

The Robust Minimal Controllability Problem [★]

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Abstract

In this paper, we address the robust minimal controllability problem, where the goal is, given a linear time-invariant system, to determine a minimal subset of state variables to be actuated to ensure controllability under additional constraints. We study the problem of characterizing the sparsest input matrices that assure controllability, when the autonomous dynamics' matrix is simple when a specified number of inputs fail. We show that this problem is NP-hard, and under the assumption that the dynamics' matrix is simple, we show that it is possible to reduce the problem to a set multi-covering problem. Additionally, under this assumption, we prove that this problem is NP-complete, and polynomial algorithms to approximate the solutions of a set multi-covering problem can be leveraged to obtain close-to-optimal solutions.

Key words: control systems design; controllability; linear systems; computational methods; control algorithms.

1 Introduction

The problem of guaranteeing that a dynamical system can be driven toward the desired state regardless of its initial position is a fundamental question studied in control systems and it is referred to as *controllability*. Several applications, for instance, control processes, multi-agents networks, control of large flexible structures, systems biology and power systems (Egerstedt, 2011; Siljak, 2007; Skogestad, 2004) rely on the notion of controllability to safeguard their proper functioning. Subsequently, it is important to identify which subsets of state vari-

ables need to be actuated, or what is the placement of actuators required, to ensure controllability (van de Wal and de Jager, 2001; Olshevsky, 2014; Pequito, Kar and Aguiar, 2016).

Moreover, actuators may malfunction over time due to the adverse nature of the environments where the actuators are deployed, e.g. due to the wear and tear of the materials, or due to external (adversarial) influence of an agent aiming to disrupt the proper functioning of the dynamical system. In fact, a classical example of such malicious attack is the Stuxnet malware incident (Langner, 2011), in which the controller's input response to a tempered measured output lead the system away from its normal operating conditions. Thus, the control designer needs to consider such scenarios, while accounting for the actuator placement (Velde and Carignan, 1984). Additionally, as the systems become larger (i.e., the dimension of their state space), we aim to identify a relatively small subset of state variables that ensure the controllability of the system, for instance, due to economic constraints (Olshevsky, 2014). Consequently, in this paper we address the following natural design question:

\mathcal{Q}_1 : *What is the minimum number of actuated state variables we need to consider to ensure the controllability of a dynamical system if a specific number of actuators failures occur?*

To formally capture \mathcal{Q}_1 , we introduce and study the *robust minimal controllability problem* (rMCP) that aims

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to determine the minimum number of state variables that need to be actuated to ensure system’s controllability, under the possible failure of a specified number of actuators. This is a generalization of the minimal controllability problem (MCP) (Olshevsky, 2014), which can be obtained as a particular case of the rMCP when no actuator fails. Therefore, the MCP is the first step to understand resilience and robustness properties of dynamical systems since it unveils which variables need to be actuated.

Finally, it is important to mention that the rMCP can be stated regarding observability, by invoking the duality between controllability and observability in LTI systems (Hespanha, 2009). In particular, Shoukry and Tabuada (2014); Chen et al. (2015); Fawzi et al. (2014) provide necessary and sufficient conditions concerning the sensor deployment to ensure that a reliable estimate of the system is recovered. More importantly, those conditions can be achieved by design, when solving the rMCP. Hence, guaranteeing the design of stable observers to properly monitor the state evolution of an LTI system. Furthermore, the results presented in this paper are for discrete-time, but they are readily applicable to continuous-time LTI systems.

Related Work: This paper follows up and subsumes previous literature by considering the deployment of actuators to ensure controllability under possible actuation failures. When no actuators fail, it extends the results available for the MCP, as we overview next. In Nabi-Abdolyousefi and Mesbahi (2013) the controllability of circulant networks is analyzed by exploring the Popov-Belevitch-Hautus eigenvalue criterion, where the eigenvalues are characterized using the Cauchy-Binet formula. The controllability in multi-agents with Laplacian dynamics was initially explored in Tanner (2004). Later, in Rahmani et al. (2009) and Egerstedt et al. (2012), necessary and sufficient conditions are given in terms of partitions of the Laplacian graph. In Parlangeli and Notarstefano (2012), the controllability is explored for paths and cycles, and later extended by the same authors to the controllability of grid graphs by means of reductions and symmetries of the graph (Notarstefano and Parlangeli, 2013), and considering dynamics that are scaled Laplacians. In Kibangou and Commault (2014) and Zhang et al. (2011), the controllability is studied for strongly regular graphs and distance-regular graphs. Recently, new insights on the controllability of Laplacian dynamics are given regarding the uncontrollable subspace, in Aguilar and Gharesifard (2015) and Chapman and Mesbahi (2014). In addition, in Pasqualetti and Zampieri (2014) the controllability of isotropic and anisotropic networks is analyzed.

Furthermore, Aguilar and Gharesifard (2015) concludes by pointing out that further study of non-symmetric dynamics and controllability is required – which we address in the present paper. Therefore, we consider a much less restrictive assumption: A is a *simple* matrix, i.e., all of

its eigenvalues are distinct. Moreover, there are several applications where A satisfies this assumption, for instance, all dynamical systems modeled as random networks of the Erdős-Rényi type (Tao and Vu, 2017), as well as several known dynamical systems used as benchmarks in control systems engineering (Ogata, 2001; Siljak, 1991, 2007).

Observe that the MCP problem presents both continuous and discrete optimization properties, captured by the controllability property and the number of non-zero entries, respectively. To avoid the nature of this problem, in Olshevsky (2014), the non-zero entries of the input matrix were randomly generated. In the present paper, we ‘decouple’ the continuous and discrete optimization properties, and show that by first solving the discrete nature of the problem, it is always possible to deterministically obtain a solution to MCP in a second phase. Besides, the first step reduces the MCP to the set covering problem – well known to be NP-hard. Nonetheless, the set covering problem is one of the most studied NP-hard problems (probably second only to the SAT problem). Subsequently, although the set covering problem is NP-hard, some subclasses of the problem are equipped with sufficient structure that can be leveraged to invoke a polynomial algorithm that approximate the solution with ‘almost’ optimality guarantees (Brönnimann and Goodrich, 1995). This contrasts with the approach proposed in Olshevsky (2014), where an approximated solution particular to the MCP problem was provided. In addition, we study the rMCP which has not been previously addressed in the literature. Similarly to the MCP, we show that the rMCP can be polynomially reduced to the *set multi-covering problem*, i.e., a set covering problem that allows the same elements to be covered a predefined number of times. Furthermore, extensions of polynomial approximation algorithms are also available with similar optimality guarantees.

Alternatively, when the parameters of the LTI system are not exactly known, and assumed to be independent, structural systems theory (Dion et al., 2003) can be used to address the MCP and rMCP while ensuring *structural controllability*, see Pequito, Kar and Aguiar (2016) and Liu et al. (2015), respectively. Notwithstanding, the tools and conditions to ensure structural controllability are quite different from those adopted in this paper, and a solution to the MSCP is not necessarily a solution to the MCP when the dynamics’ matrix is simple (Pequito, Ramos, Kar, Aguiar and Ramos, 2016). ◦

Main Contributions of the present paper are as follows: (i) we characterize the exact solutions to the MCP; (ii) we show that for a given dynamics’ matrix almost all input vectors satisfying a specified structure are solutions to the MCP; (iii) we show that the rMCP is an NP-hard problem; (iv) we characterize the exact solutions to the rMCP; (v) we prove that the decision version of both MCPs are NP-complete; (vi) we provide approximated solutions to the rMCPs and discuss their

optimality guarantees; and, finally, in (vii) we discuss the limitations of the proposed methodology. \circ

The remainder of this paper is organized as follows. In Section 2, we formally state the rMCP addressed in this paper. Next, in Section 3, we review some concepts required to prove the main results of this paper. In Section 4, we present the main results of this paper, i.e., we characterize the solutions to the rMCP, its complexity, and a polynomial algorithm that approximates the solutions. Finally, in Section 5, we provide some examples that illustrate the main results of the paper and discuss the limitations of the proposed methodology.

Notation: We denote vectors by small font letters such as v, w, b and its corresponding entries by subscripts. A collection of vectors is denoted by $\{v^j\}_{j \in \mathcal{J}}$, where the superscript indicates an enumeration of the vectors using indices from a set such as $\mathcal{I}, \mathcal{J} \subset \mathbb{N}$. The number of elements of a set \mathcal{S} is denoted by $|\mathcal{S}|$. We denote by I_n the n -dimensional identity matrix. Given a matrix A , $\sigma(A)$ denotes the set of eigenvalues of A , the *spectrum* of A . Given two matrices $M_1 \in \mathbb{C}^{n \times m_1}$ and $M_2 \in \mathbb{C}^{n \times m_2}$, the matrix $[M_1 \ M_2]$ is the $n \times (m_1 + m_2)$ concatenated complex matrix. The structural pattern of a vector/matrix or a *structural vector/matrix* have their entries in $\{0, \star\}$, where \star denotes a non-zero entry, and they are denoted by a vector/matrix with a bar on top of it. We denote by A^\top the transpose of A . The function $\cdot : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ denotes the usual inner product in \mathbb{C}^n , i.e., $v \cdot w = v^\top w$, where v^\top denotes the adjoint of v (the conjugate of v^\top). With some abuse of notation, $\cdot : \{0, \star\}^n \times \{0, \star\}^n \rightarrow \{0, \star\}$ also denotes the map where $\bar{v} \cdot \bar{w} \neq 0$, with $\bar{v}, \bar{w} \in \{0, \star\}^n$ if and only if there exists $i \in \{1, \dots, n\}$ such that $\bar{v}_i = \bar{w}_i = \star$. Additionally, $\|v\|_0$ denotes the number of non-zero entries of the vector v in either $\{0, \star\}^n$ or \mathbb{R}^n . Given a subspace $\mathcal{H} \subset \mathbb{C}^n$ we denote by \mathcal{H}^c its complement with respect to \mathbb{C} , i.e., $\mathcal{H}^c = \mathbb{C}^n \setminus \mathcal{H}$. With abuse of notation, we will use inequalities involving structural vectors as well – for instance, we say $\bar{v} \geq \bar{w}$ for two structural vectors \bar{v} and \bar{w} if and only if the following two conditions hold: (i) if $\bar{w}_i = 0$, then $\bar{v}_i \in \{0, \star\}$, and (ii) if $\bar{w}_i = \star$ then $\bar{v}_i = \star$.

2 Problems Statement

Under the adverse scenarios of failure or malicious temper of the actuators, the dynamics of the system can be modeled by

$$x(k+1) = Ax(k) + B_{\mathcal{M} \setminus \mathcal{A}} u(k), \quad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state of the system, $u(k) \in \mathbb{R}^p$ is the input signal exerted by the actuators, and $k \in \mathbb{N}$ denotes the time instance. The matrix $A \in \mathbb{R}^{n \times n}$, which is referred to as the system dynamics' matrix, describes the coupling between state variables. In addition, $B_{\mathcal{M} \setminus \mathcal{A}}$ consists of the subset of columns with indices in $\mathcal{M} \setminus \mathcal{A}$, the set $\mathcal{M} = \{1, \dots, p\}$ is the set of inputs' labeling indices and \mathcal{A} the set of indices of malfunctioning actuators. Therefore, an extra set of actuators should be in place to ensure that it is still possible to control the sys-

tem if some inputs fail. By identifying the system (1) with the pair $(A, B_{\mathcal{M} \setminus \mathcal{A}})$, we aim to ensure that this pair is controllable, so the rMCP can be posed as follows.

\mathcal{P} : Given a dynamics' matrix $A \in \mathbb{R}^{n \times n}$ and the number of possible input failures s , determine the matrix $B^* \in \mathbb{R}^{n \times (s+1)n}$ such that

$$B^* = \arg \min_{B \in \mathbb{R}^{n \times (s+1)n}} \|B\|_0 \quad (2)$$

s.t. $(A, B_{\mathcal{M} \setminus \mathcal{A}})$ is controllable,
 $|\mathcal{A}| \leq s, \mathcal{A} \subset \mathcal{M}$,

where $\mathcal{M} \subset \{1, \dots, n\}$ are the indices of the non-zero columns of the matrix B . Notice that the dimension of B is $n \times (s+1)n$, to ensure that a solution always exist. In particular, in the worst case scenario the matrix B that concatenates s times the identity matrix is a feasible solution. In practice, only the non-zero columns of B matter, which we refer to as *effective inputs*. Notice that when $s = 0$, we recover the MCP problem, so we first provide the solution to the MCP, which we later extend to characterize the solution to the rMCP.

The main assumptions in this paper are as follows:

Assumption 1: The dynamics' matrix is *simple*, i.e., all the eigenvalues of A are distinct. \circ

Observe that Assumption 1 is not very restrictive since there are several applications where A satisfy this assumption. For example, dynamical systems modeled as random networks of the Erdős-Rényi type (Tao and Vu, 2017), as well as known dynamical systems used as benchmarks in control systems engineering (Ogata, 2001; Siljak, 1991, 2007).

Assumption 2: A *left-eigenbasis* of A is available, i.e., the eigenbasis consisting of left-eigenvectors of A . \circ

The second assumption is a technical requirement, since an eigenbasis is determined using numerical methods. Therefore, in practice, it may be composed of approximated eigenvectors up to a given floating-point error – see Section 4.1 for further discussion.

3 Preliminaries and Terminology

In this section, we introduce some basic concepts of computational complexity required to characterize the rMCP using the following NP-hard problem.

Definition 1 ((Chekuri et al., 2012)) (*Minimum Set Multi-covering Problem*) Given a set of m elements $\mathcal{U} = \{1, 2, \dots, m\}$ referred to as universe, a collection of n sets $\mathcal{S} = \{\mathcal{S}_1, \dots, \mathcal{S}_n\}$, with $\mathcal{S}_j \subset \mathcal{U}$, with $j \in \{1, \dots, n\}$, $\bigcup_{j=1}^n \mathcal{S}_j = \mathcal{U}$, and a demand function $d : \mathcal{U} \rightarrow \mathbb{N}$ that indicates the number of times an element i needs to be covered. In other words, $d(i)$ is the minimum number of sets in \mathcal{S} that need to be consider such that i is member of all of this sets. The minimum set multi-covering problem consists of finding a set of indices $\mathcal{J}^* \subseteq \{1, 2, \dots, n\}$ corresponding to the minimum number of sets covering \mathcal{U} , where every element $i \in \mathcal{U}$ is covered at least $d(i)$ times, i.e.,

$$\mathcal{J}^* = \arg \min_{\mathcal{J} \subseteq \{1, 2, \dots, n\}} |\mathcal{J}|$$

$$\text{s.t. } |\{j \in \mathcal{J} : i \in \mathcal{S}_j\}| \geq d(i).$$

In particular, if $d(i) = 1$ for all $i \in \{1, \dots, n\}$, then we obtain the minimum set covering problem. \diamond

The minimum set multi-covering problem plays a double role in this paper: (i) we reduce the rMCP to a minimum set multi-covering problem; and (ii) by *polynomially reducing* (Garey and Johnson, 1979) it to the rMCP, we show the latter to be NP-hard. Such reduction is useful to determine the qualitative complexity class a particular problem belongs to, see (Garey and Johnson, 1979) for an introduction to the topic.

4 Robust Minimum Controllability Problem

In this section, we propound the main results of this paper. First, notice that when there are no input failures (i.e., $s = 0$) in the rMCP, we recover the MCP problem. Therefore, we first provide the solution to the MCP, which we later extend to provide the characterization of the solution to the rMCP.

To obtain the solution to the MCP, we perform the following two steps: (i) we polynomially reduce the structural optimization problem in (3) to a set-covering problem using Algorithm 1, and (ii) we determine a numerical parametrization of an input matrix with a specific input structure in a deterministic polynomial fashion, by solving (4). Simply speaking, by performing these two steps, we are ‘decoupling’ the discrete and continuous properties of the MCP without losing optimality. In fact, in Theorem 1, we provide a generic characterization of the solutions to the MCP, and a particular instance can be found using Theorem 2.

Next, we design a similar procedure to that used to solve MCP to obtain the solution to the rMCP, which we show to be NP-hard (Theorem 4). Specifically, we determine the sparsity of an input matrix, by polynomially reducing the problem to a minimum set multi-covering problem (see Theorem 5), which is later used to characterize the solutions to the rMCP (Theorem 6).

Complementary to the solutions to the MCPs, in what follows, we show that (under Assumption 1) the decision versions of the rMCP is NP-complete (Theorem 7). Subsequently, we provide a polynomial approximation algorithm (see Algorithm 2), which solution is feasible (see Theorem 8) and has sub-optimality guarantees (see Theorem 9). Finally, in Section 4.1, we explore numerical implications of waiving Assumption 2.

Let us start by considering the MCP and only one input, i.e., instead of an input matrix B , we only consider an input vector b . The first set of results provides necessary conditions on the structure that an input vector b must satisfy to ensure that (A, b) is controllable, and a polynomial complexity procedure (Algorithm 1) that reduces the problem of obtaining such necessary structural patterns to a minimum set covering problem.

Lemma 1 *Given a collection of non-zero vectors $\{\bar{v}^j\}_{j \in \mathcal{J}}$ with $\bar{v}^j \in \{0, \star\}^n$, the procedure of finding $\bar{b}^* \in \{0, \star\}^n$ such that*

$$\bar{b}^* = \arg \min_{\bar{b} \in \{0, \star\}^n} \|\bar{b}\|_0 \quad (3)$$

$$\text{s.t. } \bar{v}^j \cdot \bar{b} \neq 0, \text{ for all } j \in \mathcal{J}$$

is polynomially (in $|\mathcal{J}|$ and n) reducible to a minimum set covering problem with universe \mathcal{U} and a collection \mathcal{S} of sets by applying Algorithm 1. \diamond

Algorithm 1 Polynomial reduction of the structural optimization problem (3) to a set-covering problem

Input: $\{\bar{v}^j\}_{j \in \mathcal{J}}$, a collection of $|\mathcal{J}|$ vectors in $\{0, \star\}^n$.

Output: $\mathcal{S} = \{\mathcal{S}_i\}_{i \in \{1, \dots, n\}}$ and \mathcal{U} , a set of n sets and the universe of the sets, respectively.

Step 1. set $\mathcal{S}_i = \{\}$ for $i = 1, \dots, n$

Step 2. for $j = 1, \dots, |\mathcal{J}|$

 for $i = 1, \dots, n$

 if $\bar{v}_i^j \neq 0$ then

$\mathcal{S}_i = \mathcal{S}_i \cup \{j\}$;

 end if

 end for

end for

Step 3. set $\mathcal{S} = \{\mathcal{S}_1, \dots, \mathcal{S}_n\}$ and $\mathcal{U} = \bigcup_{i=1}^n \mathcal{S}_i$.

Next, we show that given the structure obtained in Lemma 1, almost all possible real numerical realizations lead to a vector $b \in \mathbb{R}^n$ that is a solution to the MCP.

Theorem 1 *Let $\{v^i\}_{i \in \mathcal{J}}$ to be the set of left-eigenvectors of A , and \bar{b} a solution to (3). Then, almost all numerical realizations b of \bar{b} are solutions to the MCP. \diamond*

Remark 1 *The generic properties that characterize structural controllability (Dion et al., 2003) imply that almost all parameters of both dynamics and input matrices satisfying a given structural pattern are controllable. Although, in Theorem 1 the dynamics’ simple matrix A is fixed, i.e., a numerical instance with specified structure, density arguments are provided to the numerical realizations of the input vector with certain structure that ensure controllability of the system. \diamond*

Although Theorem 1 ensures that almost all parameterizations provide a feasible solution to the MCP, we need to determine one parameterization that guarantees controllability, which can be determined by solving the following optimization problem.

$$B^* = \arg \min_{B \in \mathbb{R}^{n \times m}} 0$$

$$\text{s.t. } B_{l,k} = 0 \text{ if } \bar{B}_{l,k} = 0, \quad l, k = 1, \dots, n. \quad (4)$$

Remark 2 *In fact, suppose the objective function in the optimization problem (4) is given by $f(B)$. Then, this can be chosen to satisfy additional design constraints. For instance, $f(B) = c^T B \mathbf{1}$, where c could capture an actuation cost, i.e., entry c_i captures how desirable is to actuate x_i , and $\mathbf{1}$ is a vector of ones with appropriate dimensions. Subsequently, one may need additional constraints such that the total actuation budget r available is bounded, for instance, $|f(B)| \leq r$ and $B_{i,j} \geq 0$ to avoid negative entries that will restrain the objective*

goal. Alternatively, $f(B)$ can also be considered nonlinear, while capturing control-theoretic properties; in particular, it can be a function of the controllability Gramian (Pasqualetti et al., 2014), with some appropriate constraints to ensure the problem to be well defined. \diamond

Next, we show that the (sparsest) pattern given by Lemma 1 with the optimization problem (4) leads to a numerical realization that is a solution to the MCP.

Lemma 2 Given $\{v^i\}_{i \in \mathcal{J}}$ with $v^i \in \mathbb{C}^n$, the procedure of finding $b^* \in \mathbb{R}^n$ that is a solution to

$$b^* = \arg \min_{b \in \mathbb{R}^n} \|b\|_0 \quad (5)$$

s.t. $v^i \cdot b \neq 0$, for all $i \in \mathcal{J}$,

is polynomially (in $|\mathcal{J}|$ and n) reducible (by Algorithm 1) to a minimum set covering problem, with numerical entries determined using the optimization problem (4). \diamond

Now, we state one of the main results of the paper.

Theorem 2 The solution to the MCP can be determined by first identifying the sparsity of the input vector as in Lemma 1, followed by determining the numerical realization of the non-zero entries as in Lemma 2. \diamond

Next, based on the previous solution to the MCP, we extend the result to find a dedicated solution to the MCP.

Theorem 3 Let $b \in \mathbb{R}^n$ be a solution to the MCP as described in Theorem 2, \bar{b} its sparsity and $\mathcal{N} \subset \{1, \dots, n\}$ the indices where \bar{b} is non-zero, i.e., $\mathcal{N} = \{i : \bar{b}_i = \star, \text{ and } i = 1, \dots, n\}$. If $\bar{B} \in \{0, \star\}^{n \times n}$ has exactly one non-zero entry in the i -th row, where $i \in \mathcal{N}$, then the output $B \in \mathbb{R}^{n \times n}$ of (4), when \bar{B} and the left-eigenbasis of A are considered, is a solution to the MCP. \diamond

Before characterizing the solutions to the rMCP, we notice that this problem is computationally challenging. Specifically, we obtain the following result which follows from noticing that a particular instance of the rMCP is the MCP (an NP-hard problem).

Theorem 4 The rMCP is NP-hard. \diamond

Therefore, without incurring in additional computational complexity and similar to the reduction proposed from MCP to the set covering problem, we can characterize the dedicated solutions to the rMCP as follows.

Theorem 5 Let v^1, \dots, v^n be a left-eigenbasis of A , and s the number of possible input failures. Further, consider the set multi-covering problem $(\{\mathcal{S}_1, \dots, \mathcal{S}_{(s+1)n}\}, \mathcal{U} \equiv \{1, \dots, n\}; d)$, where the demand is $d(i) = s+1$ for $i \in \mathcal{U}$, and $\mathcal{S}_k = \{j : [v^j]_l \neq 0, \text{ and } l-1 = k \pmod n\}$ for $k \in \mathcal{K} \equiv \{1, \dots, (s+1)n\}$. Then, the following statements are equivalent:

- (i) \mathcal{M}^* is a solution to the set multi-covering problem $(\{\mathcal{S}_1, \dots, \mathcal{S}_{(s+1)n}\}, \mathcal{U} \equiv \{1, \dots, n\}; d)$;
- (ii) $B_n(\mathcal{M}^*)$ is a dedicated solution to rMCP, where $[B_n(\mathcal{M}^*)]_{i,l} = 1$ for $l = i \pmod n$ and $i \in \mathcal{M}^* \subset \mathcal{K}$, and zero otherwise. \diamond

Remark 3 A matrix $B_n(\mathcal{M}')$ described by the concatenation of $(s+1)$ solutions to the MCP achieves feasibility

to the rMCP, but it is not necessarily an optimal solution – see Section 5 for a counterexample. \diamond

Next, we characterize the solutions of the rMCP, i.e., not only the ones that are dedicated. Towards this goal, we introduce the following *merging* procedure. Let two distinct effective inputs i and j , associated with two non-zero columns of the input matrix, b^i and b^j , be such that they do not share non-zero entries k , i.e., $[b^i]_k \neq [b^j]_k$ for $k \in \{1, \dots, n\}$. These two inputs are said to be merged into one input $b^{i'}$, where $[b^{i'}]_k = [b^i]_k$ when $[b^i]_k \neq 0$, and $[b^{i'}]_k = [b^j]_k$ when $[b^j]_k \neq 0$, for $k \in \{1, \dots, n\}$. Further, we implicitly assume that $b^{i'}$ takes the place of b^i , and b^j is set to zero. In other words, the effective input i is associated with $b^{i'}$ and the effective input j is discarded.

Theorem 6 Let $B_n(\mathcal{M}^*) \in \mathbb{R}^{n \times (s+1)n}$ be a dedicated solution to the rMCP as described in Theorem 5. In addition, let $\bar{B} \in \{0, \star\}^{n \times (s+1)n}$ be the sparsity of the matrix resulting of the merging procedure between any of the effective inputs in $B_n(\mathcal{M}^*)$. Then, the matrix $B \in \mathbb{R}^{n \times n}$ obtained using the optimization problem (4), with \bar{B} and the left-eigenbasis of A , is a solution to the rMCP. \diamond

Although we reduced the rMCP to a set multi-covering problem, it is interesting to notice that these are ‘equivalent’ in the sense that the decision version of the rMCP is NP-complete.

Theorem 7 The MCP and rMCP are NP-complete. \diamond Therefore, from Theorem 7, we have the following observation.

Remark 4 A solution of the MCP almost always coincides with a numerical realization of a solution to the associated minimal structural controllability. Combining this with the fact that the MCP is NP-complete when the eigenvalues of A are simple (see Theorem 7), it follows that the set of simple dynamics’ matrices that lead to NP-complete problems has zero Lebesgue measure. \diamond

Also, we notice that if a problem is NP-hard, then it does not mean that all instances are not polynomially solvable; notwithstanding, these can be solved exactly (Hua et al., 2009, 2010).

Remark 5 The NP-completeness, stated in Theorem 7, allows us to consider the subclasses of the set multi-covering problem that are known to be polynomially solvable, to identify polynomially solvable subclasses of the rMCP. This enables a new characterization of solutions to the question posed in Aguilar and Gharesifard (2015), regarding the existence of polynomial algorithms to determine controllable graph structures. \diamond

Additionally, by the construction proposed in Theorem 5 and the result in Theorem 7, if the set multi-covering problem obtained possess additional structure, then this can be leveraged to use polynomial algorithms to approximate the solutions with close-to-optimal solutions (see Algorithm 2).

Furthermore, Algorithm 2 leverages the submodularity properties (Bach, 2011) of the set multi-covering prop-

erties to obtain a dedicated solution to the rMCP. Submodularity properties ensure that the associated polynomial greedy algorithms have sub-optimality guarantees while performing well in practice (Bach, 2011). Subsequently, we can obtain the following result.

Algorithm 2 Approximate Solution to the rMCP

Input: Left-eigenbasis v^1, \dots, v^n associated with $A \in \mathbb{R}^{n \times n}$ and the number s of possible input failures.

Output: Dedicated solution $B_n(\mathcal{M}') \in \mathbb{R}^{n \times (s+1)n}$.

Step 1. Let $\mathcal{S}_1, \dots, \mathcal{S}_{(s+1)n}$, where $\mathcal{S}_k = \{j : [v^j]_l \neq 0, \text{ and } l-1 = k \bmod n\}$ for $k \in \mathcal{K} \equiv \{1, \dots, n(s+1)\}$.

Step 2. set $\mathcal{U}^i = \emptyset$, with $i = 1, \dots, s$ denote the indices in \mathcal{U} that are covered i times and the indices of the sets covering them, respectively.

Step 3. set $\mathcal{J} = \emptyset$

Step 4. for $i = 1, \dots, s+1$

set $\mathcal{U}^i = \{k : |\{k \in \mathcal{U} : k \in \mathcal{S}_j, j \in \mathcal{J}\}| \geq i\}$ denote the indices that are already covered by at least i sets

Step 5. while $\mathcal{U}^i \neq \mathcal{U}$

select \mathcal{S}_j with largest number of indices in $\mathcal{U} \setminus \mathcal{U}^i$

set $\mathcal{J} \leftarrow \mathcal{J} \cup \{j\}$

set $\mathcal{U}^i \leftarrow \mathcal{U}^i \cup \mathcal{S}_j$

end while; end for

set $\mathcal{M}' \leftarrow \mathcal{J}$

Step 6. set $B_n(\mathcal{M}')$, where $[B_n(\mathcal{M}')]_{i,l} = 1$ for

$l = i \bmod n$ and $i \in \mathcal{M}' \subset \mathcal{K}$, and zero otherwise.

Theorem 8 *The matrix $B_n(\mathcal{M}')$ obtained using Algorithm 2, with \bar{B} and the left-eigenbasis of A , is a feasible solution to the rMCP. Further, the computational complexity of Algorithm 2 is $\mathcal{O}(sn)$, and it ensures an approximation optimality bound of $\mathcal{O}(\log n)$. \diamond*

Finally, by invoking Theorem 6 and Theorem 8, we obtain the following result.

Theorem 9 *Let $B_n(\mathcal{M}') \in \mathbb{R}^{n \times (s+1)n}$ be a dedicated solution to the rMCP as described in Theorem 8. In addition, let $\bar{B} \in \{0, \star\}^{n \times (s+1)n}$ be the sparsity of the matrix resulting of the merging procedure between any of the effective inputs in $B_n(\mathcal{M}')$. Then, the matrix $B \in \mathbb{R}^{n \times n}$ obtained using the optimization problem (4), with \bar{B} and the left-eigenbasis of A , achieves feasibility to the rMCP and is computed in polynomial time. \diamond*

4.1 Numerical and Computational Remarks

Now, for the sake of completeness, we discuss the implications of waiving Assumption 2 and the impact on the input vector in the MCP. The results readily extend to the general solution to the rMCP. Towards this goal, we need the following result.

Theorem 10 ((Pan and Chen, 1999)) *Let $A \in \mathbb{C}^{n \times n}$ be a matrix with simple eigenvalues. The deterministic arithmetic complexity of finding the eigenvalues and the eigenvectors of A is bounded by $\mathcal{O}(n^3) + t(n, m)$ operations, where $t(n, m) = \mathcal{O}((n \log^2 n)(\log m + \log^2 n))$, for a required upper bound of $2^{-m}\|A\|$ on the absolute output error of the approximation of the eigenvalues and eigenvectors of A and for any fixed matrix norm $\|\cdot\|$. \diamond*

More precisely, Theorem 10 states that in practice, only a numerical approximation of the left-eigenbasis is pos-

sible in polynomial time. In this case, let $\varepsilon = 2^{-m}\|A\|$ be as in Theorem 10, then the results stated in Lemma 1 and Lemma 2 (see also Algorithm 1 and the optimization problem (4)) can only be used in an ε -approximation of the left-eigenbasis of the dynamics' matrix. Therefore, the ε -approximation of the left-eigenbasis may lead to the following issues:

(i) an entry in the left-eigenvector is considered as zero, where in fact it can be some non-zero value that (in norm) is smaller than ε . Consequently, the sets generated using Algorithm 1 (see also Lemma 1) do not contain the indices associated with those non-zero entries. Thus, additional sets need to be considered to the minimum set covering, which implies that the structure of the input vector may contain more non-zero entries than the sparsest input vector that is a solution to the MCP. In other words, we obtain an over-approximation of the sparsest input vector that is a solution to the MCP.

(ii) an entry of the ε -approximation in a left-eigenvector of the left-eigenbasis is non-zero. Then, it does not represent an issue when computing the structure of the input vector as described in Lemma 1 (see also Algorithm 1), but it can represent a problem when determining the numerical realization by resorting to the optimization problem (4). Nonetheless, by Theorem 1 it follows that such issue is unlikely to occur.

To undertake a deeper understanding of which entries fall in the first issue presented above, several methods to compute eigenvectors can be used and solutions posteriorly compared, see Demmel et al. (2000) for a survey on different methods and computational issues associated with those.

5 Illustrative examples

To illustrate the first main result of this paper, to find a solution to the MCP, consider the dynamics' matrix A

$$A = \begin{bmatrix} 6 & -3 & 3 & 2 & -1 \\ 0 & 8 & 0 & 0 & 0 \\ 4 & 3 & 7 & 2 & 1 \\ 0 & 0 & 0 & 6 & 0 \\ -4 & -3 & -3 & -2 & 3 \end{bmatrix}, \quad (6)$$

where $\sigma(A) = \{2, 4, 6, 8, 10\}$ consists of distinct eigenvalues, so the matrix A is simple and our results are applicable. Consequently, to obtain the solution to the MCP, we first compute the left-eigenvectors of A that are as follows: $v^1 = [11001]^T$, $v^2 = [00101]^T$, $v^3 = [00010]^T$, $v^4 = [01000]^T$ and $v^5 = [10110]^T$. Using Algorithm 1, since \bar{v}_i for $i = 1, \dots, 5$, we obtain $\{\mathcal{S}_j\}_{j=1, \dots, 5}$, where the j -th set corresponds to the set of indices of the left-eigenvector which have a non-zero entry on the j -th position. In particular, we obtain $\mathcal{S}_1 = \{1, 5\}$, $\mathcal{S}_2 = \{1, 4\}$, $\mathcal{S}_3 = \{2, 5\}$, $\mathcal{S}_4 = \{3, 5\}$, $\mathcal{S}_5 = \{1, 2\}$, and the universe set is given by $\mathcal{U} = \{1, 2, 3, 4, 5\}$. Now, it is easy to see that a solution to this minimum set covering problem is the set of indices $\mathcal{I}^* = \{2, 3, 4\}$, since $\mathcal{U} = \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$ and there is no pair of sets, i.e., $\mathcal{I}' = \{i, i'\}$ with $i, i' \in \{1, \dots, 5\}$ such that $\mathcal{U} = \mathcal{S}_i \cup \mathcal{S}_{i'}$. Therefore, a possible structure of the vector b that is a solution to the MCP is

$$\bar{b} = [0 \star \star \star 0]^T. \quad (7)$$

Additionally, to find the numerical parametrization of b , under the sparsity pattern of \bar{b} , we have to solve the following system with three unknowns: $b_2, b_3, b_4 \neq 0$ and $b_3 + b_4 \neq 0$. By inspection, a possible choice is $b = [0 \ 1 \ 1 \ 1 \ 0]^\top$, but the numerical parametrization can be obtained by invoking the optimization problem (4), with the set of left-eigenvectors of A given by $\{v^j\}_{j \in \{1, \dots, 5\}}$ and the structure of b given by \bar{b} in (7). For the sake of completeness, we, the controllability matrix is given by

$$C = [b \ Ab \ A^2b \ A^3b \ A^4b] = \begin{bmatrix} 0 & 2 & 44 & 608 & 7184 \\ 1 & 8 & 64 & 512 & 4096 \\ 1 & 12 & 120 & 1176 & 11520 \\ 1 & 6 & 36 & 216 & 1296 \\ 0 & -8 & -104 & -1112 & -11264 \end{bmatrix},$$

and the $\text{rank}(C) = 5$, implying that (A, b) is controllable.

Observe that the single-input solution obtained with $b = [0 \ 1 \ 1 \ 1 \ 0]^\top$, can be immediately translated into a solution with two effective inputs, by Theorem 3. In particular, two possible solutions are $B = [b^1 \ b^2]$ with $b^1 = [0 \ 1 \ 1 \ 0 \ 0]^\top$ and $b^2 = [0 \ 0 \ 0 \ 1 \ 0]^\top$, and $B = [b^1 \ b^2 \ b^3]$ with $b^1 = [0 \ 1 \ 0 \ 0 \ 0]^\top$, $b^2 = [0 \ 0 \ 1 \ 0 \ 0]^\top$ and $b^3 = [0 \ 0 \ 0 \ 1 \ 0]^\top$, where the latter is a dedicated solution. Alternatively, if we consider, for instance, $B = [b^1 \ b^2]$ with $b^1 = [0 \ 1 \ 0 \ 0 \ 0]^\top$ and $b^2 = [0 \ 0 \ -1 \ 1 \ 0]^\top$, then $v^\top B = 0$ for the left-eigenvector $v = [1 \ 0 \ 1 \ 1 \ 0]^\top$, and the pair (A, B) is uncontrollable. Thus, as prescribed in Theorem 3, by the optimization problem (4), one can obtain a new realization of B that ensures controllability of (A, B) ; e.g., the same b^1 , and $b^2 = [0 \ 0 \ \frac{10}{3} \ 1 \ 0]^\top$.

Notice that a systematic polynomial approximation to the MCP can be obtained by considering the rMCP with the number of input failures $s = 0$. By doing so, we obtain the same sparsity to b , i.e., \bar{b} , as in the aforementioned example, and the subsequent analysis follows. We also observe that the approximate solution is a solution to the MCP.

Now, we illustrate how to find a solution to \mathcal{P} . Let us apply the developments of Section 4, when we consider the dynamics' matrix in (6). First, if we consider that at most one input fails, we use Algorithm 1, where a set multi-covering problem is considered with the sets as in Section 4, universe $\mathcal{U} = \{1, \dots, 5\}$ and with a demand function $d(i) = 2$ for $i = 1, \dots, 5$, i.e., each element must be covered twice. Subsequently, by inspection, we conclude that the sets \mathcal{S}_2 and \mathcal{S}_4 need to be considered twice, since the elements 5 and 4 only appear in these sets, respectively. After this, we need to cover the element 2 and to this end we can choose \mathcal{S}_3 or \mathcal{S}_5 or twice one of them, so a possible solution to the multi-set covering problem is $\mathcal{M}^* = \{2, 3, 4, 2, 3, 4\}$. Therefore, $B_n(\mathcal{M}^*)$ is a solution to the rMCP, and, in particular, the solution is the same as concatenating twice a dedicated solution to the MCP, see Remark 3. Further, Algorithm 2 produces an optimal solution as often occurs in practice.

In fact, if we apply our results when s inputs are allowed to fail, i.e., $d(i) = s + 1$ for $i = 1, \dots, 5$, we notice that the sets \mathcal{S}_2 and \mathcal{S}_4 need to be considered $s + 1$ times since the elements 5 and 4 only appear in these sets, respectively. Besides, we need to cover the element 2, so

we can choose either \mathcal{S}_3 or \mathcal{S}_5 $s + 1$ times, which implies that $B(\mathcal{M}^*)$, with $\mathcal{M}^* = \{2, 3, 4, \dots, 2, 3, 4\}$ where the elements 2, 3 and 4 appear $s + 1$ times, is a solution. Similarly, the solution consists of concatenating $s + 1$ times a dedicated solution to the MCP, and the same remarks are applicable, i.e., Remark 3.

However, the concatenation of $s + 1$ solutions to the MCP is not always a solution to the rMCP when at most s inputs are allowed to fail. Let us consider the dynamics' matrix and associated left-eigenvectors as follows:

$$A = \begin{bmatrix} 4 & -2 & 2 \\ -1 & 3 & 1 \\ 1 & -1 & 5 \end{bmatrix}; V = \begin{bmatrix} | & | & | \\ v^1 & v^2 & v^3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}. \quad (8)$$

First, we note that $\sigma(A) = \{2, 4, 6\}$, so A is simple, and we can apply our results. Secondly, the structure of the left-eigenvectors of A is given by $\bar{v}^1 = [\star \ \star \ 0]^\top$, $\bar{v}^2 = [0 \ \star \ \star]^\top$ and $\bar{v}^3 = [\star \ 0 \ \star]^\top$. Further, we consider that at most one input failure is likely to occur, i.e., $s = 1$. Then, we can invoke Algorithm 1 to build the sets for the set multi-covering problem, which are as follows: $\mathcal{S} = \{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3\}$, with $\mathcal{S}_1 = \{1, 2\}$, $\mathcal{S}_2 = \{2, 3\}$ and $\mathcal{S}_3 = \{1, 3\}$, and $\mathcal{U} = \bigcup_{i=1}^3 \mathcal{S}_i = \{1, 2, 3\}$. By inspection, we obtain that $\mathcal{M}' = \{1, 2, 3\}$ is the optimal solution, where the indices cover each element of \mathcal{U} twice. Further, observe that a solution to the dedicated input MCP always has size equal to two, and in this case, the concatenation of two solutions lead to a solution that has one more input than the optimal solution obtained. Observe that this is a small dimensional example that incurs into a solution that is already 33% worst than the optimal. Alternatively, if we apply Algorithm 2 to approximate the solution to the rMCP, we obtain one that is optimal, i.e., $B(\mathcal{M}')$ where $\mathcal{M}' = \{1, 2, 3\}$.

6 Conclusions and Further Research

In this paper, we addressed two minimal controllability problems, with the goal of characterizing the input configurations that actuate the minimal subset of variables yielding controllability, under a specified number of failures. The problems explored were shown to be NP-complete, and a polynomial reduction of these to a set multi-covering problem was provided. In particular, the strategies followed by us separate the discrete and continuous nature of the minimal controllability problems. Subsequently, we discussed greedy solutions to the minimal controllability problems that yields feasible (but sub-optimal) solutions to rMCP.

Directions for future research in this line of work include the use of the obtained inputs' structure and consider methods such as coordinate gradient descent to minimize an energy cost, and to consider the case where the model is not exactly known. Additionally, it would be interesting to assess the computational complexity of the rMCP without the assumption on the spectrum of the dynamics' matrix, as well as to provide polynomial algorithms to obtain approximated solutions with suboptimal guarantees.

Appendix

Proof of Lemma 1: Consider the sets \mathcal{S} and \mathcal{U} obtained in Algorithm 1. The following equivalences hold: let $\mathcal{I} \subset \{1, \dots, n\}$ be a set of indices and $\bar{b}_{\mathcal{I}}$ the structural vector whose i -th component is non-zero if and only if $i \in \mathcal{I}$. Then, the collection of sets $\{\mathcal{S}_i\}_{i \in \mathcal{I}}$ in \mathcal{S} covers \mathcal{U} if and only if $\forall j \in \mathcal{J}, \exists k \in \mathcal{I}$ such that $j \in \mathcal{S}_k$, which is the same as $\forall j \in \mathcal{J}, \exists k \in \mathcal{I}$ such that $\bar{v}_k^j \neq 0$ and $\bar{b}_k \neq 0$, this can be rewritten as $\forall j \in \mathcal{J}, \exists k \in \mathcal{I}$ such that $\bar{v}_k^j \bar{b}_k \neq 0$ and therefore $\forall j \in \mathcal{J} \quad \bar{v}^j \cdot \bar{b} \neq 0$. In summary, $\bar{b}_{\mathcal{I}}$ is a feasible solution to the problem in (3). In addition, it can be seen that by such reduction, the optimal solution \bar{b}^* of (3) corresponds to the structural vector $\bar{b}_{\mathcal{I}^*}$, where $\{\mathcal{S}_i\}_{i \in \mathcal{I}^*}$ is the minimal collection of sets that cover \mathcal{U} , i.e., \mathcal{I}^* solves the minimum set covering problem associated with \mathcal{S} and \mathcal{U} . Hence, the result follows by observing that Algorithm 1 has polynomial complexity, namely $\mathcal{O}(\max\{|\mathcal{J}|, n\}^3)$. ■

Proof of Theorem 1: The proof follows by showing that if $\{v^i\}_{i \in \mathcal{J}}$ with countable \mathcal{J} such that $v^i \neq 0$ for all $i \in \mathcal{J}$ and \bar{b} a solution to (3), then the set $\Omega = \{b \in \mathbb{R}^n : v^i \cdot b = 0 \text{ for some } i \in \mathcal{J}, \text{ and } b \text{ is a numerical instance of } \bar{b}\}$ has zero Lebesgue measure. The proof follows similar steps to those proposed in Wonham (1985), but due to the additional sparsity constraint we devise an independent proof. Let $\{v^i\}_{i \in \mathcal{J}}$, with countable \mathcal{J} , be given and let \bar{b} be a solution to problem (5). For $b \in \mathbb{R}^n$, the equation $v^i \cdot b = 0$ represents a hyperplane $\mathcal{H}^i \subset \mathbb{C}^n$ (provided $v^i \neq 0$ for all i), thus the equation $v^i \cdot b \neq 0$ defines the space $\mathbb{C}^n \setminus \mathcal{H}^i$. Therefore, the set of b that satisfies $v^i \cdot b \neq 0$ for all $i \in \mathcal{J}$, is given by $\bigcap_{i \in \mathcal{J}} (\mathbb{C}^n \setminus \mathcal{H}^i) = \mathbb{C}^n \setminus \left(\bigcup_{i \in \mathcal{J}} \mathcal{H}^i \right)$ and the set Ω of values which does not verify the equations is the complement, i.e., $\left(\mathbb{C}^n \setminus \bigcup_{i \in \mathcal{J}} \mathcal{H}^i \right)^c = \bigcup_{i \in \mathcal{J}} \mathcal{H}^i$, which is a set with zero Lebesgue measure in \mathbb{C}^n , since $|\mathcal{J}|$ is countable.

Now, if $\{v^i\}_{i \in \mathcal{J}}$ is taken to be the set of left-eigenvectors of A and \bar{b} the corresponding solution to problem (5), each member of the set Ω constitutes a solution to (5) and hence the MCP. Since, by the preceding arguments, Ω has Lebesgue measure zero in \mathbb{C}^n , it follows readily that almost all numerical instances of \bar{b} are solutions to the MCP. ■

Proof of Lemma 2: By Lemma 1, given $\{\bar{v}^i\}_{i \in \mathcal{J}}$, problem (5) is polynomially (in $|\mathcal{J}|$ and n) reducible to a minimum set covering problem. Now, given a solution \bar{b} to (3), the optimization problem (4) can be used to obtain a numerical instantiation b with the same structure as \bar{b} such that $v^i \cdot b \neq 0$ for all $i \in \mathcal{J}$, which incurs polynomial complexity (in $|\mathcal{J}|$ and n). Furthermore, it is readily seen that any feasible solution b' to (5) satisfies $\|b'\|_0 \geq \|\bar{b}\|_0 = \|b\|_0$. Hence, b obtained by the above recipe is a solution to (5) and the desired assertion follows by observing that all steps in the construction have polynomial complexity (in $|\mathcal{J}|$ and n). ■

Proof of Theorem 2: The proof follows by invoking the PBH eigenvector test. The left-eigenbasis is available by Assumption 1, the problem in (5) is a restatement of the MCP. ■

Proof of Theorem 3: The feasibility of the solution is ensured by proceeding similarly to Theorem 1, when the left-eigenbasis of the dynamics' matrix is considered to invoke the PHB eigenvector criterion. The optimality follows similar steps to those presented in Lemma 2. ■

Proof of Theorem 5: First, we observe that, by construction of the sets $\{\mathcal{S}_1, \dots, \mathcal{S}_{(s+1)n}\}$ and the demand function $d(i)$, for $i \in \{1, \dots, n\}$, there exists always $s+1$ entries matching every non-zero entry of the vectors in a left-eigenbasis. This implies that if at most s sensors fail, at least one entry of a column c of B is such that for each left-eigenvector $v \cdot c \neq 0$, implying $v^i{}^\top B \neq 0$ for $i \in \{1, \dots, n\}$. Hence, the system is controllable by the PBH eigenvector test, and we have a feasible solution. Now we need to show that the solution is optimal, i.e., there is not another solution with less dedicated inputs to the rMCP. We will proceed by contradiction, so assume that there is a solution to a demand function $d(i) = w$ for $i \in \{1, \dots, n\}$ and some $w < s+1$. Then, for some entry of a left-eigenvector v it is only ensured the existence of w columns in B whose inner product is not zero. Therefore, if w dedicated inputs fails, i.e., the corresponding columns of B are now zero, then B is such that $v^\top B = 0$, for some eigenvector v . Thus, contradicting the assumption that there is a sparser solution to the rMCP. ■

Proof of Theorem 6: The proof follows similar steps to those presented in Theorem 3. In particular, recall the merging procedure, and the guarantees obtained in Theorem 5. ■

Proof of Theorem 7: From Olshevsky (2014), we have that the MCP is NP-hard, and, in particular, the minimum set covering problem can be polynomially reduced to it. Therefore, we just need to show that the MCP assuming that A comprises only simple eigenvalues and the left-eigenbasis is known, i.e., under our assumptions, can be reduced polynomially to the minimum set covering problem.

To this end, note that, given the set $\{v^i\}_{i \in \mathcal{J}}$ of left-eigenvectors of A , the MCP is equivalent to problem (5), the latter being polynomially (in $|\mathcal{J}|$ and n) reducible to the minimum set covering problem (see Lemma 2). Since $|\mathcal{J}| = n$, the overall reduction to the minimum set covering problem is polynomial in n .

Similar arguments hold for the rMCP. It was shown to be NP-hard, in Theorem 4, and a reduction to the minimum set multi-covering problem can be obtained by Theorem 5. ■

Proof of Theorem 8: Algorithm 2 terminates when each element of the universe set \mathcal{U} is covered $s+1$ times (steps 4-5) by the sets of the set multi-covering problem indexed by \mathcal{J} . In other words, it terminates when we obtain a solution to the set multi-covering problem. By designing $B_n(\mathcal{M}')$, with $\mathcal{M}' = \mathcal{J}$, we build a matrix that corresponds to dedicated inputs. Thus, using Theorem 5, since \mathcal{J} is a solution to the set multi-covering problem, then $B_n(\mathcal{M}')$ is a dedicated solution to the rMCP.

The computational complexity of Algorithm 2 is obtained by the overall complexity of steps 1, 4 and 5. In step 1, we need to compute $(s+1)n$ sets, in step 5 we need to consider at most n sets, and, in step 4, $(s+1)$ iterations are performed, each with the number of steps of step 5, yielding $(s+1)n$ computational steps. Summing up the complexity of each step, Algorithm 2 has, in the worst case, complexity of order $\mathcal{O}(sn)$. In addition, notice that the performance attained in a multi-set covering problem is the same as in the rMCP, as a consequence of Theorem 7. Furthermore, the solution obtained incurs in an optimality gap of at most $\mathcal{O}(\log n)$ since the algorithm implements the greedy algorithm associated with submodular functions, as it is the case of the multi-set covering problem, and the result follows. ■

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