On Convergence and Performance Certification of a Continuous-Time Economic Model Predictive Control Scheme with Time-Varying Performance Index

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Abstract

This paper addresses the design of convergence and performance certified sampled-data model predictive control (MPC) laws with a time-dependent economic performance index. More precisely, using a dissipativity property of the system, we provide a set of sufficient conditions that guarantee convergence of the closed-loop state trajectory to a, possibly time-varying, average economically optimal state trajectory. Moreover, the average performance of the closed-loop system is shown to be no worse than the one obtained by operating the system at the average economically optimal state trajectory. Constructive methods to design an appropriate terminal set and terminal cost that satisfy the proposed sufficient conditions are presented and illustrated with numerical examples.

Key words: Predictive control; nonlinear control; convergence analysis; continuous-time systems; constraint satisfaction problems.

1 Introduction

A key factor that allowed Model Predictive Control (MPC) schemes to gain an important role in many practical applications lies in their ability to explicitly optimize over a desired performance index, while satisfying state and input constraints. Ideally one would wish to minimize the infinite horizon time integral of a predefined stage cost evaluated along the constrained state and input trajectories. Although, since this problem is generally intractable, MPC is often used to approximate the infinite horizon optimization with a sequence of easier finite horizon optimizations. Depending on the meaning of the chosen stage cost, we can distinguish between classic MPC schemes, also termed Tracking MPC, and Economic MPC schemes.

The main objective of a Tracking MPC controller is to steer the state of a system to a desired steady-state or state trajectory. Toward this goal, the stage cost is properly designed to penalize the distance from the current state to the desired one. This approach has been widely investigated in the literature and, depending on the methodology used to approximate the infinite horizon optimal control problem, we can identify two families of schemes: the ones that utilize the so-called terminal sets and terminal costs and the terminal-set-free schemes. For the first family we refer the reader to, e.g., [21,20,24] for the discrete-time case and [7,16,11] for the continuous-time. Similarly, for the second family, we refer to [13] and [25] for the discrete-time and continuous-time case, respectively.

In recent years, a growing attention has been devoted to Economic MPC schemes, where the main objective is the minimization of a performance index associated with a given economic stage cost. Here, the term economic is utilized to emphasize that such function is not designed to penalize the distance of the current state to the desired one, but it rather represents an index of interest to be minimized, e.g., an economic index. Such generality gives rise to many interesting applications. Similarly to

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the Tracking MPC case, also in this case the infinite horizon problem can be approximated using a terminal set and a terminal cost or, in a terminal-set-free approach, by properly selecting a long horizon. In the works [8,5,6], the economically optimal steady-state is precomputed and used to constrain the terminal state of the prediction with a terminal equality [8,5] or inequality [6] constraint. Sufficient conditions for stability of the economically optimal steady-state were initially provided in [8], and later generalized by [5,6] using a dissipativity property of the system. In [12], the author provides conditions on the horizon length and stage cost for closed-loop convergence to an arbitrarily small neighborhood of the optimal steady-state. Stability and recursive feasibility properties of the Economic MPC controller for the case of changes in the economic stage cost are addressed in [10].

Although the use of dissipativity properties of a system provides an elegant approach for analysis and design of Economic MPC schemes, alternative methods have been proposed. The works [9,22] employ a generalized terminal set, consisting of all the feasible steady-states, and a constraint on the increase of the terminal cost to guarantee convergence to *a* steady-state. In [14], the authors discuss the use of a given control Lyapunov function (CLF), defined over the whole desired region of attraction, to design a dual mode scheme. Here, initially the economic optimization is performed while enforcing the state within a level set of the CLF, and then, at a given point triggered at an arbitrary time, a Lyapunov decrease is enforced driving the state to the desired equilibrium point. In the works [19,2,4] a combination of a classic stage cost and an economic stage cost is adopted. In this case, assumptions on the magnitude of the economic stage cost are introduced in order to preserve stability [19], convergence [2], and ultimate boundedness [4] guarantees.

To the best of our knowledge, almost all the research devoted to Economic MPC is developed for discrete-time systems, even if the applications, addressed via discretization, are often stemming from continuous-time dynamical models. One exception is the work [14] that has the restriction of requiring a CLF defined over the whole desired region of attraction. Moreover, while a vast effort was dedicated to the analysis and synthesis of MPC schemes leading to the convergence of the closed-loop state trajectory to a steady-state, few results address the interesting scenario of time-varying stage cost and convergence to potentially time-varying state trajectories. See, for instance, the works [6,26,18], for the case of closed-loop convergence to periodic orbits in the discrete-time setting. Furthermore, only [5,23] address the relaxation from terminal equality to terminal inequality in the dissipativity-based approach.

Inspired by these observations, this work addresses the design of a continuous-time Economic MPC with terminal constraint for time-varying continuous-time systems with time-dependent stage cost and with convergence and performance guarantees. The main dissipativity-based results introduced in the time-invariant discrete-time case are extended to the time-varying setting for continuous-time systems. Moreover, an average performance analysis of the MPC controller is performed and constructive methods for the computation of a suitable terminal set and a terminal cost are presented. We build on our previous result in [3], extending it to the case of time-varying systems, constraints, performance indexes, and dissipative functions. The differentiability assumption on dissipativity function and terminal cost is dropped. Moreover, convergence to a time-varying state trajectory, rather than a steady-state, and the associated extension on the performance analysis is presented.

The remainder of this paper is organized as follows: Section 2 contains the problem definition. Section 3 and Section 4, similarly to [5] but for the continuous-time case, address the convergence properties and the performance analysis of the closed-loop system, respectively. Design methodologies for a suitable terminal set and terminal cost are presented in Section 5, followed by Section 6 with some numerical examples. Section 7 closes the paper with some conclusions. All the proofs are reported in the appendix.

Notation. For a generic continuous-time trajectory $\boldsymbol{x}$, the term $\boldsymbol{x}([t_1, t_2])$ denotes the trajectory considered in the time interval $[t_1, t_2]$ and $x(t)$ the trajectory evaluated at a specific time t. The notation $x(\tau; t, z)$ is used whenever we want to make explicit the dependence of $x(\tau)$ on the optimization problem parameters t and z. For a generic scalar function $g : \mathbb{R}_{\geq t_0} \to \mathbb{R}$ and time t_0 we denote by $\mathrm{Av}[g(t), t_0]$ the set $\mathrm{Av}[g(t), t_0] := \left[\liminf_{\delta\to+\infty} \frac{\int_{t_0}^{t_0+\delta} g(t)dt}{\delta},\ \limsup_{\delta\to+\infty} \frac{\int_{t_0}^{t_0+\delta} g(t)dt}{\delta}\right]$ where if the limit exists, then we have $\mathrm{Av}[g(t), t_0] = \{\lim_{\delta\to+\infty} \frac{1}{\delta}\int_{t_0}^{t_0+\delta} g(t)dt\}$. Moreover, for a generic function $g : \mathbb{R}^n \to \mathbb{R}$, with n being a positive integer, the terms $g_x(\hat{x})$ and $g_{xx}(\hat{x})$ denote the Jacobian and the Hessian, respectively, of $g(\cdot)$ with respect to the vector $x \in \mathbb{R}^n$ evaluated at $\hat{x} \in \mathbb{R}^n$. For a given matrix A, $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote the minimum and the maximum real valued eigenvalue of A. The notation $A \succ 0$ is used to denote that A is a positive definite matrix. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}_\infty$, or to be a class-$\mathcal{K}_\infty$ function, if it is zero at zero, strictly increasing and radially unbounded, i.e., $\alpha(x) \to \infty$ as $x \to \infty$. For a given function $g : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ and scalars r and t we denote by $\mathcal{L}(t; g, r)$ the r-sub-level-set $\mathcal{L}(t; g, r) := \{x : g(t, x) \leq r\}$ parametrized with t. For a generic set $\mathcal{A} \subseteq \mathbb{R}^n$ we denote by $\operatorname{int}\mathcal{A}$ the interior of $\mathcal{A}$. The generic closed ball of radius r is denoted by $\mathcal{B}(r) := \{x : \|x\| \leq r\}$. Given two generic continuous-time trajectories $\boldsymbol{x}$ and $\boldsymbol{y}$ we say that

$\boldsymbol{x}$ asymptotically converges to $\boldsymbol{y}$ if $\|x(t)-y(t)\| \to 0$ as $t \to +\infty$. The term $\mathcal{PC}(a,b)$ denotes the space of piecewise continuous trajectories defined over $[a,b]$. For the sake of simplicity, the dependence on time and parameters is dropped whenever it is clear from the context.

2 Problem Definition

Consider the continuous-time time-varying dynamical system

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(t_0) = x_0, \qquad t \geq t_0 \tag{1}$$

and let the state and input vectors $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ be constrained as

$$(x(t), u(t)) \in \mathcal{X}(t) \times \mathcal{U}(t), \qquad t \geq t_0, \tag{2}$$

where the set-valued maps $\mathcal{X} : \mathbb{R} \rightrightarrows \mathbb{R}^n$ and $\mathcal{U} : \mathbb{R} \rightrightarrows \mathbb{R}^m$ denote the time-varying state and input constraint sets, and t_0 and $x_0 = x(t_0)$ are the initial time and state, respectively.

Definition 1 (Open-Loop MPC Problem) *Given a pair* $(t, z) \in \mathbb{R}_{\geq t_0} \times \mathcal{X}(t)$ *and a horizon length* $T > 0$, *the open-loop MPC optimization problem* $\mathcal{P}(t, z)$ *consists of finding the optimal control signal* $\bar{\boldsymbol{u}}^* \in \mathcal{PC}(t, t+T)$ *that solves*

$$\begin{aligned} J_T^*(t,z) = & \min_{\bar{\boldsymbol{u}} \in \mathcal{PC}(t,t+T)} J_T(t,z,\bar{\boldsymbol{u}}) \\ & s.t. \quad \dot{\bar{x}}(\tau) = f(\tau, \bar{x}(\tau), \bar{u}(\tau)), \qquad \forall \tau \in [t, t+T] \\ & \qquad (\bar{x}(\tau), \bar{u}(\tau)) \in \mathcal{X}(\tau) \times \mathcal{U}(\tau), \quad \forall \tau \in [t, t+T] \\ & \qquad \bar{x}(t) = z, \\ & \qquad \bar{x}(t+T) \in \mathcal{X}_{aux}(t+T) \end{aligned}$$

with
$J_T(t,z,\bar{\boldsymbol{u}}) := \int_t^{t+T} l(\tau, \bar{x}(\tau), \bar{u}(\tau)) d\tau + m(t+T, \bar{x}(t+T))$, *where the* finite horizon cost $J_T(\cdot)$, *which corresponds to the* performance index *of the MPC controller, is composed of the* stage cost $l : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ *and the* terminal cost $m : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, *where the last is defined over the time-varying* terminal set $\mathcal{X}_{aux} : \mathbb{R} \rightrightarrows \mathbb{R}^n$. □

We denote by $k_{aux} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ an *auxiliary control law* defined over the terminal set with $k_{aux}(t, x(t)) \in \mathcal{U}(t)$ for all $x(t) \in \mathcal{X}_{aux}(t)$ and $t \geq t_0$. In a *sampled-data MPC* approach, the control input is computed at the discrete-time samples $\mathcal{T} := \{t_0, t_1, \dots\}$, where $t_i > t_j$ for $i > j$. Concatenating the solution of the MPC optimization problem with the output of the auxiliary control law results in the following notion of extended state and input trajectories.

Definition 2 (Extended Trajectories) *The* extended input trajectory $u_{e_i}(t)$ *for* $t \in [t_i, \infty)$ *at* (t_i, x_i), *with* $x_i := x(t_i)$, *is defined to be the concatenation of the open-loop MPC solution* $\bar{u}^*(t; t_i, x_i)$ *with the output of the auxiliary control law* $k_{aux}(\cdot)$ *as follows:* $u_{e_i}(t) = k_{ext}(t, x(t); t_i, x_i)$ *where*

$$k_{ext}(t, x; t_i, x_i) := \begin{cases} \bar{u}^*(t; t_i, x_i), & t \in [t_i, t_i + T] \\ k_{aux}(t, x), & t > t_i + T \end{cases}$$

and the extended state trajectory $x_{e_i}(t)$ *is the associated state trajectory starting at* (t_i, x_i). □

The sampled-data MPC control law is defined to be the extended input trajectory computed using the last solution of the open-loop MPC optimization problem, i.e.,

$$u(t) = k_{MPC}(t, x(t)) := k_{ext}(t, x(t); \lfloor t \rfloor, x(\lfloor t \rfloor)), \tag{4}$$

where $\lfloor t \rfloor = \max_i \{t_i \in \mathcal{T} : t_i \leq t\}$ is the maximum sampling time instant $t_i \in \mathcal{T}$ smaller than or equal to t.

Remark 3 *Note that the extended trajectories, and therefore the control input* (4), *is well defined even when the distance of two consecutive sampling instants of time is greater than the horizon length, i.e.,* $t_{i+1} - t_i > T$. □

Definition 4 (Feasible Trajectory Set) *We denote by* $\mathcal{S}$ *the set of all state and input trajectories that satisfy the system constraints, i.e.,* $\mathcal{S} := \{(\boldsymbol{x}, \boldsymbol{u}) : (1), (2)\}$, *for every initial condition* $x_0 \in \mathcal{X}(t_0)$ *and for a fixed* t_0. □

In order to asses the economic performance of the MPC scheme we consider the following notions of average economically optimal trajectory and economically optimal control law.

Definition 5 (Economically Optimal Trajectory) *Consider the feasible trajectory set* $\mathcal{S}$ *from Definition 4. The pair* $(\boldsymbol{x}_e, \boldsymbol{u}_e) \in \mathcal{S}$ *is said to be an average economically optimal trajectory if*

$$\mathrm{Av}[l(t, x(t), u(t)) - l(t, x_e(t), u_e(t)), t_0] \subseteq [0, +\infty) \tag{5}$$

for all $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{S}$. □

Definition 6 (Economically Optimal Controller) *Consider the feasible trajectory set* $\mathcal{S}$ *and let* $(\boldsymbol{x}_e, \boldsymbol{u}_e) \in \mathcal{S}$ *be an average economically optimal trajectory pair. Moreover, let* $(\boldsymbol{x}, \boldsymbol{u}) \in \mathcal{S}$ *be the closed-loop state and input trajectories associated with a control law* $u(t) = k(t, x(t))$ *and the initial state* x_0. *Then, the controller* $k(\cdot)$ *is said to be economically optimal if*

$$\mathrm{Av}[l(t, x_e(t), u_e(t)) - l(t, x(t), u(t)), t_0] \subseteq [0, +\infty) \tag{6}$$

for all initial conditions $x_0 \in \mathcal{X}(t_0)$. □

Remark 7 *Let the limits in the definition of* $Av[\cdot]$ *exist, then* (5) *and* (6) *imply that, for a system controlled by* *and economically optimal controller, the following holds:* $\lim_{\delta\to+\infty} \frac{1}{\delta}\int_{t_0}^{t_0+\delta} l(t,x_e,u_e) - l(t,x,u)dt = 0.$ □

Assumption 8 *Consider the Definitions 2 and 6. The closed-loop systems* (1) *with* $u(t) = k_{aux}(t,x(t))$ *and* (1) *with* $u(t) = k(t,x(t))$ *admit a unique solution defined for all* $t \geq t_0 + T$ *and* $t \geq t_0$*, respectively.* □

In the view of the above, given the constrained system (1)-(2) in closed-loop with the MPC control law (4), the goals of this paper are to:

(1) Answer under which (sufficient) conditions the state and input trajectories of the closed-loop system (1) with (4) converge to an average economically optimal trajectory pair $(\boldsymbol{x}_e, \boldsymbol{u}_e) \in \mathcal{S}$.
(2) Provide constructive methods to design a terminal set and a terminal cost that satisfy such sufficient conditions.
(3) Demonstrate that the resulting MPC control law is an economically optimal controller.

3 Convergence Analysis

This section is dedicated to the understanding of the behavior of the system in closed-loop with the MPC control law defined in (4). In particular we provide a set of sufficient conditions under which the sampled-data MPC controller drives the state and input signals to an average economically optimal trajectory pair $(\boldsymbol{x}_e, \boldsymbol{u}_e) \in \mathcal{S}$. To this end, we start by introducing the following standard assumptions from the MPC literature.

Assumption 9 *The function* $f(\cdot)$ *in* (1) *is locally Lipschitz in* x*, piecewise continuous in* t *and* u*, and bounded for bounded* x *in the region of interest, i. e., the set* $\{\|f(t,x,u)\| : t \geq t_0, x \in \bar{\mathcal{X}}, u \in \mathcal{U}(t)\}$ *is bounded for any bounded* $\bar{\mathcal{X}} \subset \mathcal{R}^n$*.* □

Assumption 9 guarantees, together with the boundedness of the x, the existence, uniqueness, and continuity of the solution of the initial value problem (1) and (4).

Assumption 10 (Initial Feasibility) *The optimization problem* $\mathcal{P}(t_0, x_0)$ *admits a feasible solution.* □

A property of the system that plays an important role in Economic MPC is the strict dissipativity property, [8,5,6], here extended for the time-varying case.

Definition 11 (Strictly Dissipative System) *System* (1) *is said to be strictly dissipative at a state trajectory* $\boldsymbol{x}_e$ *with respect to the* supply rate $s : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}$ *if there exists a* dissipativity function $\lambda : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ *such that*

$$\lambda(t+\delta, x_e(t+\delta)) = \lambda(t, x_e(t)) \tag{7}$$

$$\lambda(t+\delta, x(t+\delta)) - \lambda(t, x(t)) \leq \int_t^{t+\delta} s(\tau, x(\tau), u(\tau)) - \alpha(\|x(\tau) - x_e(\tau)\|)d\tau \tag{8}$$

for all $t \geq t_0$*,* $\delta > 0$*, and with* $\alpha(\cdot)$ *being a class-*$\mathcal{K}_\infty$ *function.* □

We can now state the following sufficient conditions for convergence.

Assumption 12 (Dissipativity of the System) *There exists a pair* $(\boldsymbol{x}_e, \boldsymbol{u}_e) \in \mathcal{S}$ *such that system* (1) *is strictly dissipative at* $\boldsymbol{x}_e$ *with supply rate* $s(t,x,u) = l(t,x,u) - l(t,x_e(t),u_e(t))$ *and with* $\boldsymbol{x}_e$ *uniformly bounded over time.* □

Later we will see that Assumption 12 implies that $(\boldsymbol{x}_e, \boldsymbol{u}_e)$ is an average economically optimal trajectory pair.

Assumption 13 (Properties of the MPC Probl.) *There exists a pair of trajectories* $(\boldsymbol{x}_e, \boldsymbol{u}_e) \in \mathcal{S}$ *such that*

(i) The state constraint set $\mathcal{X}(\cdot)$ *and the terminal set* $\mathcal{X}_{aux}(t) \subseteq \mathcal{X}(t)$ *are closed, connected, and contain* $\boldsymbol{x}_e$ *for all time, i.e.,* $x_e(t) \subseteq \mathcal{X}_{aux}(t)$ *for all* $t \geq t_0$*. Moreover, the input constraint set* $\mathcal{U}(t)$ *is compact and contains* $u_e(t) \in \mathcal{U}(t)$ *for all* $t \geq t_0$*.*
(ii) There exists a feasible auxiliary control law $k_{aux} : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^m$*, defined over the terminal set* $\mathcal{X}_{aux}(\cdot)$*, such that, for the closed-loop system* (1) *with*

$$u(t) = k_{aux}(t, x(t)), \tag{9}$$

$x(t) \to x_e(t)$ *as* $t \to \infty$*,* $x(t) \in \mathcal{X}_{aux}(t)$*, and* $u(t) \in \mathcal{U}(t)$ *for all* $t \geq t_0 + T$*.*
(iii) The terminal cost $m(\cdot)$ *is zeros when evaluated at* $\boldsymbol{x}_e$*, i.e.,* $m(t, x_e(t)) = 0$ *for all* $t \geq t_0$*, and the following condition on the terminal cost holds:*

$$m(\hat{t}+\delta, x(\hat{t}+\delta)) - m(\hat{t}, x(\hat{t})) \leq -\int_{\hat{t}}^{\hat{t}+\delta} l(t,x,k_{aux}(t,x)) - l(t,x_e,u_e)dt \tag{10}$$

for all $\delta > 0$ *and for all* $\hat{t} \geq t_0 + T$ *and* $x(\hat{t}) \in \mathcal{X}_{aux}(\hat{t})$*.*
(iv) For any given state and input vector pair $(\hat{x}, \hat{u}) \in \mathcal{X}(t)\times\mathcal{U}(t)$*, the functions* $l(t,\hat{x},\hat{u})$ *and* $m(t,\hat{x})$ *are uniformly bounded over time* $t \geq t_0$*.* □

It is worth noticing that the existence of the auxiliary control law $k_{aux}(\cdot)$ in Assumption 13 (ii) is enough to guarantee the recursive feasibility of the optimization problem $\mathcal{P}(\cdot)$, i.e., if there exists a solution for

$\mathcal{P}(t_i, x(t_i))$, then the extended trajectories $(\boldsymbol{x}_{e_i}, \boldsymbol{u}_{e_i})$ from Definition 2, evaluated in $[t_{i+1}, t_{i+1} + T]$, are feasible for the optimization problem $\mathcal{P}(t_{i+1}, x(t_{i+1}))$.

It can be shown that the function $J_T^*(\cdot)$, commonly used in Tracking MPC as a value function, in Economic MPC is in general not decreasing and therefore cannot be used to establish the attractiveness of $\boldsymbol{x}_e$. To solve this problem, we follow the similar approach as in [5], by introducing the rotated performance index $J_T^r(\cdot)$ that, as will be made clear later, shares the same minimizer of $J_T(\cdot)$ but, in contrast, possess the desired properties of the value function used in Tracking MPC.

Definition 14 (Rotated Open-Loop MPC Probl.) *Consider the problem $\mathcal{P}(\cdot)$ from Definition 1 and a dissipativity function $\lambda(\cdot)$ from Definition 11. Then, the rotated open-loop MPC problem $\mathcal{P}^r(\cdot)$ is obtained from $\mathcal{P}(\cdot)$ by replacing the performance index $J_T(\cdot)$ with the* rotated performance index

$$J_T^r(t, z, \bar{\boldsymbol{u}}) := \underset{[t,t+T]}{\mathrm{L}}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) + M(t+T, \bar{x}(t+T)) \quad (11)$$

where

$$\begin{aligned} \underset{[t,t+T]}{\mathrm{L}}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}) &:= \int_t^{t+T} l(\tau, \bar{x}, \bar{u}) - l(\tau, x_e, u_e) d\tau \\ &\quad + \lambda(t, z) - \lambda(t+T, \bar{x}(t+T)) \quad (12) \\ M(t, \bar{x}) &:= \lambda(t, \bar{x}) + m(t, \bar{x}) - \lambda(t, x_e(t)). \quad (13) \end{aligned}$$

□

Lemma 15 (Properties of $\mathcal{P}^r(\cdot)$) *Consider the rotated MPC problem from Definition 14 and let Assumptions 12-13 hold with the same trajectory pair $(\boldsymbol{x}_e, \boldsymbol{u}_e)$. The following properties hold:*

$$\underset{[t,t+\delta]}{\mathrm{L}}(\boldsymbol{x}_e, \boldsymbol{u}_e) = 0, \quad (14)$$

$$\underset{[t,t+\delta]}{\mathrm{L}}(\boldsymbol{x}, \boldsymbol{u}) \geq \int_t^{t+\delta} \alpha(\|x(\tau) - x_e(\tau)\|) d\tau \quad (15)$$

$$\begin{aligned} &\underset{[t+\delta,t+T+\delta]}{\mathrm{L}}(\boldsymbol{x}, \boldsymbol{u}) - \underset{[t,t+T]}{\mathrm{L}}(\boldsymbol{x}, \boldsymbol{u}) \\ &\quad \leq -\underset{[t,t+\delta]}{\mathrm{L}}(\boldsymbol{x}, \boldsymbol{u}) + \underset{[t+T,t+T+\delta]}{\mathrm{L}}(\boldsymbol{x}, \boldsymbol{u}) \quad (16) \end{aligned}$$

for all $t \geq t_0$, $T \geq \delta > 0$, and where $\alpha(\cdot)$ is the class-$\mathcal{K}_\infty$ function introduced in Definition 11. Moreover, for the closed-loop system (1) with (9) and for any initial condition $\hat{x} \in \mathcal{X}_{aux}(\hat{t})$, with $\hat{t} \geq t_0 + T$, the properties

$$M(\hat{t} + \delta, x(\hat{t} + \delta)) - M(\hat{t}, \hat{x}) \leq -\underset{[\hat{t},\hat{t}+\delta]}{\mathrm{L}}(\boldsymbol{x}, \boldsymbol{u}) \quad (17)$$

$$M(t, x(t)) \geq 0, \qquad M(t, x_e(t)) = 0$$

hold for all $\delta > 0$ and $t \geq \hat{t}$. □

Remark 16 (Equivalence of $\mathcal{P}^r(\cdot)$ and $\mathcal{P}(\cdot)$) *The MPC optimization problems $\mathcal{P}(\cdot)$ and $\mathcal{P}^r(\cdot)$ share the same set of minimizers. In fact, the performance indexes $J_T(\cdot)$ and $J_T^r(\cdot)$ differ by only the term $\lambda(t, z) - \lambda(t+T, x_e(t+T))$ that is not a function of the optimization variables.* □

Using the rotated MPC optimization problem it is possible to establish convergence of the state trajectory of the closed-loop system (1) with (4) to $\boldsymbol{x}_e$, as is stated in the following result.

Theorem 17 (Convergence of Economic MPC) *Consider the constrained system (1)-(2) in closed-loop with the MPC control law defined in (4) and suppose that Assumptions 8-13 hold with the same trajectory pair $(\boldsymbol{x}_e, \boldsymbol{u}_e)$. Then, the vector $x(t) - x_e(t)$ is uniformly bounded over time and, as $t \to \infty$, the state vector $x(t)$ converges to $x_e(t)$. The region of attraction associated with the controller (4) consists of the set of states x for which $\mathcal{P}(t_0, x)$ admits a feasible solution.* □

It is worth noticing that the terminal set, and therefore the region of attraction of the MPC controller, is in general time-varying.

Remark 18 (Tracking MPC) *Theorem 17 generalizes the standard sufficient condition of Tracking MPC for convergence to desired state and input trajectory pair $(\boldsymbol{x}_d, \boldsymbol{u}_d) \in \mathcal{S}$. Indeed, in Tracking MPC the stage cost is chosen to be representative of the distance from the current to the desired state, i.e.,*

$$l(t, x_d, u_d) = 0, \qquad l(t, x, u) \geq \alpha(\|x - x_d\|). \quad (18)$$

Considering the proposed Economic MPC scheme with $l(t, x_e(t), u_e(t)) = 0$ and $\lambda(t, x) = 0$, Assumption 12 results in the condition (18) for $(\boldsymbol{x}_d, \boldsymbol{u}_d) = (\boldsymbol{x}_e, \boldsymbol{u}_e)$. □

Remark 19 *While there might be multiple economically optimal state trajectories, the MPC controller in Theorem 17 is designed on a specific one. Therefore, it addresses the convergence to a specific (time-parametrized) trajectory, e.g., the periodic trajectory in the example of Section 6.2, and not the convergence to a geometric path (without explicit time parametrization) like, for instance, a circular orbit.* □

4 Performance Analysis

In the previous section, we presented sufficient conditions for convergence to a feasible state and input trajectory pair $(\boldsymbol{x}_e, \boldsymbol{u}_e)$. In this section, we show that strict dissipativity of system (1) implies that $(\boldsymbol{x}_e, \boldsymbol{u}_e)$ is an average economically optimal trajectory pair. Moreover, the proposed MPC control law is shown to be an economically optimal controller.

Proposition 20 (Optimality of $(\boldsymbol{x}_e, \boldsymbol{u}_e)$) *Consider the system* (1)-(2), *with the associated feasible trajectory set $\mathcal{S}$ from Definition 4. If Assumption 12 holds and $\lambda(t, x)$ is uniformly bounded over time, then $(\boldsymbol{x}_e, \boldsymbol{u}_e)$ is an average economically optimal trajectory pair, as defined in Definition 5.* □

Remark 21 (Boundedness of $\lambda(\cdot)$) *The boundedness of $\lambda(\cdot)$, required by Proposition 20, is not a very restrictive assumption. For instance, it can be obtained considering $\mathcal{X}(\cdot)$ compact and $\lambda(\cdot, \hat{x})$ to be continuous in $\hat{x}$ and uniformly bounded over time for any given $\hat{x} \in \mathbb{R}^n$.* □

Next, we look specifically at the comparison between the closed-loop trajectory associated with the Economic MPC controller and the pair $(\boldsymbol{x}_e, \boldsymbol{u}_e)$.

Theorem 22 (Performance of MPC Scheme) *Consider Definition 6 and let Assumptions 8-10 and 13 hold. Moreover, let $\boldsymbol{x}_e$ and $\boldsymbol{x}$ of the closed-loop system* (1)-(4) *be uniformly bounded over time (e.g., using a bounded $\mathcal{X}(\cdot)$). Then, the MPC controller* (4) *is economically optimal.* □

Remark 23 *Note that Theorem 22 does not require the system to be dissipative. Although, if Assumption 12 holds, the boundedness of the state trajectory is given from Theorem 17 and form the boundedness of $\boldsymbol{x}_e$.* □

5 Computation of the Terminal Set and Terminal Cost

This section addresses the computation of a terminal set and a terminal cost to satisfy Assumption 13. First, we consider the general case where a suitable auxiliary control law is given. Then, if such law is unknown, we provide two constructive procedures for the case of a stabilizable linearization around a constant x_e.

5.1 Known Auxiliary Control Law

The methods proposed in this section are based on the following result.

Lemma 24 (Known Auxiliary Control Law) *Consider the closed-loop system $\dot{x}_{aux} = f(\tau, x_{aux}, u_{aux})$ with $u_{aux} = k_{aux}(\tau, x_{aux}) \in \mathcal{U}(\tau)$ starting at time $\hat{t} \geq t_0$ from the state $x_{aux}(\hat{t}) = \hat{x} \in \mathcal{X}_{aux}(\hat{t})$, and a trajectory pair $(\boldsymbol{x}_e, \boldsymbol{u}_e) \in \mathcal{S}$. Assume that the set $\mathcal{X}_{aux}(t)$ is positively invariant for all $t \geq \hat{t}$ and that $x_{aux}(\tau) \to x_e(\tau)$ as $\tau \to \infty$. Moreover, consider a time-dependent bound $\hat{l}(\tau; \hat{t}, \hat{x})$, parametrized with the initial time $\hat{t} \geq t_0$ and state $\hat{x} = x(\hat{t})$, such that*

$$\hat{l}(\tau; \hat{t}, \hat{x}) \geq l(\tau, x_{aux}, u_{aux}) - l(\tau, x_e, u_e) \tag{19a}$$
$$\hat{l}(\tau; \hat{t} + \delta, x_{aux}(\hat{t} + \delta)) \leq \hat{l}(\tau; \hat{t}, \hat{x}) \tag{19b}$$
$$\lim_{\tau \to +\infty} \hat{l}(\tau; \hat{t}, \hat{x}) = 0. \tag{19c}$$

for all $\tau \geq \hat{t} + \delta$ and $\delta \geq 0$. Then, the terminal cost

$$m(\hat{t}, \hat{x}) = \int_{\hat{t}}^{+\infty} \hat{l}(\tau; \hat{t}, \hat{x}) d\tau, \tag{20}$$

satisfies the inequality (10). □

It is worth noticing that the auxiliary law in Lemma 24 can be computed using standard control design methods, see, e.g., [17]. Next, we focus on the common case of exponentially stabilizing auxiliary control laws and stage costs with a polynomial upper bound.

Assumption 25 (Known Auxiliary Control Law) *Consider a trajectory pair $(\boldsymbol{x}_e, \boldsymbol{u}_e) \in \mathcal{S}$ and suppose the following elements to be given:*

- *A control law $k_{aux} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ that asymptotically drives $x(t)$ to $x_e(t)$ through feasible state and input trajectories, i.e., for the closed-loop system* (1) *with $u(t) = k_{aux}(t, x(t)) \in \mathcal{U}(t)$, $x(t) \in \mathcal{X}(t)$.*
- *A continuously differentiable Lyapunov function $V_{aux} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, with the positive constants k_1, k_2, k_3, and a, such that*

$$k_1 \|x - x_e\|^a \leq V_{aux}(t, x) \leq k_2 \|x - x_e\|^a \tag{21a}$$
$$\dot{V}_{aux}(t, x) \leq -k_3 \|x - x_e\|^a \tag{21b}$$

hold for all $x \in \mathcal{X}_{aux}(t)$, $t \geq t_0$, and for a set $\mathcal{X}_{aux}(t) = \mathcal{L}(t; V_{aux}, r) \subseteq \mathcal{X}(t)$, with $r \geq 0$. □

Assumption 26 (Bound on Stage Cost) *The control law $k_{aux}(\cdot)$ from Assumption 25 and the stage cost $l(\cdot)$ are such that*

$$l(t, x(t), k_{aux}(t, x(t))) - l(t, x_e(t), u_e(t)) \leq \sum_{i=1}^{v} a_i \|x(t) - x_e(t)\|^i, \quad \forall x(t) \in \mathcal{X}_{aux}(t) \tag{22}$$

for some constants $v \in \mathbb{N}_{>0}$, $a_i \in \mathbb{R}$, $i = 1, \ldots, v$, and for a trajectory pair $(\boldsymbol{x}_e, \boldsymbol{u}_e) \in \mathcal{S}$. □

Assumption 26 captures a wide range of economic stage costs and auxiliary control laws, e.g., any stage cost and auxiliary control law with an upper bound that is polynomial in $\|x - x_e\|$ and tight, i.e., zero, at $x = x_e(t)$.

Proposition 27 *Consider the set $\mathcal{X}_{aux}(\cdot)$ and the system* (1) *in closed-loop with the auxiliary control law from*

Assumption 25. Moreover, let Assumptions 25-26 hold with the same trajectory pair $(\boldsymbol{x}_e, \boldsymbol{u}_e) \in \mathcal{S}$*. Then, the terminal cost function*

$$m(t,x) = \frac{ak_2}{k_3} \sum_{i=1}^{v} \frac{a_i}{k_1^{i/a} i} V_{aux}(t,x)^{i/a} \tag{23}$$

and the terminal set $\mathcal{X}_{aux}(\cdot)$ *satisfy Assumption 13.* □

5.2 Unknown Auxiliary Control Law and Stabilizable Linearization

This subsection considers the time invariant case with stabilizable linearization around the desired equilibrium point, common in the MPC literature. In this case the steady-state (x_e, u_e) is constant and, without loss of generality, it is assumed to be at the origin. Two linearization based algorithms to compute suitable terminal sets and terminal costs are presented. Due to space limitation, the proofs in this section are omitted. They can be found in [1].

Assumption 28 (Stabilizable Linearization) *Given system* (1)*, the function* $f(\cdot)$ *is twice continuously differentiable on* $\mathbb{R}^n \times \mathbb{R}^m$*, and the origin of the linearized system* $\dot{x} = Ax + Bu$*, with* $A := f_x(0,0)$ *and* $B := f_u(0,0)$*, is stabilizable.* □

This assumption allows the design of an auxiliary control law that locally stabilizes $x_e = 0$ with

$$k_{aux}(t, x(t)) = Kx, \qquad A_{cl} := A + BK \tag{24}$$

where K is any matrix that makes A_{cl} Hurwitz, which by Assumption 28, always exists.

For the sake of clarity, we define the following operator:

Definition 29 (The Operator $\Omega(\cdot)$) *Let Assumption 28 hold and consider the constrained system* (1)*-*(2) *with* $0 \in \mathcal{X}$ *and* $0 \in \mathcal{U}$*, and the auxiliary control law* (24)*. The operator* $\Omega(\cdot)$ *takes as input a set* $\mathcal{A} \subseteq \mathbb{R}^n$ *with* $0 \in \mathcal{A}$*, a positive definite matrix* $0 \prec Q \in \mathbb{R}^{n\times n}$*, a vector* $q \in \mathbb{R}^n$*, and positive scalar* $\beta \in (0, -\lambda_{max}(A_{cl}))$*, and returns the triplet*

$$(P, p, \mathcal{X}^*) \;=\; \Omega(\mathcal{A}, Q, q, \beta).$$

The matrix P *uniquely solves the Lyapunov equation* $(A_{cl} + \beta I)'P + P(A_{cl} + \beta I) = Q$*, the vector* p *is defined as* $p := -(A_{cl} + \beta I)'^{-1} q$*, and the set* $\mathcal{X}^*$ *is chosen to be* $\mathcal{X}^* := \mathcal{L}(V, r^*)$ *with* $V(x) := x'Px$ *and where* r^* *is the optimal solution of the optimization problem*

$$\begin{aligned}
& r^* = \max_r r \;\; s.t. && (25a)\\
& 2x'Pe + p'e \le 2\beta x'Px + \beta p'x,\; \forall x \in \mathcal{L}(V, r) && (25b)\\
& Kx \subseteq \mathcal{U},\; \forall x \in \mathcal{L}(V, r) && (25c)\\
& \mathcal{L}(V, r) \subseteq \mathcal{A} && (25d)
\end{aligned}$$

with $e := f(x, Kx) - A_{cl}x$*.* □

This operator $\Omega(\cdot)$ is used to compute the terminal set for the economic MPC controller. As detailed in the proof of the following lemma, i) condition (25b) is used to guarantee that $\mathcal{X}^*$ is contained in the region where the origin of the closed-loop system (1) with $u(t) = Kx(t)$ is stable, ii) condition (25c) guarantees the associated control input to be feasible, and iii) (25d) constraints $\mathcal{X}^*$ within $\mathcal{A}$. The following properties hold:

Lemma 30 (Properties of $\Omega(\cdot)$) *Let Assumption 28 hold and consider the constrained system* (1)*-*(2) *in closed-loop with the auxiliary control law* (24)*. Let*

$$W(x) := x'Px + p'x$$

with $(P, p, \mathcal{X}^*) = \Omega(\mathcal{A}, Q, q, \beta)$ *for a set* $\mathcal{A} \subseteq \mathbb{R}^n$ *with* $0 \in \mathcal{A}$*, any positive definite matrix* $0 \prec Q \in \mathbb{R}^{n\times n}$*, vector* $q \in \mathbb{R}^n$*, and positive scalar* $\beta \in (0, -\lambda_{max}(A_{cl}))$*. Then, the following is true:*

(i) The set $\mathcal{X}^*$ *is always non-empty and contained in* $\mathcal{A}$*, i.e.,* $\mathcal{X}^* \subseteq \mathcal{A}$*.*
(ii) For any $x \in \mathcal{X}^*$ *we have* $Kx \in \mathcal{U}$*, and the following inequality holds* $\dot{W}(x) \le -x'Qx - x'q$*.*

Moreover, if in addition, $q = 0$*,* $0 \in \operatorname{int}\mathcal{A}$*, and* $0 \in \operatorname{int}\mathcal{U}$*, then*

(iii) the set $\mathcal{X}^*$ *is compact, positively invariant, and with non-empty interior.* □

Note that by using Lemma 30 with $q = 0$ the operator $\Omega(\cdot)$ returns the standard ellipsoidal terminal set commonly used in the linearization based design of Tracking MPC, originally proposed in [7]. Moreover, it represents the continuous-time counterpart of [5]. Next, a quadratic upper bound on the stage cost, needed for the computation of the terminal cost, is obtained using the following regularity assumption.

Assumption 31 (Regularity of the Stage Cost) *The stage cost function* $l(\cdot)$ *is twice continuous differentiable on* $\mathbb{R}^n \times \mathbb{R}^m$*.* □

Lemma 32 (Quadratic Bound on the Stage Cost) *Let Assumptions 28-31 hold and consider the auxiliary* *control law* (24). *Then, for any compact set* $\mathcal{C}$ *the optimization problem*

$$\lambda^* = \min_{\lambda} \lambda \text{ s.t. } l(x, Kx) \le l_q(x),\ \forall x \in \mathcal{C} \tag{26a}$$

$$l_q(x) := \lambda x'x + l_x(0,0)x + l(0,0) \tag{26b}$$

admits a feasible solution. □

Now, we are ready to state the following result.

Theorem 33 *Let Assumptions 28-31 hold and consider the constrained system* (1)-(2) *with* $0 \in \operatorname{int}\mathcal{X}$ *and* $0 \in \operatorname{int}\mathcal{U}$ *and the auxiliary control law* (24). *Then, the following algorithm provides a terminal set* $\mathcal{X}_{aux}$ *and a terminal cost* $m(\cdot)$ *that satisfy Assumption 13:*

(1) Compute $(\star, \star, \mathcal{X}_{lin}) = \Omega(\mathcal{X}, Q, 0, \beta_{lin})$ *for a* $\beta_{lin} \in (0, -\lambda_{max}(A_{cl}))$ *and a positive definite matrix* $Q \succ 0$.
(2) Compute λ^* *by solving the optimization problem* (26) *with* $\mathcal{C} = \mathcal{X}_{lin}$.
(3) Compute the terminal cost $m(x) = x'P_{aux}x + p'_{aux}x$ *and the terminal set* $\mathcal{X}_{aux}$ *by solving*

$$(P_{aux}, p_{aux}, \mathcal{X}_{cost}) = \Omega(\mathcal{X}_{lin}, \lambda^* I, l_x(0,0)', \beta_{aux})$$
$$(\star, \star, \mathcal{X}_{aux}) = \Omega(\mathcal{X}_{cost}, Q, 0, \beta_{lin})$$

for some $\beta_{aux} \in (0, -\lambda_{max}(A_{cl}))$. □

Next, a different method, based on the Proposition 27, is proposed.

Theorem 34 (MPC with Terminal Inequality) *Let Assumptions 28-31 hold and consider the constrained system* (1)-(2) *with* $0 \in \operatorname{int}\mathcal{X}$ *and* $0 \in \operatorname{int}\mathcal{U}$ *and the auxiliary control law* (24). *Then, the following algorithm provides a terminal set* $\mathcal{X}_{aux}$ *and a terminal cost* $m(\cdot)$ *that satisfy Assumption 13:*

(1) Compute $(P_{lin}, \star, \mathcal{X}_{lin}) = \Omega(\mathcal{X}, Q, 0, \beta_{lin})$ *for a* $\beta_{lin} \in (0, -\lambda_{max}(A_{cl}))$ *and a positive definite matrix* $Q \succ 0$.
(2) Compute λ^* *solving the optimization problem* (26) *with* $\mathcal{C} = \mathcal{X}_{lin}$.
(3) Choose $m(\cdot)$ *as in* (23) *with* $k_1 = \lambda_{min}(P_{lin})$, $k_2 = \lambda_{max}(P_{lin})$, $k_3 = \lambda_{min}(Q)$, $a = 2$, $v = 2$, $a_1 = \|l_x(0,0)\|$, *and* $a_2 = \lambda^*$. □

Remark 35 (Comparison of Design Methods) *The linearization based method provided in Theorem 34 guarantees a non-empty interior of the terminal set. This is in contrast to the method proposed in Theorem 33, where the operator* $\Omega(\cdot)$ *used in point three might return a terminal set consisting in only the origin. While in terms of size of the terminal set the method in Theorem 34 outperforms the one in Theorem 33, the terminal cost is case dependent, and depends on the specific economic stage cost and auxiliary control law employed.* □

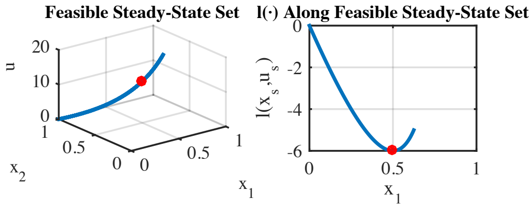

Fig. 1. The line denotes the set of feasible steady-state (left) and the associated stage cost (right). The dot represents the economically optimal steady-state.

Remark 36 (Unknown Dissipativity Function) *Note that the dissipativity function* $\lambda(\cdot)$ *is not used in the design of the MPC controllers.* □

6 Examples

The examples presented in this section are solved using the ACADO Toolbox [15] with a uniform time sampling $\mathcal{T} = \{0.05i,\ i \in \mathbb{N}_{\ge 0}\}$.

6.1 Example 1: Isothermal CSTR

This example is taken from [8], and i) directly solved in the continuous-time framework presented in this paper ii) where the terminal equality is replaced with a suitable terminal set and a terminal cost designed using Theorem 34. It addresses the economic control of an isothermal CSTR modeled by the following constrained dynamical system

$$\dot{x} = \begin{pmatrix} 0.1u(1-x_1) - 1.2x_1 \\ -0.1u\,x_2 + 1.2x_1 \end{pmatrix}, \tag{27}$$

where the state vector $x := (x_1\ \ x_2)'$, constrained as $x_i \in [0,\ 1]$, $i = 1, 2$, is a vector of molar concentrations of the materials of the reaction and $u \in [0, 20]$, the constrained control input, is the flow through the reactor. The economic stage cost that we wish to minimize, associated with the production and separation costs, is $l(x, u) = -(2ux_2 - 0.5u)$. Fig. 1 shows the set of the feasible steady-states, given by $\left\{(x, u) :\ x_2 = 1 - x_1,\ u = \frac{12x_1}{(1-x_1)}\right\}$, and the stage cost evaluated along such set, where the red dot identifies the economically optimal steady-state $(x_e, u_e) = \left(\begin{pmatrix}0.5\ 0.5\end{pmatrix}', 12\right)$. In order to guarantee convergence to the economically optimal steady-state, a regularization term is added to the stage cost, resulting in $l(x, u) = -(2ux_2 - 0.5u) + 0.505(\|x - x_e\|^2 + \|u - u_e\|^2)$. It can be shown (see [8]) that the dissipativity function $\lambda(x) = [10, 20]'x$ satisfies Assumption 12 at (x_e, u_e) for the regularized stage cost. Since system dynamic, stage cost, and pair (x_s, u_s) are time invariant, we can use the constructive methods of Section 5.2 to design

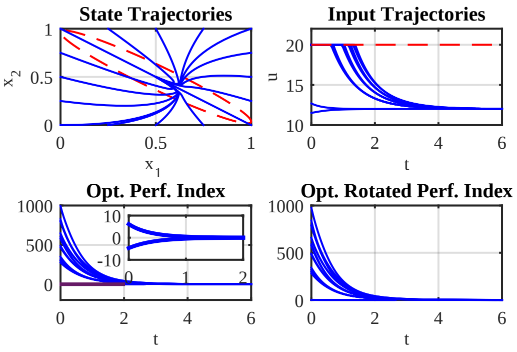

Fig. 2. Closed-loop state (top-left) and input (top-right) trajectories, starting at different initial conditions, with the associated optimal performance index (bottom-left) and optimal rotated performance index (bottom-right). The dashed lines represent the input constraints in the top-right figure and the terminal constraint in the top-left figure.

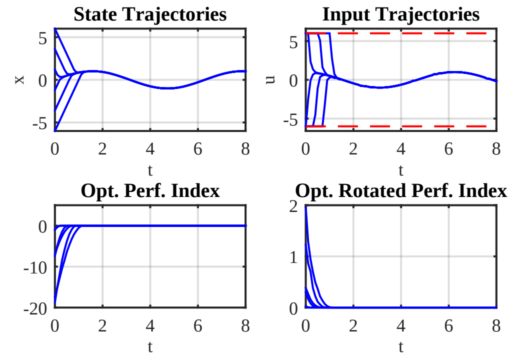

Fig. 3. Closed-loop state (top-left) and input (top-right) trajectories, starting at different initial conditions, with the associated optimal performance index (bottom-left) and optimal rotated performance index (bottom-right). The dashed lines denote the input constraints.

the Economic MPC controller. For this specific case, the algorithm proposed by Theorem 33 provides a terminal *equality* constraint, while the second method, proposed in Theorem 34, delivers the promised terminal set with not-empty interior and is therefore chosen for the design. Fig. 2 shows that the rotated optimal performance index, in contrast to the standard one, manifests a monotonic decrease for all the initial conditions and convergence to the economically optimal trajectory is achieved.

6.2 Example 2: Time-Varying Stage Cost

In this example, a simple time-varying economic optimization problem is considered in order to illustrate the use of the design method from Proposition 27. Consider the single integrator model $\dot{x}(t) = u(t)$ with state and input constraints $|x| \leq 6$ and $|u| \leq 6$, respectively, and the nonlinear time-varying stage cost $l(t,x,u) = (u-\cos(t))(x-\sin(t))+0.1(x-\sin(t))^2$. Note that the dissipativity function $\lambda(x,t) = 0.5(x - x_e(t))^2$ certifies the dissipativity property of the system at the feasible state and input trajectory pair $x_e(t) = \sin(t)$ and $u_e(t) = \cos(t)$. The auxiliary control law $u = k_{aux}(t,x(t)) = -(x - x_e(t)) + u_e(t)$ drives x to $x_e(t)$ exponentially quickly with a certificate of exponential stability given by $V_{aux}(t,x) = 0.5(x - x_e(t))^2$ where, for the system in closed-loop, we have $\dot{V}_{aux}(t,x) = -(x - x_e(t))^2$. For the design of a suitable terminal set it is enough to notice that $\|k_{aux}(t,x(t))\| \leq \|u_e(t))\| + \|x - x_e(t)\| \leq 1 + \|x - x_e(t)\|$ and therefore, for each $x \in \mathcal{L}(t; V_{aux}, 0.5\ 5^2)$, the auxiliary control law is feasible. Noting that $l(t,x,k_{aux}(t,x)) - l(t,x_e,u_e) = -0.9(x - x_e)^2$, the condition (10) it is always satisfied with $m(t,x) = 0$. Clearly, the decrease (17) on the terminal cost of the rotated optimization problem still holds due to the existence of the dissipatifity function. Similarly to the previous example, Fig. 3 shows that the rotated optimal performance index, in contrast to the standard one, manifests a monotonic decrease for all the initial conditions and convergence to the economically optimal trajectory is achieved.

7 Conclusion

This paper introduced an Economic MPC scheme with terminal penalty for time-varying continuous-time systems. The dissipativity properties of the system are used to certify the asymptotic convergence of the closed-loop trajectories to the time-dependent state and input trajectory pair $(\boldsymbol{x}_e, \boldsymbol{u}_e)$, which is shown to be an average economically optimal trajectory pair. Moreover, the Economic MPC controller is proven to possess the same average performance obtained by operating the system at $(\boldsymbol{x}_e, \boldsymbol{u}_e)$. Constructive methods for the design of a suitable terminal set and terminal cost are presented and illustrated via numerical examples.

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A Appendix

Lemma 37 *Consider the system (1)-(2), and let* $\{\|f(t,x,u)\| : t \geq t_0, x \in \bar{\mathcal{X}}, u \in \mathcal{U}(t)\}$ *is bounded for any bounded* $\bar{\mathcal{X}} \subset \mathcal{R}^n$*. Then, for any class-*$\mathcal{K}_\infty$ *function* $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *and constant* $\delta > 0$ *there exists a class-*$\mathcal{K}_\infty$ *function* $\alpha_v : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *such that*

$$\int_t^{t+\delta} \alpha(\|x(\tau)\|)d\tau \geq \alpha_v(\|x(t)\|)$$

holds for all $x(t) \in \mathcal{X}(t)$ and $t \geq t_0$. □

PROOF. Next follows a sketch of the proof, the full version can be found in [1]. Consider the compact ball set $\mathcal{B}(r)$ of radius r. Then, using the fact that $\|f(t,x,u)\|$ is compact on compact x, we can write

$$\|f(t,x,u)\| \leq b(r), \forall t \geq t_0,\ u \in \mathcal{U}(t),\ x \in \mathcal{B}(r) \quad \text{(A.1)}$$

for a function $b : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is, without loss of generality, monotonically increasing and greater than 0 away from zero. Now, consider the set of constant samples $r_i := \delta_r i,\ i \in \mathbb{N}_{\geq 0}$ with $\delta_r > 0$. Then, evaluating (A.1) in the intervals $[r_i, r_{i+1}]$ results in a piecewise affine lower-bound $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\|x(\tau)\| \geq \beta(\|x(t)\|, \tau - t)$ where

$$\beta(r,\tau) := \begin{cases} \alpha_s(\alpha_s^{-1}(r) - \tau), & \text{if } \alpha_s^{-1}(r) \geq \tau \\ 0 & \text{otherwise} \end{cases}$$
$$\alpha_s(s) := r_i + (s - s_i) b(r_{i+1}),\ \forall s \in [s_i, s_{i+1}]$$

with $s_{i+1} = s_i + \delta_r / b(r_i)$ and $s_0 = 0$. Defining, $\bar{\beta}(r,s) := \alpha(\beta(r,s))$, for any $t \geq t_0$ and $\delta > 0$ we can write $\int_t^{t+\delta} \alpha(\|x(\tau)\|) d\tau \geq \int_t^{t+\delta} \bar{\beta}(\|x(t)\|, \tau - t) d\tau = \int_0^{\delta} \bar{\beta}(\|x(t)\|, \tau) d\tau =: \alpha_v(\|x(t)\|)$ where, it is possible to show that the function $\alpha_v(\cdot)$ belongs to class $\mathcal{K}_\infty$. ■

A.1 Proof of Lemma 15

The conditions (14) and (16) follow from the definition (12) of $L(\cdot)$ and the fact that, from (7), $\lambda(t, x_e(t))$ is constant. Combining the strict disipativity Assumption 12 with (12), results in (15). From the definition of $M(\cdot)$ in (13), (7), and the decrease (10) on $m(\cdot)$ in Assumption 13, we obtain (17). Combining the latter with (15) and taking the limit of δ to infinity, results in $M(t, x(t)) \geq 0$. Last, Assumption 13 (iii) results in $M(t, x_e(t)) = 0$, which concludes the proof. ■

A.2 Proof of Theorem 17

Consider the *value function* defined as

$$V(t,z) := \min_{\bar{\boldsymbol{u}} \in \mathcal{PC}(t, t+T)} J_T^r(t, z, \bar{\boldsymbol{u}}). \quad \text{(A.2)}$$

Next, we prove that, for any state $x_i = x(t_i) = x_{e_i}(t_i)$, such that the optimization problem $\mathcal{P}(t_i, x_i)$ admits a feasible solution, and for any $\delta \geq 0$, the following cost inequality holds along the extended state trajectory:

$$V(t_i + \delta, x_{e_i}(t_i + \delta)) \leq V(t_i, x_i) - \int_{t_i}^{t_i + \delta} \alpha(\|x_{e_i} - x_e\|) d\tau.$$

To this end, consider first $\delta \leq T$. Then, from (11)

$$\begin{aligned} V(t_i + \delta, x_{e_i}(t_i + \delta)) &\leq \underset{[t_i+\delta, t_i+\delta+T]}{\mathrm{L}}(\boldsymbol{x}_{e_i}, \boldsymbol{u}_{e_i}) \\ &+ M(t_i + T + \delta, x_{e_i}(t_i + T + \delta)) \\ &= V(t_i, x_{e_i}(t_i)) - \underset{[t_i, t_i+\delta]}{\mathrm{L}}(\boldsymbol{x}_{e_i}, \boldsymbol{u}_{e_i}) \\ &+ \underset{[t_i+T, t_i+T+\delta]}{\mathrm{L}}(\boldsymbol{x}_{e_i}, \boldsymbol{u}_{e_i}) - M(t_i + T, x_{e_i}(t_i + T)) \\ &+ M(t_i + T + \delta, x_{e_i}(t_i + T + \delta)) \\ &\leq V(t_i, x_{e_i}(t_i)) - \underset{[t_i, t_i+\delta]}{\mathrm{L}}(\boldsymbol{x}_{e_i}, \boldsymbol{u}_{e_i}) \\ &\leq V(t_i, x_{e_i}(t_i)) - \int_{t_i}^{t_i+\delta} \alpha(\|x(\tau) - x_e(\tau)\|) d\tau \quad \text{(A.3)} \end{aligned}$$

where the first inequality arises from the fact that i) the extended trajectory is not optimal, and ii) the minimizer of $J_T(\cdot)$ corresponds to the minimizer of $V(\cdot)$, as observed in the Remark 16, and the second and third inequalities come from the property (17) and (15), respectively. The same reasoning holds for the case $\delta > T$. The inequality (A.3) also applies to the closed-loop (1) with (4), i.e., using recursively the MPC optimization problem in a receding horizon fashion. In fact, the associated state trajectory $\boldsymbol{x}$ is a concatenation of pieces of extended trajectories, and therefore

$$\begin{aligned} &V(t, x(t)) - V(t_0, x_0) \\ &\leq -\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \alpha(\|x_{e_j} - x_e\|) d\tau - \int_{t_i}^{t} \alpha(\|x_{e_i} - x_e\|) d\tau \\ &= -\int_{t_0}^{t} \alpha(\|x - x_e\|) d\tau \quad \text{(A.4)} \end{aligned}$$

with $t_i = \lfloor t \rfloor$. Note that the optimization problem $\mathcal{P}(t_i, x(t_i))$ is recursively feasible since, by Assumption 10, it is feasible at (t_0, x_0) and the extended input keeps it feasible for all $\delta > 0$ and, specifically, for the generic interval from t_i to t_{i+1}. Moreover, from (A.2), the Definition 14 of rotated MPC optimization problem and Lemma 37 (applied to $x - x_e$, where we notice that $f(t,x,u) - f(t, x_e, u_e)$ is bounded for bounded $x - x_e$ from Assumption 9 and from $\boldsymbol{x}_e$ being bounded) any level set of $V(t,\cdot)$ is uniformly bounded over time and, from (A.4), positively invariant. Thus, the closed-loop trajectory $\boldsymbol{x}$ is confined within the time-varying bounded set $\mathcal{L}(t; V, V(t_0, x_0))$. At this point, in order to prove convergence to $\boldsymbol{x}_e$, we can use Barbalat's lemma (see, e.g., Lemma 8.2 in [17]). In fact, $\alpha(\|x(t) - x_e(t)\|)$ is a uniformly continuous function of t because i) the function $\alpha(\|x - x_e\|)$ is continuous in $x - x_e$ with $x - x_e$ bounded from the invariance of $\mathcal{L}(t; V, V(t_0, x_0))$ and ii) $x - x_e$ is uniformly continuous in t from Assumption 9. Moreover, from (A.4) we have that

$$\lim_{\delta \to \infty} \int_{t_0}^{t_0+\delta} \alpha(\|x(\tau) - x_e(\tau)\|) d\tau \leq V(t_0, x_0) < +\infty$$

where the limit exists since the function $\int_{t_0}^{t_0+\delta} \alpha(\|x - x_e\|)d\tau$ is non decreasing in δ, from $\alpha(\cdot)$ being non negative, and from (A.4), it is upper bounded by $V(t_0, x_0)$. Thus, by Barbalat's lemma, $\alpha(\|x(t) - x_e(t)\|) \to 0$ as $t \to +\infty$ and, by the positive-definitiveness of $\alpha(\cdot)$, the state vector $x(t)$ converges to $x_e(t)$ with $t \to +\infty$, which concludes the proof. ■

A.3 Proof of Proposition 20

Dividing the dissipativity inequality (8) by δ and taking the limit with δ that goes to infinity results in
$0 = \lim_{\delta\to\infty} \frac{1}{\delta}(\ \lambda(t_0+\delta, x(t_0+\delta)) - \lambda(t_0, x(t_0)) \ )$
$\leq \lim_{\delta\to\infty} \frac{1}{\delta} \int_{t_0}^{t_0+\delta} l(\tau, x, u) - l(\tau, x_e, u_e)d\tau$
where we use the facts that $\lambda(\cdot)$ is bounded and $\alpha(\|x - x_e\|) \geq 0$. This concludes the proof. ■

A.4 Proof of Theorem 22

Using the optimality of the solution of $\mathcal{P}(\cdot)$, the feasibility of the associated extended trajectories introduced in Definition 2, and the properties (10) it is possible to show that $J_T^*(t+\delta) - J_T^*(t) \leq -\int_t^{t+\delta} l(\tau)d\tau + \int_{t+T}^{t+T+\delta} l_e(\tau)d\tau$ for any $t \geq t_0$ and $\delta > 0$, where we write in a compact from $l(\tau) = l(\tau, x, u)$, $l_e(\tau) = l(\tau, x_e, u_e)$, $J_T^*(t_0) = J_T^*(t_0, x_0)$. Choosing $t = t_0$, dividing both sides by δ and taking the limit of δ that goes to infinity we have
$\lim_{\delta\to+\infty} \frac{1}{\delta}(J_T^*(t+\delta) - J_T^*(t)) \leq \lim_{\delta\to+\infty} \frac{1}{\delta}(\int_{t+T}^{t+T+\delta} l_e(\tau)d\tau$
$- \int_t^{t+\delta} l(\tau)d\tau) = \lim_{\delta\to+\infty} \frac{1}{\delta}(\int_t^{t+\delta} l_e(\tau) - l(\tau)d\tau)$ where the last equality comes from the boundedness of $l_e(\cdot)$, and therefore the fact that
$\lim_{\delta\to+\infty} \frac{1}{\delta}(\int_t^{t+\delta} l_e(\tau)d\tau - \int_{t+T}^{t+T+\delta} l_e(\tau)d\tau) = 0$. The term $J_T^*(\cdot)$ is uniformly bounded over time from the boundedness of the vectors x and u and the consequent boundedness of the terms $l(\cdot)$ and $m(\cdot)$, which makes the term on the left-hand side to go to zero. Choosing $t = t_0$ concludes the proof. ■

A.5 Proof of Lemma 24

This lemma follows immediately from the choice of the terminal set (20) and the properties in (19). Notice that
$m(t_0+\delta, x_0+\delta) - m(t_0, x_0) = -\int_{t_0}^{t_0+\delta} \hat{l}(\tau; t_0, x_0)d\tau$
$+ \int_{t_0+\delta}^{+\infty} \hat{l}(\tau; t_0+\delta, x_0+\delta) - \hat{l}(\tau; t_0, x_0)dt$
$\leq -\int_{t_0}^{t_0+\delta} \hat{l}(\tau; t_0, x_0)d\tau$
$\leq -\int_{t_0}^{t_0+\delta} l(\tau, x_{aux}, u_{aux}) - l(\tau, x_e(\tau), u_e(\tau))d\tau$
where the last two inequalities arise from (19b) and (19a), respectively. ■

A.6 Proof of Proposition 27

The extended proof can be found in [1] or, for the case $x_e(t) = 0$, in Proposition 3 of [3] . ■

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