A Framework for Actuator Placement in Large Scale Power Systems: Minimal Strong Structural Controllability

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***Abstract*—This paper addresses the problem of minimal placement of actuators in large-scale linear time invariant (LTI) systems, such as large-scale power systems, for dynamic controller design. A novel sufficient and necessary condition to ensure a strong structurally controllable (SSC) system is proposed. Specifically, the paper addresses the problem of obtaining the minimal number of *dedicated* inputs, i.e., inputs which actuate only a single state variable, and the respective state variables they should be assigned to, such that the LTI system is SSC. In addition, an efficient and scalable algorithm, with polynomial implementation complexity, to achieve such minimal placement of dedicated inputs is proposed. An illustration of the proposed design methodology is provided on the IEEE 5-bus test system, thereby identifying the minimal number of physical state variables to be actuated for ensuring strong structural controllability.**

I. Introduction

The last decade has witnessed the introduction of novel generation technologies in large-scale power systems. Of particular interests are the unconventional renewable generation resources, such as wind and solar farms, given their high volatility and low degree of controllability. To compensate for their low (or lack of) controllability, a fundamental challenge for the power industry is to determine the optimal placement, size and type of controllers. For example, large-scale batteries, cluster of electric vehicles, an array of flywheels, pumped hydro storage. Primarily deployed in large-scale power systems for frequency regulation, these storage devices require high initial investment. The objective is to ensure high efficacy of the controllers for frequency regulation. A formal mathematical framework is therefore required for their selection as well as placement. The classical approach towards modeling a large-scale power system involves a set of differential and algebraic equations [1], where the loads are modeled as constant impedances. However, to achieve real-time dynamic control, an improved modeling approach is essential to preserve the structural properties of the system. For example in [2], a structure preserving modeling approach is proposed where the aggregate loads are modeled as dynamic components. Nevertheless, the fundamental question regarding actuator placement in a large-scale power system remains unresolved, i.e., where the storage devices be placed to ensure system controllability. In general, finding such minimal placement to ensure controllability or to achieve pre-specified control performance is an NP-hard problem, see [3]. Alternative approaches that lead to efficient and scalable algorithms, i.e., with polynomial time complexity, have been proposed in [4]. The proposed approaches are based on structural systems reformulation (see [5]) and provide optimal placement of actuators to ensure structural controllability of the system. It involves analyzing only the sparsity (zero/non-zero pattern) of the dynamical interaction placement configurations and controllability is ensured in a *structural* sense. Specifically, such structural system theory based methods ensure structural controllability, i.e., provide actuator placement configurations that ensures system controllability for *almost all* numerical realizations of the system parameters. This approach is especially suitable for power systems where the exact values of the system parameters are not available in general due to, either, numerical inaccuracy resulting from the linearization or the unknown numerical parameters of the system components. However, the claim that controllability is ensured for *almost all* realizations of the system parameters, does not rule out the possibility of those realizations for which the system is uncontrollable. Moreover, such uncontrollable realizations are likely to occur while modeling large-scale power systems. It is due to the similar modeling of the system components, such as generators and loads, as well as the coupling induced by energy conservation laws (the algebraic network constraints). Consequently a set of *numerically similar* parameters may be observed, thereby resulting in an uncontrollable realization, even though the system may be structurally controllable. This motivates the requirement for stronger notions of controllability, namely that of strong structural controllability [6] (see also [7] for survey), which seeks conditions under which *all* numerical realizations of the system are controllable, see also [8] for an application. Specifically, given the structural pattern of the dynamical system, in this paper, we are interested in obtaining minimal dedicated controller placement configurations that ensure such strong structural controllability. Note that the design of the dedicated controllers (storage devices) supplements the conventional governor control of the generators.

In this paper, we focus on a more fundamental problem: given a dynamic system's structure, find the minimal subset of state variables that require dedicated inputs (i.e., inputs that are assigned only to a single state variable) to ensure a strong structurally controllable system. Formally, we consider the following problem.

Problem Statement

Let a linear time-invariant system be modeled as

$$\dot{x} = Ax, \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the system state and A the system matrix governing the autonomous dynamics. Let $\bar{A}$ denote the structural matrix encoding the sparsity pattern of A, i.e., the entries of $\bar{A}$ are either X's or 0's according to whether the corresponding entries of A are non-zeroes or zeroes. The objective is to design the input structural matrix $\bar{B} \in \{0, \times\}^{n\times p}$ with the minimal number of columns p and where each column of $\bar{B}$ has exactly one non-zero entry $\times$, such that the pair $(\bar{A}, \bar{B})$ is strong structurally controllable[1,2]. ◇

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[1]Note that by restricting $\bar{B}$ to contain exactly one non-zero entry per column, we are considering the *dedicated input* design problem, i.e., in which each control input may only control a single state variable. Further, the design objective is to obtain the $\bar{B}$ with the minimal number of columns p, i.e., in other words, we are interested in the minimal placement of dedicated inputs such that the pair $(\bar{A}, \bar{B})$ is SSC.

[2]We say that the structural pair $(\bar{A}, \bar{B})$, where $\bar{A} \in \{0, \times\}^{n\times n}$ and $\bar{B} \in \{0, \times\}^{n\times p}$, is strong structurally controllable, if and only if the linear dynamical system

$$\dot{x} = Ax + Bu$$

is controllable for **all** numerical realizations (A, B) with the same structural pattern as $(\bar{A}, \bar{B})$.

Most recently, in [9], the problem of the minimum number of inputs selection (non necessarily dedicated) that ensure strong structural controllability has been shown to be NP-complete if an initial set of possible inputs is provided. This is so, because the problem reduces to the analysis of a constrained maximum matching, that incurs in the existence of an order[3] between the edges in some maximum matching.

The main contribution of this paper is twofold: first, we derive a simplified necessary and sufficient condition for strong structural controllability. Second, based on this new characterization we propose an efficient algorithm (with cubic complexity in the number of state variables). The rest of this paper is organized as follows. In Section II we introduce standard terminology in structural systems theory, followed by the main results in Section III. Finally, we provide an illustration on th IEEE 5-bus test system, where we identify the minimal placement of dedicated inputs to achieve strong structural controllability and discuss the physical implications of such design.

II. Preliminaries and Terminology

In this section we introduce some terminology and recall some basic results from linear algebra. Given a matrix M of dimension $m_1 \times m_2$, we refer to m_1 and m_2 as the height and length of the matrix, respectively. A permutation matrix is a square matrix that is obtained by permuting some rows/columns of the identity matrix. Recall that for a given matrix, multiplication on the left by a permutation matrix permutes its rows, whereas the multiplication on the right, permutes the columns. Also, recall that the rank and determinant of a matrix are invariant to row-column permutations; these properties will be used throughout the paper, often without explicit mention. A lower (upper) triangular matrix is a matrix that has only zeros above (below) the diagonal, whereas, a lower (upper) block-triangular matrix is one that has only zeros above (bellow) the block-diagonal. We now introduce the notion of *stair matrix* as follows, see [11].

Definition 1 (Stair matrix): Let M be a $m_1 \times m_2$ matrix with entries that are either 0 (zero), $\times$ (non-zero) or $\otimes$ (an arbitrary value, including zero). The matrix M is said to be a *stair matrix* if it is of the form

$$M_{m_1\times m_2} = \begin{bmatrix} S^1_{n_1^1\times n_2^1} & \mathbf{0}_{n_1^1\times(m_2-n_2^1)} \\ S^2_{n_1^2\times n_2^2} & \mathbf{0}_{n_1^2\times(m_2-n_2^2)} \\ \ddots & \ddots \\ S^k_{n_1^k\times n_2^k} & \mathbf{0}_{n_1^k\times(m_2-n_2^k)} \end{bmatrix}$$

where $\mathbf{0}$ denotes the zero matrix of appropriate dimensions and each $n_1^i \times n_2^i$ matrix $S^i_{n_1^i\times n_2^i}$ denotes the i-th *step* ($i = 1,\cdots,k$) such that $n_2^i < n_2^{i+1}$ for $i = 1,\cdots,k-1$. In addition, when designating the submatrices $S^i_{n_1^i\times n_2^i}$ of a stair matrix M, we assume that M is in the *maximal stair form*, i.e., there exist no permutation matrices P_1^M, P_2^M such that $P_1^M M P_2^M$ has more steps or zero matrices with larger length in any given step than M. ⋄

Remark, that the steps in a stair matrix M are ordered by length, from the smallest $S^1_{n_1^1\times n_2^1}$ to the largest $S^k_{n_1^k\times n_2^k}$. Now, given a stair matrix we introduce the notion of *step difference*.

Definition 2 (Step difference): Let M be a stair matrix with k steps. A *step difference*, denoted by Δ^i_{i+1}, between two adjacent steps $S^i_{n_i\times n_i}, S^{i+1}_{n_{i+1}\times n_{i+1}}$, for $i = 1,\cdots,k-1$, is the submatrix of $S^{i+1}_{n_{i+1}\times n_{i+1}}$ comprising the same rows of $S^{i+1}_{n_{i+1}\times n_{i+1}}$ and only the columns from $n_2^i + 1$ to n_2^{i+1}, i.e.,

$$\begin{matrix} S^i_{n_1^i\times n_2^i} & & \mathbf{0}_{n_1^i\times(m_2-n_2^i)} \\ S^{i+1}_{n_1^{i+1}\times n_2^{i+1}} & \Delta^i_{i+1} & \mathbf{0}_{n_1^{i+1}\times(m_2-n_2^{i+1})} \end{matrix}$$

and by definition $\Delta^0_1 = S^1_{n_1\times n_1}$. ⋄

Additionally, consider the following characterization of step differences.

Definition 3 (Pivot in a step difference): Given a stair matrix, a pivot is a non-zero entry in the left-top most entry of a step difference. A step difference Δ^i_{i+1} of a stair matrix M has a *pivot* if there exist two permutation matrices P_1^Δ, P_2^Δ such that $\Delta_P = P_1^\Delta \Delta^i_{i+1} P_2^\Delta$ has $\times$ as its left-top most entry, i.e., the entry in the first column and row of Δ_P is non-zero. ⋄

Moreover, since there exists at most one pivot in each step difference, we can order (and name) the pivots by the induced order of the steps. Specifically, we say that two pivots k_1, k_2 are consecutive if there exists no other pivot k' such that $k_1 < k' < k_2$.

Definition 3 motivates the normalization of step differences as presented next.

Definition 4 (Normalized forms): We say that a step difference is in its *normal form* if it has a non-zero in its left-top most entry. Moreover, we say that a stair matrix is in its *normal form* if all step differences with pivots are in their normalized form. ⋄

Finally, we introduce the notion of a *ramp matrix*.

Definition 5: A *ramp matrix* $M \in \{0, \times, \otimes\}^{m_1\times m_2}$ is a stair matrix with m_1 steps where each step difference has a pivot. ⋄

Remark 1: First note that, by definition, each step of a ramp matrix M is a row vector. Definition 5 also implies that, for a ramp matrix M, there exists a lower-triangular sub-matrix with non-zero entries in its diagonal, given by the columns of M that have the pivots of the normalized step differences.

III. Main Results

In this section we state the main results of this paper. First, we introduce a (new) simplified necessary and sufficient condition for strong structural controllability, that, in particular, relies on the satisfability of a single criterion, instead of two as in [7], [9]. Second, we present an algorithm to obtain an input matrix that corresponds to the minimum dedicated input assignment ensuring strong structural controllability. In addition, from the existence of possible pivots to the step differences, follows that several solutions to our problem are possible (see Section IV for further discussion).

Theorem 1 (SSC Theorem): The structural pair $(\bar{A}, \bar{B})$ is strong structurally controllable if and only if for each $\lambda \in \mathbb{C}$ the matrix $[\bar{A}^\lambda \ \bar{B}]$ with $\bar{A}^\lambda = \bar{A} - \lambda\mathbf{I}$ and $\mathbf{I}$ denoting the identity matrix of appropriate dimensions, can be transformed into a ramp matrix. □

Theorem 1 motivates our design approach, Algorithm 1, where given the system structure $\bar{A}$, the design of the optimal input structural matrix $\bar{B}$ is essentially achieved, by introducing the minimum number of columns (in $\bar{B}$) with only a single non-zero entry such that $[\bar{A}^\lambda \ \bar{B}]$, for all $\lambda \in \mathbb{C}$, is transformable to a ramp matrix. The correctness and complexity of Algorithm 1 is analyzed as follows:

[3] The need to select an order is a well know cause of NP-completeness, see for instance [10].

ALGORITHM 1: Compute a minimal placement of dedicated inputs that achieve strong structural controllability

Input: Dynamic matrix structure $\bar{A}$
Output: Input matrix $\bar{B}$ representing the minimal placement of dedicated inputs that achieve strong structural controllability

1) Find permutation matrices P_1 and P_2 such that $M = P_1 \bar{A}^\lambda P_2$ (for all $\lambda \in \mathbb{C}$) is a stair matrix with k steps, in its normalized form. In addition, let $p_1, \cdots, p_{k'}$ correspond to the k' pivots of the step differences, and the row and column entry of the α-th pivot in M, be denoted by p^r_α and p^c_α, respectively.

2) Let

$$\bar{B}^{P_1} = \mathbb{I}^{\mathcal{J}}_{n\times n},$$

where $\mathcal{J} = \{1, \cdots, n\} \setminus \bigcup_{\alpha=1}^{k'} p^r_\alpha$ corresponds to the set of indices of rows without pivot for the step differences, $\mathbb{I}_{n\times n}$ is the diagonal matrix with non-zeros entries and $\mathbb{I}^{\mathcal{J}}_{n\times n}$ is the matrix resulting from only keeping the columns of $\mathcal{I}_{n\times n}$ with nonzero entry in the rows indexed by $\mathcal{J}$.

3) Set $\bar{B} = P_1^{-1} \bar{B}^{P_1}$.

Theorem 2 (Correctness of Algorithm 1): Algorithm 1 is correct, i.e., the output of Algorithm 1 is a structural input matrix $\bar{B}$ corresponding to the minimum number of dedicated inputs such that the pair $(\bar{A}, \bar{B})$ is SSC. ◇

Theorem 3 (Complexity of Algorithm 1): Algorithm 1 has computational complexity $\mathcal{O}(n^3)$, where n denotes the dimension of the state-space, i.e., the number of state variables. ◇

IV. ILLUSTRATION: A 5-BUS POWER SYSTEM

The power system in Fig. 1 consists of five dynamical components interconnected through transmission lines. These are two coal-based generators C_1, C_2, one gas based combustion turbine G_3 and the aggregate loads D_4, D_5.

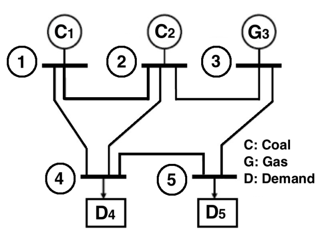

Fig. 1. IEEE 5-bus test system (*1,2,3,4,5* represents the bus number)

The generators C_1, C_2 and G_3 are modeled as linearized governor control, see Appendix-A of [12] for details. The aggregate loads D_4,D_5, modeled as dynamic components, are represented by swing equations [2], [13]. Subsequently, the generators and the loads are electrically coupled through the differentiated linearized real-power flow equations[4]. The 5-bus power system is represented as a 16^{th} order LTI system (1), where, for each $\lambda \in \mathbb{C}$, the structural matrix $\bar{A}^\lambda_{5\text{bus}}$ is of the form:

$$\bar{A}^\lambda_{5\text{bus}} = \begin{bmatrix}
\otimes & \times & \times & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \times & 0 & 0 & 0 & 0 \\
0 & \otimes & \times & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\times & 0 & \otimes & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \otimes & \times & \times & 0 & 0 & 0 & 0 & 0 & 0 & \times & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \otimes & \times & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \times & 0 & \otimes & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \otimes & \times & 0 & 0 & 0 & 0 & 0 & \times & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \otimes & \times & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \times & 0 & \otimes & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \otimes & 0 & 0 & 0 & 0 & \times & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \otimes & 0 & 0 & 0 & 0 & \times \\
\times & 0 & 0 & \times & 0 & 0 & 0 & 0 & 0 & \times & 0 & \otimes & 0 & 0 & 0 & 0 \\
\times & 0 & 0 & \times & 0 & 0 & \times & 0 & 0 & \times & 0 & 0 & \otimes & 0 & 0 & 0 \\
0 & 0 & 0 & \times & 0 & 0 & \times & 0 & 0 & 0 & \times & 0 & 0 & \otimes & 0 & 0 \\
\times & 0 & 0 & \times & 0 & 0 & 0 & 0 & 0 & \times & \times & 0 & 0 & 0 & \otimes & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \times & 0 & 0 & \times & \times & 0 & 0 & 0 & 0 & \otimes
\end{bmatrix}$$

The state variables are described in Table I.

TABLE I. 5-BUS POWER SYSTEM'S STATE VARIABLES

States	Description
x_1	C_1's prime mover frequency
x_2	C_1's turbine mechanical power output
x_3	C_1's governor valve position
x_4	C_2's prime mover frequency
x_5	C_2's turbine mechanical power output
x_6	C_2's governor valve position
x_7	G_3's turbine fuel flow
x_8	G_3's prime mover frequency
x_9	G_3's governor valve position
x_{10}	frequency at bus-4
x_{11}	frequency at bus-5
x_{12}	power injection by C_1
x_{13}	power injection by C_2
x_{14}	power injection by G_3
x_{15}	power consumption by D_4
x_{16}	power consumption by D_5

Next, Algorithm 1 is implemented to obtain the minimal dedicated actuation configuration that ensures strong structural controllability for the 5-bus power system. The normalized stair matrix $M_{5\text{bus}}$ (see Algorithm 1 for notation) is given as:

$$\begin{bmatrix}
\times & \otimes & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\otimes & 0 & \times & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \otimes & \times & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \times & \otimes & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \otimes & 0 & \times & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \times & \times & \otimes & \times & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \times & \otimes & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \otimes & 0 & \times & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \times & \otimes & 0 & 0 & \times & 0 & 0 & 0 & 0 & \times & 0 & 0 & 0 & 0 \\
0 & 0 & \times & 0 & 0 & 0 & \times & \otimes & 0 & 0 & \times & \times & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \times & \otimes & 0 & \times & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \otimes & 0 & \times & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \times & 0 & 0 & 0 & \times & 0 & \otimes & 0 & \times & 0 \\
0 & 0 & \times & 0 & 0 & 0 & \times & 0 & 0 & 0 & 0 & \times & 0 & \otimes & \times & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \otimes & \times \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \times & \times & 0 & 0 & \times & \otimes
\end{bmatrix}$$

with permutation matrices

$$P_1 = [e_2\ e_3\ e_1\ e_5\ e_6\ e_4\ e_8\ e_9\ e_{12}\ e_{13}\ e_7\ e_{10}\ e_{14}\ e_{15}\ e_{11}\ e_{16}],$$
$$P_2 = [e_3\ e_2\ e_1\ e_{12}\ e_6\ e_5\ e_4\ e_{13}\ e_9\ e_8\ e_7\ e_{10}\ e_{14}\ e_{15}\ e_{11}\ e_{16}],$$

written in terms of vectors $e_i \in \{0, \times, \otimes\}^{15}$ with non-zero ith entry and zero elsewhere.

The rectangles and squares in $M_{5\text{bus}}$ denote the step differences. Each has a pivot corresponding to a non-zero entry in the top-left most entry. For the blue rectangles, only one normalized step difference is possible. It must be noted that there exist no pivot in lines $10, 14, 16$. This implies that we need to keep the corresponding columns in P_1 to obtain $\bar{B} = [e_{13}\ e_{15}\ e_{16}]$.

We now demonstrate that the minimal solution is not unique and specifically, alternative minimal placements may be obtained by considering different permutation matrices that transform $\bar{A}^\lambda_{5bus}$ to ramp forms. As an example, consider the

[4]The coupling constraints are the linearized power flow equations. For the purpose of real-power balancing, decoupling between real and reactive power flow is assumed, see [1].

squared step difference highlighted with red in $M_{5\text{bus}}$. A new normalized stair matrix is obtained with the permutation of rows corresponding to red-marked rectangles. This results in an alternative input matrix that ensures SSC for the 5-bus power system. In more detail, considering

$$P_1'=[\,e_2\ e_3\ e_1\ e_5\ e_6\ e_4\ e_8\ e_9\ \mathbf{e_{12}}\ \mathbf{e_{13}}\ e_7\ e_{10}\ \mathbf{e_{14}}\ \mathbf{e_{15}}\ e_{11}\ e_{16}\,],$$

where the bold face canonical vectors highlight P_1''s dissimilarity to P_1, an alternative input matrix that ensures strong structural controllability is given by $\bar{B}' = [e_{13}\ e_{14}\ e_{16}]$ associated with the permutation pair (P_1', P_2). In the same fashion we could also obtain $\bar{B}'' = [e_{12}\ e_{15}\ e_{16}]$ and $\bar{B}''' = [e_{12}\ e_{14}\ e_{16}]$ as other candidates for the minimal placement design.

Discussion of results: As per the four alternative input matrices achieving the minimal design, the state variables to be actuated are subsets of the nodal power injections/consumptions. The states x_{16} always require a dedicated input, i.e., the power consumption by the aggregate load D_5. The second actuator can be placed at bus-4 (x_{15}), or, at bus-3 (x_{14}). The third actuator can be placed at bus-4 (x_{15}), or, at bus-3 (x_{14}). All possible input matrices are physically feasible. However, based on performance objective, for example cost of actuators, some of the input matrices to achieve the condition of strong structural controllability for the 5-bus power system may be more suitable.

V. Conclusions

In this paper we have provided a systematic method with polynomial implementation complexity, in terms of number of state variables, to obtain the minimal placement of dedicated inputs ensuring strong structural controllability of a given LTI system. We have shown that our method yields the globally optimal dedicated input placement. By duality, the results extend to the corresponding strong structural observability output design under similar constraints. Additionally, we have illustrated our design approach by providing the optimal placement of actuators in an IEEE 5-bus test system.

Appendix

Proof of Theorem 1: [$\Leftarrow$] Let (A, B) be a pair (with complex entries) such that (A, B) is a numerical realization of $(\bar{A}, \bar{B})$. Consider for each $\lambda \in \mathbb{C}$, the matrix $[A^\lambda\ B]$ which, by hypothesis, can be transformed into a ramp matrix. By Remark 1, it then follows that the resulting ramp matrix contains a lower triangular submatrix with non-zero diagonal entries, and, hence,

$$rank[A^\lambda,\ B] = n, \quad \forall \lambda \in \mathbb{C}.$$

Thus, by the Popov-Belevitch-Hautus (PBH) test for controllability [14] it follows that (A, B) is controllable; since, the above holds for all numerical realizations of $(\bar{A}, \bar{B})$ we conclude that $(\bar{A}, \bar{B})$ is SSC.

[$\Rightarrow$] On the contrary, suppose that the matrix $[\bar{A}^\lambda\ \bar{B}]$ cannot be transformed into a ramp matrix for all choices of $\lambda \in \mathbb{C}$. Hence, any matrix composed of n columns of $[\bar{A}^\lambda\ \bar{B}]$, can be rearranged as a stair matrix where there is at least one step difference, with size at least 2×2. Such a stair matrix is clearly block lower triangular. Now considering the diagonal block which contains the step difference of size at least 2×2, it follows that there exists a numerical parametrization (realization) of the entries that makes the determinant of that block equal to zero. In other words, there exists a numerical realization (A, B) and $\lambda \in \mathbb{C}$ such that any matrix composed of n columns of $[A^\lambda\ B]$ is rank deficient, which implies that $rank([A^\lambda\ B]) < n$. Thus, by the PBH test for controllability [14] it follows that the realization (A, B) is not controllable. This contradicts with the hypothesis that $(\bar{A}, \bar{B})$ is SSC and the assertion follows. ∎

Proof of Theorem 2: Let $C_{p_\alpha^c}$ denote the column in M containing the α-th pivot. First, remark that $\bar{B}$ is a feasible solution, since the matrix $[\bar{A} - \lambda \mathbf{I}\ \bar{B}]$ can be transformed in a ramp matrix, where the matrix

$$[C_{p_1^c}\ \mathcal{I}^{\mathcal{J}(p_1^c, p_2^c)}\ C_{p_2^c}\ \cdots\ C_{p_{k'-1}^c}\ \mathcal{I}^{\mathcal{J}(p_{k'-1}^c, p_{k'}^c)}\ C_{p_{k'}^c}]$$

where $\mathcal{J}(p_i^c, p_{i+1}^c) = \{j \in \mathbb{N} :\ p_i^c < j < p_{i+1}^c\}\}$, is a lower-triangular matrix. Second, the minimality is obtained by noticing that a step matrix leads to the maximum number of steps, and the subset of columns associated with the pivots corresponds to the maximum number of linear independent columns with respect to all possible parameterizations (which follows by similar reasoning used in the proof of Theorem 1). Finally, remark that there are as many columns in $\bar{B}$ as necessary to complete the rank n, in addition, these columns are linearly independent since they correspond to a single non-zero entry (in different positions) columns. ∎

Proof of Theorem 3: Step 1 can be implemented in $\mathcal{O}(n^3)$, see for instance [15] for a discussion on obtaining two permutation matrices, such that: first, the maximum number of zeros are shifted to the top-right to ensure a lower-block triangular matrix structure, and second to move the maximum number of zeros in each diagonal block to the top-right, which leads to a stair matrix. Step 2 can be implemented with linear complexity, as well as Step 3, since $P_1^{-1} = P_1^T$. ∎

References

[1] M. Ilić and J. Zaborszky, *Dynamics and Control of Large Electric Power Systems*, ser. A Wiley-Interscience publication. Wiley, 2000. [Online]. Available: http://books.google.pt/books?id=i1JjQgAACAAJ

[2] M. D. Ilić, L. Xie, U. A. Khan, and J. M. F. Moura, "Modeling of future cyber-physical energy systems for distributed sensing and control," *IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans*, vol. 40, no. 4, pp. 825–838, July 2010.

[3] A. Olshevsky, "The minimal controllability problem," *Arxiv*. [Online]. Available: http://www.arxiv.org/abs/1304.3071v2

[4] S. Pequito, S. Kar, and A. Aguiar, "A structured systems approach for optimal actuator-sensor placement in linear time-invariant systems," *Proceedings of American Control Conference 2013*.

[5] J.-M. Dion, C. Commault, and J. V. der Woude, "Generic properties and control of linear structured systems: a survey." *Automatica*, pp. 1125–1144, 2003.

[6] H. Mayeda and T. Yamada, "Strong structural controllability," *SIAM J. Control and Optimization*, vol. 17, pp. 123–138, 1979.

[7] F. S. Jan Christian Jarczyk and B. Alt, "Strong structural controllability of linear systems revisited," *Proc. of the 50th IEEE Conference on Decision and Control*, pp. 1213–1218, 2011.

[8] C. Bowden, W. Holderbaum, and V. Becerra, "Strong structural controllability and the multilink inverted pendulum," *Automatic Control, IEEE Transactions on*, vol. 57, no. 11, pp. 2891–2896, 2012.

[9] A. Chapman and M. Mesbahi, "On strong structural controllability of networked systems: A constrained matching approach," *Proceedings of American Control Conference 2013*.

[10] M. R. Garey and D. S. Johnson, *Computers and Intractability; A Guide to the Theory of NP-Completeness*. New York, NY, USA: W. H. Freeman & Co., 1990.

[11] H. Lu, "Stair matrices and their generalizations with applications to iterative methods I: A generalization of the successive overrelaxation method," *SIAM J. Numer. Anal.*, vol. 37, no. 1, pp. 1–17, Nov. 1999.

[12] J. Cardell, *Control Strategies and Dynamic Pricing for Small Scale Distributed Generation in a Deregulated Market*. Doctoral Dissertation, MIT, 1997.

[13] M. Ilic, N. Popli, J.-Y. Joo, and Y. Hou, "A possible engineering and economic framework for implementing demand side participation in frequency regulation at value," in *Power and Energy Society General Meeting, 2011 IEEE*, july 2011, pp. 1 –7.

[14] J. Hespanha, *Linear Systems Theory*. Princeton University Press, 2009.

[15] A. Pothen and C.-J. Fan, "Computing the block triangular form of a sparse matrix," *ACM Trans. Math. Softw.*, vol. 16, pp. 303–324, December 1990. [Online]. Available: http://doi.acm.org/10.1145/98267.98287