with α sources and β terminals, in an instance of a DRN. Let $\epsilon = \sqrt{\frac{2(n-2)d\ln(n-2)}{\mu^2}}$ with $d > 0$, and $\mu = \pi r_S^2 + \pi p^2(r_L^2 - r_S^2)$. Then

$$C_{\min}(\alpha,\beta) > (1-\epsilon)(n-1)\mu$$

with probability $1 - \mathrm{O}\left(\frac{\alpha\beta}{n^{2d}}\right)$, and

$$C_{\min}(\alpha,\beta) < (1+\epsilon)\alpha(n-\alpha)\mu$$

with probability $1 - \mathrm{O}\left(\frac{1}{n^{4nd}}\right)$.

Proof: Recall that, for a single-source single terminal transmission, $N = n-2$. Therefore $\mathcal{P}(C_{\min}(s_1 \to t_1) \le (1-\epsilon)(n-1)\mu) = \mathcal{P}(C_{\min}(s_1 \to t_1) \le (1-\epsilon)(N+1)\mu)$. Thus, replacing ϵ in *Corollary 2* by the expression $\sqrt{\frac{2(n-2)d\ln(n-2)}{\mu^2}} = \sqrt{\frac{2Nd\ln N}{\mu^2}}$, we have that:

$$\begin{aligned}\mathcal{P}(C_{\min}(s_1 \to t_1) \le (1-\epsilon)(n-1)\mu) &\le 2e^{\frac{-2dN\mu^2\ln N}{N\mu^2}} \cdot [1+e^{\frac{-2dN\mu^2\ln N}{2N\mu^2}}]^N \\ &\le \frac{2}{N^{2d}} \cdot \left[1 + \frac{1}{N^d}\right]^N.\end{aligned}$$

We have that $(x+y)^N = \sum_{k=0}^{N} \binom{N}{k} x^k y^{N-k}$, thus

$$\left[1 + \frac{1}{N^d}\right]^N = \sum_{k=0}^{N} \binom{N}{k} \left(\frac{1}{N^d}\right)^k.$$

Therefore, we have that:

$$\begin{aligned}\mathcal{P}(C_{\min}(s_1 \to t_1) \le (1-\epsilon)(n-1)\mu) &\le \frac{2}{N^{2d}} \cdot \sum_{k=0}^{N} \binom{N}{k} \left(\frac{1}{N^d}\right)^k \\ &\overset{(a)}{\le} \frac{2}{N^{2d}} \cdot \sum_{k=0}^{\infty} \left(\frac{N}{N^d}\right)^k \\ &\overset{(b)}{\le} \frac{2}{N^{2d} - N^{d+1}} \\ &\approx O\left(\frac{1}{N^{2d}}\right) \\ &= O\left(\frac{1}{n^{2d}}\right)\end{aligned}$$

where:

- (a) follows from the fact that $\binom{N}{k} = \frac{N!}{(N-k)!k!} = \frac{N \times (N-1) \times \cdots \times (N-k+1)}{k!}$, thus $\binom{N}{k} \le N \times (N-1) \times \cdots \times (N-k+1) \le N^k$;
- (b) follows from the fact that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $|x| < 1$, therefore
$$\sum_{k=0}^{\infty} \left(\frac{N}{N^d}\right)^k = \frac{1}{1 - N^{1-d}},$$
which implies that $\frac{2}{N^{2d}} \cdot \sum_{k=0}^{\infty} \left(\frac{N}{N^d}\right)^k = \frac{2}{N^{2d} - N^{d+1}}$

Now, back to the multiple-source multiple-terminal transmission, we have that

$$\begin{aligned}&\mathcal{P}(\ C_{\min}(\alpha,\beta) \le (1-\epsilon)(n-1)\mu) = \\ &\quad = \mathcal{P}\left(\cup_{i=1}^{\alpha} \cup_{j=1}^{\beta} \{C_{\min}(s_i \to t_j) \le (1-\epsilon)(n-1)\mu\}\right).\end{aligned}$$

Therefore, by the union bound,

$$\begin{aligned}&\mathcal{P}(\ C_{\min}(\alpha,\beta) \le (1-\epsilon)(n-1)\mu) \le \\ &\quad \le \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} \mathcal{P}\left(C_{\min}(s_i \to t_j) \le (1-\epsilon)(n-1)\mu\right).\end{aligned}$$

From the fact that, as we derive in *Corollary 2*, $\mathcal{P}\left(C_{\min}(s_i \to t_j) \le (1-\epsilon)(n-1)\mu\right)$ does not depend on nodes i and j, we have that $\mathcal{P}(C_{\min}(\alpha,\beta) \le (1-\epsilon)(n-1)\mu) \le \alpha\beta\mathcal{P}\left(C_{\min}(s_1 \to t_1) \le (1-\epsilon)(n-1)\mu\right)$. Therefore, we have that

$$\begin{aligned}&\mathcal{P}(\ C_{\min}(\alpha,\beta) \ge (1-\epsilon)(n-1)\mu) \ge \\ &\quad \ge 1 - \alpha\beta \cdot \mathcal{P}\left(C_{\min}(s_1 \to t_1) \le (1-\epsilon)(n-1)\mu\right)\end{aligned}$$

and, because we already proved that $\mathcal{P}(C_{\min}(s_1 \to t_1) \le (1-\epsilon)(n-1)\mu) = O\left(\frac{1}{n^{2d}}\right)$, the first part of the theorem follows.

Now, to compute the upper bound on $\mathcal{P}(C_{\min}(\alpha,\beta) \ge (1+\epsilon)\alpha(n-\alpha)\mu)$, notice that, by definition, any cut (that contains in one partition the source nodes and in the other partition the terminal nodes) is greater or equal to $C_{\min}(\alpha,\beta)$. Thus, the value of the cut in which one of the partitions consists of source nodes only (denoted by $C^*(\alpha,\beta)$) is greater or equal to $C_{\min}(\alpha,\beta)$. This means that, if $C_{\min}(\alpha,\beta) \ge (1+\epsilon)\alpha(n-\alpha)\mu$, then $C^*(\alpha,\beta) \ge (1+\epsilon)\alpha(n-\alpha)\mu$. Therefore, because $\mathcal{P}\left(C_{\min}(\alpha,\beta) \ge (1+\epsilon)\alpha(n-\alpha)\mu\right) = \mathcal{P}\left(C_{\min}(\alpha,\beta) \ge (1+\epsilon)\alpha(N+\beta)\mu\right)$, we have that

$$\begin{aligned}&\mathcal{P}(\ C_{\min}(\alpha,\beta) \ge (1+\epsilon)\alpha(n-\alpha)\mu) \le \\ &\quad \le \mathcal{P}(C^*(\alpha,\beta) \ge (1+\epsilon)\alpha(N+\beta)\mu),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\mathcal{P}(\ C_{\min}(\alpha,\beta) \ge (1+\epsilon)\alpha(n-\alpha)\mu) \le \\ &\quad \le \mathcal{P}(C^*(\alpha,\beta) - \alpha(N+\beta)\mu \ge \epsilon\alpha(N+\beta)\mu).\end{aligned}$$

Noticing that $C^*(\alpha,\beta) \in [0, \alpha(N+\beta)]$, $E(C^*(\alpha,\beta)) = \alpha(N+\beta)\mu$, and applying Hoeffding's inequality [16], we have that

$$\begin{aligned}&\mathcal{P}(\ C^*(\alpha,\beta) - \alpha(N+\beta)\mu \ge \epsilon\alpha(N+\beta)\mu) \le \\ &\quad \le \exp\left(-\frac{2\epsilon^2\alpha^2(N+\beta)^2\mu^2}{\alpha^2(N+\beta)^2}\right).\end{aligned}$$

Therefore

$$\mathcal{P}\left(C_{\min}(\alpha,\beta) \geq (1+\epsilon)\alpha(n-\alpha)\mu\right) \leq \exp\left(-2\epsilon^2\mu^2\right) \stackrel{(a)}{\leq} \frac{1}{N^{4Nd}}$$

where (a) follows from the substitution of ϵ by $\sqrt{\frac{2(n-2)d\ln(n-2)}{\mu^2}}$. Thus, we have that

$$\mathcal{P}\left(C_{\min}(\alpha,\beta) \geq (1+\epsilon)\alpha(n-\alpha)\mu\right) = \mathrm{O}\left(\frac{1}{n^{4nd}}\right)$$

and the result follows. ■

This result shows that the capacity for a multiple-source multiple-terminal transmission grows quadratically in function of the parameter p, which represents the percentage of nodes with two wireless technologies. Thus, this result shows that there is a significant benefit (in terms of capacity) by using dual-radio schemes in wireless systems.

Setting $\alpha = \beta = 1$ in *Theorem 2*, we obtain the following bounds for the capacity of a single-source single-terminal transmission:

Corollary 3: Let $C_{\min}$ be the global minimum cut for a single-source single-terminal transmission in an instance of a DRN. Let $\epsilon = \sqrt{\frac{2(n-2)d\ln(n-2)}{\mu^2}}$, and $\mu = \pi r_S^2 + \pi p^2(r_L^2 - r_S^2)$. Then

$$C_{\min} > (1-\epsilon)(n-1)\mu$$

with probability $1 - \mathrm{O}\left(\frac{1}{n^{2d}}\right)$, and

$$C_{\min} < (1+\epsilon)(n-1)\mu$$

with probability $1 - \mathrm{O}\left(\frac{1}{n^{4nd}}\right)$.

V. Conclusions

We defined a class of random geometric graphs that models a wireless network in which all devices share the same short-range radio capability, and some of them have a secondary long-range wireless interface. For this class of networks, we provided upper and lower bounds on the probability of its connectivity. We also provided bounds for the capacity of this class of networks, evidencing that the use of dual radio technologies can improve the capacity of the network. Specifically, we showed that the capacity of our model grows quadratically with the fraction of devices with two wireless interfaces. As part of our ongoing work, we are analyzing the diameter and the clustering coefficient of dual radio networks and explore their relationship with small world networks.

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