A new algorithm for linearization up to multi-output and multi-input injection for a class of systems with implicitly defined outputs

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***Abstract*— Given a class of nonlinear systems with implicitly defined outputs, we provide a new algorithm to find appropriate local coordinates, in which the resulting system takes a desired target form that is state-affine, up to output and input injection. Once in the target form, it is possible to construct a state-space observer with linear, possibly time-varying, error dynamics.**

I. Introduction

Over the last few decades, a number of research work has addressed the problem of obtaining sub-classes in the general class of continuous-time nonlinear systems

$$\begin{array}{rcl} \dot{x} & = & f_u(x) := f(u,\,x) \\ y & = & h_u(x) := h(u,\,x) \end{array} \tag{1}$$

for which there exists, at least locally, an observer with linear error dynamics [9], [11], [12], [14], [16]–[18]. The problem can be formulated as follows: given system (1) where f and h are sufficiently smooth functions, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is an input signal and $y \in \mathbb{R}^q$ is the measured output, find (if it exists) a smooth change of coordinates $z = \Theta(x)$ such that in the new coordinates it is possible to construct a state observer $\hat{z}$ so that the estimation error $\tilde{z} = \hat{z} - z$ is governed by an asymptotically stable linear (possibly time varying) dynamical system.

The first results in this area includes the work by Krener [16], where the linearization of a nonlinear system is addressed with no reference to any output; and Krener and Isidori [17], where the linearization is studied up to output injection, that is, the aim is to find $z = \Theta(x)$ for a nonlinear system $\dot{x} = f(x),\ y = h(x)$ that leads into a linear system, up to an output injection,

$$\dot{z} = Az + \Phi(y), \quad y = Cz$$

where A and C are linear maps and the vector field $\Phi(y)$ only depends on the known output signal y. Note that a Luenberger type observer $\dot{\hat{z}} = A\hat{z} + \Phi(y) + L(y - C\hat{x})$ reaches the linear error dynamics $\dot{\tilde{z}} = (A - LC)\tilde{z}$, where $\tilde{z}(t) \to 0$ as $t \to +\infty$ provided that L is selected so that $A - LC$ is Hurwitz .

Using tools from Differential Geometry, Hammouri and Gauthier in [11], [12], Hammouri and Kinnaert in [14], and Xia and Gao in [21] extended the linearization problem to systems in form (1) to obtain the following *target system*,

$$\dot{z} = A_u z + \Phi_u(y),\ y = P^q z \tag{2}$$

which is a time-varying linear system up to output and input injection and where $P^q z = [z^1\, z^2\, \ldots\, z^q]^\top \in \mathbb{R}^q$ collects the first q coordinates[1] of $z = [z^1\, z^2\, \ldots\, z^n]^\top \in \mathbb{R}^n$, $q \le n$.

More recently, in [20] we have considered target systems in the more general form

$$\dot{z} = A_u z + \Phi_u(y) \tag{3a}$$

$$0 = C_u P^s z + (P^s z)^\top D_u y + E_u y + F_u \tag{3b}$$

where $s \le n$, $A_u \in \mathcal{M}_{n\times n}(\mathbb{R})$, $C_u \in \mathcal{M}_{p\times s}(\mathbb{R})$, $E_u \in \mathcal{M}_{p\times q}(\mathbb{R})$, and $F_u \in \mathcal{M}_{1\times p}(\mathbb{R})$ are matrices with real entries, and $D_u = \mathrm{col}_{\mathcal{M}}(D_u^1,\, D_u^2,\, \ldots,\, D_u^p) \in \mathcal{M}_{1\times p}(\mathcal{M}_{s\times q}(\mathbb{R}))$ is a column matrix whose entries are the matrices $D_u^i \in \mathcal{M}_{s\times q}(\mathbb{R})$; $i = 1, \ldots, p$. In this case, we define $(P^s z)^\top D_u y := [(P^s z)^\top D_u^1 y \quad (P^s z)^\top D_u^2 y \quad \ldots \quad (P^s z)^\top D_u^p y]^\top \in \mathbb{R}^p$. As a simple illustration of the notation, $\mathrm{col}_{\mathbb{R}}(D_u^1,\, D_u^2)$ and $\mathrm{col}_{\mathcal{M}}(D_u^1,\, D_u^2)$ stand, respectively, for $\begin{bmatrix} a_{11} & a_{12} \\ b_{11} & b_{12} \end{bmatrix}$ and $\begin{bmatrix} \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \\ \begin{bmatrix} b_{11} & b_{12} \end{bmatrix} \end{bmatrix}$, for given matrices $D_u^1 = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$ and $D_u^2 = \begin{bmatrix} b_{11} & b_{12} \end{bmatrix}$. The main motivation for considering target systems in the form of (3) is that there exists an observer with linear error dynamics for this class of systems (under some suitable observability conditions, involving the matrices appearing in (3), see details in [2]). For example, for the following perspective output system

$$\dot{x} = f_u(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ y+u \end{bmatrix}, \quad x^1 y = x^2 \tag{4}$$

the output equation can be rewritten in the form of equation (3b): $\begin{bmatrix} 0 & -1 \end{bmatrix} x + x^\top \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} y = 0$, because $x = P^2 x$. Thus, system (4) can be rewritten in the form of system (3) but not in the form of (2). Resorting to the results in [2, section 3], we can derive the following robust H_∞ type optimal observer with dynamics $\dot{\hat{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ y+u \end{bmatrix} - \gamma^2 Q \begin{bmatrix} y^2 & -y \\ -y & 1 \end{bmatrix} \hat{x}$, where $\gamma > 0$ is a given gain level. The

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[1] For a given matrix M, $M^\top$ denotes its transpose. Notice that we use superscripts to denote the coordinates of a vector $v = [v^1\, v^2\, \ldots\, v^k]^\top \in \mathbb{R}^k$. The reason of this is because we will use some tools from Differential Geometry where often that notation is convenient. Further, collecting either the first or the last q coordinates lead to completely equivalent problems.

matrix Q is the solution of $\dot{Q} = \begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} Q + Q \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} - \gamma^2 Q \begin{bmatrix} y^2 - \gamma^{-2} & -y \\ -y & 1 - \gamma^{-2} \end{bmatrix} Q$, $Q(0) = Q_0$, with $Q_0^{-1} > 0$, where $\lambda \geq 0$ denotes a forgetting factor. It follows that the estimation error $\tilde{x} = \hat{x} - x$ satisfies the (time-dependent) linear dynamics $\dot{\tilde{x}} = \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \gamma^2 Q \begin{bmatrix} y^2 & -y \\ -y & 1 \end{bmatrix} \right) \tilde{x}$. See [2] for details. On the other hand, as shown in [20], there is no change of coordinates transforming system (4) into the form (2), in a neighborhood of the point $x_0 = [1\ 0]^\top$. The example shows the importance of taking into consideration also the implicit output notation, that is not covered by the explicit framework proposed previously in [11], [12], [14]. Notice that we can also show that our framework is not covered by the ones proposed more recently in [4], [5], [7], [8], [10]; see [20].

In [20], we have also provided an algorithm to obtain the desired coordinate transformation $z = \Theta(x)$ that transforms the given nonlinear system

$$\dot{x} = f_u(x) \tag{5a}$$

$$0 = C_u g(x) + g(x)^\top D_u y + E_u y + F_u \tag{5b}$$

into the target form (3), with $P^s z = g(x) = g(\Theta^{-1}(z))$. Comparing the algorithms in [11] and in [14], for the explicit single-output case, the one in [14] looks more elegant. On the other hand the algorithm presented in [20], for the implicit multi-output case, is closer to a generalization of the one in [11] than to the one in [14]. A natural question is whether is it possible to generalize the results in [14] to the multi-output case. In this paper, we give a positive answer by restricting appropriately the sub-class of nonlinear systems. Both the algorithm in [20] and the one we present here, aim to find a suitable s-tuple of vector fields from which we can obtain the desired change of coordinates (if such a change of coordinates does exist). While the algorithm in [20] provides a set of equations that certain suitable s-tuples must do satisfy, the one we present here gives us directly those s-tuples.

The rest of the paper is organized as follows: in Section II we recall the necessary and sufficient conditions to be able to rewrite the original system (5) in the desired target form (3). In Section III we recall the algorithm in [20], and present the new one, for the restricted sub-class of systems. Section IV illustrates the contribution of the paper. Brief conclusions are discussed in Section V.

II. Linearization up to output and input injection

A. Notation and definitions

We assume that the reader has some familiarity with basic concepts of Differential Geometry and Control Theory. We briefly recall some terminology. For a more complete discussion on what follows we suggest the works [1], [6], [15], see also [19].

Given a system of coordinates $(x^1, x^2, \ldots, x^n)$, we consider the vector fields in $\mathbb{R}^n$, ${\partial}/{\partial x^k}$ defined by ${\partial}/{\partial x^k}(x) = [\delta_k^1\ \delta_k^2\ \ldots\ \delta_k^n]^\top \in T_x\mathbb{R}^n \sim \mathbb{R}^n$, where δ_i^j is the Kronecker delta function, for $i, j \in \mathbb{N}$ nonnegative integers. We denote by $\mathcal{V}(\Omega)$ the $C^\infty(\Omega)$-module of smooth vector fields in Ω and, for $k \in \mathbb{N}$, by $\Lambda^k(\Omega)$ the $C^\infty(\Omega)$-module of (differential) k-forms defined in the Cartesian product $\mathcal{V}(\Omega)^k$ for $k > 0$, and $\Lambda^0(\Omega) := C^\infty(\Omega)$. Further, we denote by $\alpha \wedge \beta$ the wedge product between the forms α and β, and by $\iota_X w$ the interior product $\iota_X w(V_1, V_2, \ldots, V_{k-1}) := w(X, V_1, V_2, \ldots, V_{k-1})$ between a vector field X and a k-form w; for a r-tuple of vector fields, with $r \leq k$, we define recursively $\iota_{(X_1, X_2, \ldots, X_r)} w := \iota_{X_r} \iota_{(X_1, X_2, \ldots, X_{r-1})} w$. The exterior derivative of a k-form w will be denoted by $\mathrm{d}w$. The n-tuple $(h^1, h^2, \ldots, h^n)$ of smooth functions is a system of coordinates in $\Omega \subseteq \mathbb{R}^n$ if $\mathrm{d}h^1 \wedge \mathrm{d}h^2 \wedge \cdots \wedge \mathrm{d}h^n|_x \neq 0$, for all the points $x \in \Omega$. Given two systems of local coordinates $(x^1, x^2, \ldots, x^n)$ and $(h^1, h^2, \ldots, h^n)$, we have

$$ {\partial}/{\partial x^k} = \sum_{j=1}^{n} {\partial h^j}/{\partial x^k} {\partial}/{\partial h^j}. \tag{6}$$

We denote by $\mathcal{L}$ the Lie derivative operator. For a vector field X, and a k-form w we have: $\mathcal{L}_X w := \iota_X \mathrm{d}w = \mathrm{d}w(X)$ for a function w, i.e., if $k = 0$, and $\mathcal{L}_X w := (\iota_X \mathrm{d} + \mathrm{d}\iota_X)w$ if $k \geq 1$. In particular, given a vector field $V = \sum_{j=1}^n V^j {\partial}/{\partial x^j}$ we have that $V^j = \mathrm{d}x^j(V) = \mathcal{L}_V x^j$.

Writing $X = \sum_{i=1}^n X^i {\partial}/{\partial x^i}$, we may also define the Lie derivative of a vector field $Y = \sum_{i=1}^n Y^i {\partial}/{\partial x^i}$ by setting, in coordinates,

$$\mathcal{L}_X Y = [X, Y] := \sum_{i=1}^{n} \sum_{j=1}^{n} \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) {\partial}/{\partial x^i}; \tag{7}$$

the vector field $[X, Y]$ is called the Lie bracket between the vector fields X and Y. The identity

$$\mathcal{L}_{[X,Y]} f = \mathcal{L}_X \mathcal{L}_Y f - \mathcal{L}_Y \mathcal{L}_X f \tag{8}$$

holds for every function $f \in \Lambda^0(\Omega)$.

B. Necessary and sufficient conditions

We now recall the conditions to the existence of a change of coordinates that carries (5) into the target form (3), with $P^s z = g(x)$. To this end, consider first the following auxiliary system

$$\dot{x} = f_u(x), \quad \bar{y} = g(x) \tag{9}$$

where f_u and g are obtained from (5) and it is assumed that (9) is observable in the rank sense (see definition, e.g., [3]) in a neighborhood of a given point x_0. This means that $\{\mathrm{d}w|_{x_0} \mid w \in \mathcal{O}\}$ is n-dimensional, where $\mathcal{O}$ stands for the smallest set containing $\{g^1, g^2, \ldots, g^s\}$, and closed under all the Lie derivatives $\{\mathcal{L}_{f_u} \mid u \text{ is a constant in } \mathbb{R}^m\}$, and $\mathrm{d}w|_{x_0}$ stands for the evaluation of $\mathrm{d}w$ at x_0.

Let $X = (X_1, X_2, \ldots, X_s)$ be a s-tuple of vector fields, and define a sequence of vector spaces as follows: set $\mathrm{d}\Gamma := \mathrm{d}g^1 \wedge \mathrm{d}g^2 \wedge \cdots \wedge \mathrm{d}g^s$ and denote by Ω_1^X the real vector space generated by the set of $(1+s)$-forms $\{\mathrm{d}\mathcal{L}_{f_u} g^j \wedge \mathrm{d}\Gamma \mid 1 \leq j \leq s \text{ and } u \in \mathbb{R}^m\}$, that is

$$\Omega_1^X := \mathrm{span}_{\mathbb{R}} \{\mathrm{d}\mathcal{L}_{f_u} g^j \wedge \mathrm{d}\Gamma \,|\, 1 \leq j \leq s \text{ and } u \in \mathbb{R}^m\}. \tag{10}$$

Recursively, for $k \geq 2$, define the real vector space

$$\Omega_k^X := \mathrm{span}_{\mathbb{R}}\{\mathcal{L}_{f_u}(\iota_X w) \wedge \mathrm{d}\Gamma \mid w \in \Omega_{k-1}^X \text{ and } u \in \mathbb{R}^m\}.$$

Define also the smallest real vector space Ω^X containing all these previous ones by

$$\Omega^X := \mathrm{span}_{\mathbb{R}}\{w \mid w \in \Omega_k^X \text{ and } k \in \mathbb{N}_0\}$$

and consider the vector space

$$\Omega[X,\, g] := \mathrm{span}_{\mathbb{R}}(\, \iota_X \Omega_X \cup \{\mathrm{d}g\}\,)$$

with $\{\mathrm{d}g\} := \{\mathrm{d}g^j \mid j = 1,\, 2,\, \dots,\, s\}$. Finally, denote $\mathrm{d}\Upsilon := \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \cdots \wedge \mathrm{d}y^q$, where $[y^1\, y^2\, \dots\, y^q]^\top := y$ are the coordinate functions of the output $y \in \mathbb{R}^q$ in system (5).

We are now ready to present the necessary and sufficient conditions to be able to transform the system (5) into the target form (3), see [20].

Theorem 2.1 ([20]): Consider a given point x_0 and suppose that in a neighborhood U of x_0, system (9) is observable in the rank sense, $\mathrm{d}\Upsilon \neq 0$, and $\mathrm{d}y^j \wedge \mathrm{d}\Gamma = 0$ for every $j = 1,\, 2,\, \dots,\, q$. Then up to a change of coordinates, system (5) can be written in the form of (3) in a sub-neighborhood $\mathcal{N} \subseteq U$ of x_0 if, and only if,

- a. $\mathrm{d}\Gamma|_{x_0} \neq 0$;
- b. there exists a s-tuple of vector fields $X = (X_1,\, X_2,\, \dots,\, X_s)$ such that:
  - i. $\mathcal{L}_{X_i} g^j = \delta_i^j$ in U;
  - ii. the real dimension of Ω^X is equal to $n - s$ in $\mathcal{N}$;
  - iii. $\mathrm{d}\iota_X \Omega^X = \{0\}$ in $\mathcal{N}$;
  - iv. $\bigwedge^{n-s} \iota_X \Omega^X \wedge \mathrm{d}\Gamma|_{x_0} \neq \{0\}$, and
  - v. $\mathrm{d}\iota_{f_u} \Omega[X,\, g] \wedge \mathrm{d}\Upsilon \subseteq \Omega[X,\, g] \wedge \mathrm{d}\Upsilon$ in $\mathcal{N}$.

Remark 2.1: To have the desired output equation (3b) we select $z^i = g^i$ for $i \in \{1,\, 2,\, \dots,\, s\}$. For the rest of the coordinate functions we may choose them from a basis $\{\mathrm{d}z^j \mid j = s+1,\, s+2,\, \dots,\, n\}$ for $\iota_X \Omega^X$. Notice, however that these last coordinates can be replaced by a family $\{\tilde{z}^j \mid j = s{+}1,\, s{+}2,\, \dots,\, n\}$, where $\tilde{z}^j \in \mathrm{span}_{\mathbb{R}}\{z^i \mid i = 1,\, 2,\, \dots,\, n\}$ and $\mathrm{d}\Gamma \wedge \mathrm{d}\tilde{z}^{s+1} \wedge \mathrm{d}\tilde{z}^{s+2} \wedge \cdots \wedge \mathrm{d}\tilde{z}^n \neq 0$. Applying this linear change preserves the form of the target system, which means that both $(z^1,\, z^2,\, \dots,\, z^n)$ and $(z^1,\, z^2,\, \dots,\, z^s,\, \tilde{z}^{s+1},\, \tilde{z}^{s+2},\, \dots,\, \tilde{z}^n)$ lead to the desired target system form. In comparison to the explicit output case addressed in [11], [12], [14] we have mainly two new conditions: $\mathrm{d}\Upsilon \neq 0$ and b.v. Further, notice that in the explicit output case we have $\mathrm{d}\Upsilon = \mathrm{d}\Gamma$, and b.v will follows from the preceding conditions and from the definitions.

III. The algorithms

From Theorem 2.1 we could conclude that the conditions to cast the original system (5) into the desired target form (3) resumes to find a suitable s-tuple of vector fields. Once we obtain this s-tuple we can find an appropriate local change of coordinates as indicated in Remark 2.1. This section addresses the problem of finding that s-tuple of vector fields. We start by recalling the general algorithm presented in [20], then we present a more elegant one for a particular sub-class of systems. To this end, we first generalize for the multi-output case the results in [14], and which proof can be done analogously as in [14], see [19, Lemma 4.1].

Lemma 3.1: Let system (9) be observable in the rank sense at x_0 with $s < n$. Denote $\{\mathrm{d}g\} := \{\mathrm{d}g^j \mid j = 1,\, 2,\, \dots,\, s\}$. Then, it is possible to construct a length-k_0 sequence of subsets $\mathcal{S}_1,\, \mathcal{S}_2,\, \dots,\, \mathcal{S}_{k_0}$ such that

- $\mathcal{S}_1 \subset \{1,\, 2,\, \dots,\, s\} \times \mathbb{R}^m$ and $\mathcal{S}_k \subset \mathcal{S}_{k-1} \times \mathbb{R}^m$, for $k = 2,\, 3,\, \dots,\, k_0$;
- for $1 \leq k_1 \leq k_0$ the family

  $$\mathcal{B}_{k_1} := \{\mathrm{d}g\} \cup \{\mathrm{d}\mathcal{L}_{f_{u_k}} \mathcal{L}_{f_{u_{k-1}}} \dots \mathcal{L}_{f_{u_1}} g^i \mid k = 1,\, 2,\, \dots,\, k_1 \text{ and } (i,\, u_1,\, u_2,\, \dots,\, u_k) \in \mathcal{S}_k\}$$

  is a basis for the module $\mathrm{span}_{C^\infty(U)}(\{\mathrm{d}g\} \cup \mathcal{A}_{k_1})$ spanned by the set $\{\mathrm{d}g\} \cup \mathcal{A}_{k_1}$, with $\mathcal{A}_{k_1}$ given by

  $$\left\{ \mathrm{d}\mathcal{L}_{f_{u_k}} \mathcal{L}_{f_{u_{k-1}}} \dots \mathcal{L}_{f_{u_1}} g^i \;\middle|\; \begin{array}{l} k = 1,\, 2,\, \dots,\, k_1 \\ i = 1,\, 2,\, \dots,\, s \\ u_1,\, u_2,\, \dots,\, u_k \in \mathbb{R}^m \end{array} \right\}$$

  in a suitable neighborhood U of x_0;
- $\mathcal{B}_{k_0}$ is a basis for $\mathrm{d}\mathcal{O}$, where $\mathcal{O}$ is the observable space.

A. The general algorithm

1) The case $s < n$: We start by noticing that, for the target form (3), we have that for each $k \in \mathbb{N}_0$

$$\begin{aligned} \Omega_k^{\tilde{X}} &= \mathrm{span}_{\mathbb{R}}\{\pi^j M_{u_1 u_2 \dots u_k} \mathrm{d}z \wedge \mathrm{d}\Gamma \mid j = 1,\, 2,\, \dots,\, s \\ &\qquad \text{and } u_1,\, u_2,\, \dots,\, u_k \in \mathbb{R}^m\} \\ &= \mathrm{span}_{\mathbb{R}}\{(\mathrm{d}\mathcal{L}_{f_{u_l}} \dots \mathcal{L}_{f_{u_2}} \mathcal{L}_{f_{u_1}} z^j \wedge \mathrm{d}\Gamma)|_{z_0} \mid 1 \leq l \leq k \\ &\qquad j = 1,\, 2,\, \dots,\, s \text{ and } u_1,\, u_2,\, \dots,\, u_l \in \mathbb{R}^m\} \end{aligned}$$

where, rewritting the matrix A_u as a block matrix $[\hat{A}_u \;\; \bar{A}_u] := A_u$ with $\hat{A}_u \in \mathcal{M}_{n\times s}(\mathbb{R})$ and $\bar{A}_u \in \mathcal{M}_{n\times(n-s)}(\mathbb{R})$, we define $Q^s z := [z^{s+1}\, z^{s+2}\, \dots\, z^n]^\top$, and $M_{u_1 u_2 \dots u_k} := \bar{A}_{u_1} Q^s \bar{A}_{u_2} Q^s \dots \bar{A}_{u_k} Q^s$. Further $\tilde{X} := (\partial/\partial z^1,\, \partial/\partial z^2,\, \dots,\, \partial/\partial z^s)$, $\mathrm{d}z := [\mathrm{d}z^1\, \mathrm{d}z^2\, \dots\, \mathrm{d}z^n]^\top$, and π^j is the row matrix (projection onto the j-th coordinate) $\begin{bmatrix}\delta_j^1 & \delta_j^2 & \dots & \delta_j^n\end{bmatrix}$. Then given a sequence $\mathcal{S}$ as in Lemma 3.1 we find, for each $k \in \{1,\, 2,\, \dots,\, k_0\}$:

$$\begin{aligned} &\mathrm{span}_{\mathbb{R}}\{\pi^j M_{u_1 u_2 \dots u_k} \mathrm{d}z \wedge \mathrm{d}\Gamma \mid j = 1,\, 2,\, \dots,\, s \\ &\qquad \text{and } u_1,\, u_2,\, \dots,\, u_k \in \mathbb{R}^m\} \\ &= \mathrm{span}_{\mathbb{R}}\{(\mathrm{d}\mathcal{L}_{f_{u_l}} \dots \mathcal{L}_{f_{u_2}} \mathcal{L}_{f_{u_1}} z^j \wedge \mathrm{d}\Gamma)|_{z_0} \mid 1 \leq l \leq k, \\ &\qquad j = 1,\, 2,\, \dots,\, s, \text{ and } (j,\, u_1,\, u_2,\, \dots,\, u_l) \in \mathcal{S}_l\} \\ &= \mathrm{span}_{\mathbb{R}}\{\pi^j M_{u_1 u_2 \dots u_l} \mathrm{d}z \wedge \mathrm{d}\Gamma \mid 1 \leq l \leq k, \\ &\qquad j = 1,\, 2,\, \dots,\, s, \text{ and } (j,\, u_1,\, u_2,\, \dots,\, u_l) \in \mathcal{S}_l\}. \end{aligned} \tag{11}$$

To find a s-tuple of vector fields satisfying the conditions of Theorem 2.1 for system (5), and supposing that such s-tuple exists, we may proceed as follows: first of all, for a s-tuple of vector fields $X = (X_1,\, X_2,\, \dots,\, X_s)$, define recursively the following $(s+1)$-forms

$$\mathcal{I}^X_{(r,\, u_1)} = \mathrm{d}\mathcal{L}_{f_{u_1}} g^r \wedge \mathrm{d}\Gamma$$

for all $r \in \{1,\, 2,\, \dots,\, s\}$ and $u_1 \in \mathbb{R}^m$; and

$$\mathcal{I}^X_{(r,\, u_1,\, u_2,\, \dots,\, u_{k-1},\, u_k)} = (\mathcal{L}_{f_{u_k}} \iota_X \mathcal{I}^X_{(r,\, u_1,\, u_2,\, \dots,\, u_{k-1})}) \wedge \mathrm{d}\Gamma$$

for all $r \in \{1, 2, \ldots, s\}$, $k \geq 2$ and $u_1, u_2, \ldots, u_k \in \mathbb{R}^m$.

Suppose that the system (9), auxiliary to system (5), is observable in the rank sense at x_0, and fix a sequence $\mathcal{S}$ as in Lemma 3.1. We look for a s-tuple $X = (X_1, X_2, \ldots, X_s)$ of vector fields, with $X_i = \sum_{j=1}^n X_i^j \partial/\partial x^j$ solving, step by step, the following conditions

1. $\mathrm{d}g^j(X_i) = \delta_i^j$ for all $i, j \in \{1, 2, \ldots, s\}$;
2. successively for $1 \leq k \leq k_0$:
   a. for all $j \in \{1, 2, \ldots, s\}$ and $v_1, v_2, \ldots, v_k \in \mathbb{R}^m$, $\mathcal{I}^X_{(j, v_1, v_2, \ldots, v_k)} \in \mathrm{span}_{\mathbb{R}}\{\mathcal{I}^X_{(r, u_1, u_2, \ldots, u_l)} \mid l \in \{1, 2, \ldots, k\}$ and $(r, u_1, u_2, \ldots, u_l) \in \mathcal{S}_l\}$;
   b. for all $(r, u_1, u_2, \ldots, u_k) \in \mathcal{S}_k$, $$\mathrm{d}\iota_X \mathcal{I}^X_{(r, u_1, u_2, \ldots, u_k)} = 0;$$
3. for all $(j, u_1, u_2, \ldots, u_{k_0}) \in \mathcal{S}_{k_0}$ and $v \in \mathbb{R}^m$, $\mathcal{I}^X_{(j, u_1, u_2, \ldots, u_{k_0}, v)} \in \mathrm{span}_{\mathbb{R}}\{\mathcal{I}^X_{(r, u_1, u_2, \ldots, u_l)} \mid l \in \{1, 2, \ldots, k_0\}$ and $(r, u_1, u_2, \ldots, u_l) \in \mathcal{S}_l\}$;
4. for all $j \in \{1, 2, \ldots, s\}$, $k \in \{1, 2, \ldots, k_0\}$ and $(r, u_1, u_2, \ldots, u_k) \in \mathcal{S}_k$, both $(\mathrm{d}\iota_{f_u}\mathrm{d}g^j) \wedge \mathrm{d}\Upsilon$ and $(\mathrm{d}\iota_{f_u}\iota_X \mathcal{I}^X_{(r, u_1, u_2, \ldots, u_k)}) \wedge \mathrm{d}\Upsilon$ are elements of the real vector space spanned by
$$\begin{aligned}&\{\mathrm{d}g^i \wedge \mathrm{d}\Upsilon \mid i = 1, 2, \ldots, s\}\\ &\cup \Big\{(\iota_X \mathcal{I}^X_{(r, u_1, u_2, \ldots, u_k)}) \wedge \mathrm{d}\Upsilon \mid k \in \{1, 2, \ldots, k_0\}\\ &\qquad \text{and } (r, u_1, u_2, \ldots, u_k) \in \mathcal{S}_k\Big\}.\end{aligned}$$

It now follows from the way the s-tuple X is computed that it will satisfy the conditions of Theorem 2.1 if, and only if, it results from the algorithm. The solution for X is not necessarily unique but we only have to select one. Note that by (11), the intrinsic condition in step 2.a of the algorithm holds for a system in target form.

2) The case $s = n$: Let system (9) be observable in the rank sense. From Theorem 2.1, the system (5) can be rewritten in target form if, and only if, a holds together with $\mathrm{d}\mathcal{L}_{f_u}\mathrm{d}g^i \wedge \mathrm{d}\Upsilon \in \mathrm{span}_{\mathbb{R}}\{\mathrm{d}g\} \wedge \mathrm{d}\Upsilon$, for all $i = 1, 2, \ldots, n$.

B. A particular algorithm

Resorting to the ideas presented in [14] that only considers the explicit single-output case, we now present for the implicit multi-output case a more elegant algorithm to find the s-tuple of vector fields for a class of systems for which there exists a special sequence among those described in Lemma 3.1.

Proposition 3.2: Let system (9) be observable in the rank sense at x_0 with $s < n$. Denote $\{\mathrm{d}g\} := \{\mathrm{d}g^j \mid j = 1, 2, \ldots, s\}$. Suppose that we can construct a length-k_0 sequence $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{k_0}$ satisfying the properties in Lemma 3.1 and such that

$$\begin{array}{l}\textit{there exist } v_1, v_2, \ldots, v_{k_0} \in \mathbb{R}^m \textit{ with}\\ (i, v_1, v_2, \ldots, v_{k_0}) \in \mathcal{S}_{k_0} \textit{ for all } i \in \{1, 2, \ldots, s\}.\end{array} \tag{12}$$

Then, for each $i \in \{1, 2, \ldots, s\}$, define the vector field Y_i by the equations

- $\mathcal{L}_{Y_i} g^j = 0$ for all $j \in \{1, 2, \ldots, s\}$,
- $\mathcal{L}_{Y_i}\mathcal{L}_{f_{u_k}}\mathcal{L}_{f_{u_{k-1}}} \ldots \mathcal{L}_{f_{u_1}} g^j = 0$ for all $\left\{\begin{array}{l} k = 1, 2, \ldots, k_0 - 1\\ (j, u_1, \ldots, u_{k-1}, u_k) \in \mathcal{S}_k\end{array}\right.$,
- $\mathcal{L}_{Y_i}\mathcal{L}_{f_{u_{k_0}}}\mathcal{L}_{f_{u_{k_0-1}}} \ldots \mathcal{L}_{f_{u_1}} g^j = 0$ for all $\left\{\begin{array}{l} \tau_{j,1,k_0} := (j, u_1, \ldots, u_{k_0-1}, u_{k_0}) \in \mathcal{S}_{k_0}\\ \tau_{j,1,k_0} \neq (i, v_1, v_2, \ldots, v_{k_0})\end{array}\right.$,
- $\mathcal{L}_{Y_i}\mathcal{L}_{f_{v_{k_0}}}\mathcal{L}_{f_{v_{k_0-1}}} \ldots \mathcal{L}_{f_{v_1}} g^i = 1$.

Finally, for X_i set the iterated Lie bracket
$$X_i := (-1)^{k_0}[f_{v_1}, [f_{v_2}, [\ldots[f_{v_{k_0-1}}, [f_{v_{k_0}}, Y_i]]\ldots]]].$$

Then, system (5) can be rewritten in the target form (3) if, and only if, the s-tuple of vector fields $X = (X_1, X_2, \ldots, X_s)$ satisfy the conditions in Theorem 2.1.

Proof: [Outline] Following a similar reasoning as in [14] we can prove that for a system in target form in coordinates z, the proposed algorithm yields constant vector fields $X_i = \partial/\partial z^i + \sum_{j=s+1}^n C_i^j \partial/\partial z^j$ for each $i = 1, 2, \ldots, s$. Then, we can easily find the change of coordinates $z^i = w^i$ for $i = 1, 2, \ldots, s$ and $z^j = w^j + \sum_{i=1}^s C_i^j w^i$ for $j = s+1, s+2, \ldots, n$. Note that, by (6), we find $\partial/\partial z^j = \partial/\partial w^j$ for $j = s+1, s+2, \ldots, n$ and $\partial/\partial z^i = \partial/\partial w^i + \sum_{j=s+1}^n C_i^j \partial/\partial w^j$ for $i = 1, 2, \ldots, s$. Thus, we can conclude that (3a), $F_u(z) = \sum_{j=1}^n \left(\sum_{i=1}^n A_u^{ji} z^i + \Phi_u^j(y)\right) \partial/\partial z^j$, in the new coordinates reads $F_u(w) = \sum_{j=1}^n \left(\sum_{i=1}^n A_u^{ji} M w + \Phi_u^j(y)\right) N[\partial/\partial w^1\, \partial/\partial w^2 \ldots \partial/\partial w^n]^\top$ for suitable matrices M and N, i.e., $F_u(w) = \widetilde{A}_u w + \tilde{\Phi}(y)$ for a suitable matrix $\widetilde{A}_u$ and a suitable function $\tilde{\Phi}$. In other words, the system is still in target form when re-written in the new coordinates $(w^1, w^2, \ldots, w^n)$. Moreover, we have the identity $X_i = \partial/\partial w^i$ and, from the proof of necessity in Theorem 2.1, we have that the s-tuple X satisfies the conditions of Theorem 2.1 in the coordinates $(w^1, w^2, \ldots, w^n)$. Since the conditions are intrinsic, they are also satisfied by the same s-tuple in the coordinates $(z^1, z^2, \ldots, z^n)$. This means that for a system in target form, the s-tuple defined in the Proposition satisfies the conditions in Theorem 2.1. Since the s-tuple is defined intrinsically, it will satisfy those conditions in any coordinates. ∎

Remark 3.1: This idea to construct the sequence $(\mathcal{S}_k)$, in Lemma 3.1, and the vector fields Y_i; $i = 1, \ldots, s$ is borrowed from [14]. We only have to perform some adjustments to the multi-output case. In fact those adjustments imply that for a system in target form we have $\mathcal{L}_{(-1)^{k_0}X_i} g^j = (-1)^{k_0}\mathcal{L}_{Y_i}\mathcal{L}_{f_{v_{k_0}}}\mathcal{L}_{f_{v_{k_0-1}}} \ldots \mathcal{L}_{f_{v_2}}\mathcal{L}_{f_{v_1}} g^j$, from which we can conclude that $\mathcal{L}_{X_i} g^j = (-1)^{2k_0}\delta_i^j = \delta_i^j$. To check this we may use the identity (8) and the definitions of Y_i and $(\mathcal{S}_k)$.

Remark 3.2: Comparing the single-output case with the multi-output one we have the new condition (12), that is always satisfied in the single-output case. We do not know whether the condition (12) can be weakened or not. The difficulty, or perhaps impossibility, to construct an analogous algorithm to the one in [14], without a restriction like (12), for the multi-output case may be related to some important differences between this case and the single-output case referred in [12], [13]; namely, the fact that in the multi-output case there are, in general, nonlinear changes of coordinates that preserve the target form.

Remark 3.3: The way to write the output equations in form (5b) is not unique. For example, $x^1y = x^2$ in (4), may be rewritten as $y = {x^2}/{x^1}$ or $x^1x^1y = x^1x^2$. This leads to different choices of candidates g to the first s coordinate functions $P^sz = g(x)$: respectively, $g = [g^1 \;\; g^2]^\top = x = [x^1 \;\; x^2]^\top$, $g = g^1 = {x^2}/{x^1}$, and $g = [g^1 \;\; g^2]^\top = [x^1x^1 \;\; x^1x^2]^\top$. Not all choices are appropriate: the latter two will "destroy" the target form of (4), in a neighborhood of $x_0 = [1 \; 0]^\top$ (cf. [20]). The algorithms will tell us if a given choice is appropriate or not.

IV. Illustrative Examples

This section illustrates the (new) particular algorithm, proposed in Section III-B. We discuss also the differences from the general one, proposed in Section III-A.

A. Illustration of the particular algorithm

Consider the system

$$\dot{x} = f_u(x) = \begin{bmatrix} x^1(u+x^3) \\ -x^1(u+x^3) + x^1x^4 + 1 + {x^2}/{x^1} \\ -x^3(u+x^3) + 1 + {(x^2+u)}/{x^1} \\ -x^4(u+x^3) + x^3 \end{bmatrix} \tag{13a}$$

$$x^1y = x^1 + x^2 \tag{13b}$$

where $x = [x^1 \;\; x^2 \;\; x^3 \;\; x^4]^\top$ is the state space, $u \in \mathbb{R}$ is the input and $y = y^1$ is the output. Our aim is to write this system in the form (3) in a neighborhood of the point $x_0 = [1 \;\; 0 \;\; 0 \;\; 0]^\top$. The output equation can be written as $\begin{bmatrix}0 & -1\end{bmatrix} g(x) + g(x)^\top \left[\begin{bmatrix}1\\0\end{bmatrix}\right] y = 0$, with $g = [g^1 \;\; g^2]^\top := [x^1 \;\; x^1{+}x^2]^\top$, which is in the form of equation (5b). In this way we obtain the candidates (g^1, g^2) for the first two new coordinates. In this case, $\mathrm{d}\Gamma := \mathrm{d}g^1 \wedge \mathrm{d}g^2 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \neq 0$ and $\mathrm{d}\Upsilon = \mathrm{d}y = \frac{1}{x^1}(\mathrm{d}x^2 + (1-y)\mathrm{d}x^1)$. We also have $\mathrm{d}y \wedge \mathrm{d}\Gamma = 0$. Next, we must check if the auxiliary system $\dot{x} = f_u(x), \quad \bar{y} = g = \begin{bmatrix} x^1 \\ x^1 + x^2 \end{bmatrix}$, is observable in the rank sense. For that we notice that $S := \{g^1, g^2, \mathcal{L}_{f_0}g^1, \mathcal{L}_{f_0}g^2\} = \{x^1, x^1{+}x^2, x^1x^3, x^1x^4 + 1 + {x^2}/{x^1}\}$ is a subset of the observable space $\mathcal{O}$ and after some straightforward computations we find that $\mathrm{d}\mathcal{L}_{f_0}g^1 = x^3\mathrm{d}x^1 + x^1\mathrm{d}x^3$, $\mathrm{d}\mathcal{L}_{f_0}g^2 = (x^4 - {x^2}/{(x^1)^2})\mathrm{d}x^1 + {1}/{x^1}\mathrm{d}x^2 + x^1\mathrm{d}x^4$ and $\mathrm{d}S|_{x_0} := \{\mathrm{d}x^1, \mathrm{d}x^1 + \mathrm{d}x^2, \mathrm{d}x^3, \mathrm{d}x^2 + \mathrm{d}x^4\}$, which has rank 4. Therefore, the system is observable in the rank sense. Notice also that the length-1 sequence $\mathcal{S}_1 = \{(1,0), (2,0)\} \subset \{1, 2\} \times \mathbb{R}$ satisfies the condition in Proposition 3.2. Thus, we are now ready to follow the algorithm described in that proposition: firstly, we must compute the pair of vector fields $Y = (Y_1, Y_2)$; writing $Y_1 = \sum_{j=1}^4 Y_1^j \partial/\partial x^j$ and $Y_2 = \sum_{j=1}^4 Y_2^j \partial/\partial x^j$, and following the algorithm, for the vector field Y_1 we find the equations

$$\begin{aligned} &Y_1^1 = 0, && x^3Y_1^1 + x^1Y_1^3 = 1, \\ &Y_1^1 + Y_1^2 = 0, && (x^4 - {x^2}/{(x^1)^2})Y_1^1 + {1}/{x^1}Y_1^2 + x^1Y_1^4 = 0, \end{aligned}$$

from which we derive $Y_1 = {1}/{x_1}\partial/\partial x^3$. For Y_2 we obtain

$$\begin{aligned} &Y_2^1 = 0, && x^3Y_2^1 + x^1Y_2^3 = 0, \\ &Y_2^1 + Y_2^2 = 0, && (x^4 - {x^2}/{(x^1)^2})Y_2^1 + {1}/{x^1}Y_2^2 + x^1Y_2^4 = 1, \end{aligned}$$

that is, $Y_2 = {1}/{x_1}\partial/\partial x^4$. Next, we compute the pair $X = (X_1, X_2)$ with $X_i = -[f_0, Y_i] = [Y_i, f_0]$. In coordinates this may be computed by the formula (7): from $f_0(x) = x^1x^3\partial/\partial x^1 + (x^1x^4 - x^1x^3 + 1 + {x^2}/{x^1})\partial/\partial x^2 + (1 - (x^3)^2 + {x^2}/{x^1})\partial/\partial x^3 + (x^3 - x^3x^4)\partial/\partial x^4$ we obtain

$$\begin{aligned} &X_1^1 = {1}/{x^1}(x^1) = 1, && X_1^3 = {-2x^3}/{x^1} + {x^1x^3}/{(x^1)^2}, \\ &X_1^2 = {1}/{x^1}(-x^1) = -1, && X_1^4 = {1}/{x^1}(1 - x^4) = {(1-x^4)}/{x^1}, \end{aligned}$$

and

$$\begin{aligned} &X_2^1 = 0, && X_2^2 = {1}/{x^1}(x^1) = 1, \\ &X_2^3 = 0, && X_1^4 = {1}/{x^1}(-x^3) - [x^1x^3(-{1}/{(x^1)^2})] = 0. \end{aligned}$$

Thus, $X_1 = \partial/\partial x^1 - \partial/\partial x^2 - \frac{x^3}{x^1}\partial/\partial x^3 + \frac{1-x^4}{x^1}\partial/\partial x^4$ and $X_2 = \partial/\partial x^2$. Now, to obtain the change of coordinates, we compute $\iota_X \mathrm{d}\mathcal{L}_{f_0}g^1 \wedge \mathrm{d}\Gamma$ and $\iota_X \mathrm{d}\mathcal{L}_{f_0}g^2 \wedge \mathrm{d}\Gamma$; from $\mathrm{d}\mathcal{L}_{f_0}g^1 \wedge \mathrm{d}\Gamma = x^1\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, $\mathrm{d}\mathcal{L}_{f_0}g^2 \wedge \mathrm{d}\Gamma = x^1\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^4$, $\det \begin{bmatrix} 1 & 0 & V^1 \\ -1 & 1 & V^2 \\ -{x^3}/{x^1} & 0 & V^3 \end{bmatrix} = {x^3}/{x^1}V^1 + V^3$, and $\det \begin{bmatrix} 1 & 0 & V^1 \\ -1 & 1 & V^2 \\ {(1-x^4)}/{x^1} & 0 & V^4 \end{bmatrix} = {(x^4-1)}/{x^1}V^1 + V^4$ we obtain $\iota_X(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3) = {x^3}/{x^1}\mathrm{d}x^1 + \mathrm{d}x^3$ and $\iota_X(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^4) = {(x^4-1)}/{x^1}\mathrm{d}x^1 + \mathrm{d}x^4$. Therefore we obtain, $\iota_X \mathrm{d}\mathcal{L}_{f_0}g^1 \wedge \mathrm{d}\Gamma = x^3\mathrm{d}x^1 + x^1\mathrm{d}x^3 = \mathrm{d}(x^1x^3)$ and $\iota_X \mathrm{d}\mathcal{L}_{f_0}g^2 \wedge \mathrm{d}\Gamma = (x^4-1)\mathrm{d}x^1 + x^1\mathrm{d}x^4 = \mathrm{d}(x^1x^4) - \mathrm{d}x^1$.

We can now construct a coordinate transformation (z^1, z^2, z^3, z^4) with $z^1 = g^1 = x^1$, $z^2 = g^2 = x^1 + x^2$ and z^3 and z^4 such that $\mathrm{d}z^3$ and $\mathrm{d}z^4$ are in the real vector space spanned by the 1-forms in $\{\mathrm{d}z^1, \mathrm{d}z^2, \iota_X \mathrm{d}\mathcal{L}_{f_0}g^1 \wedge \mathrm{d}\Gamma, \iota_X \mathrm{d}\mathcal{L}_{f_0}g^2 \wedge \mathrm{d}\Gamma\}$ and $\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 \wedge \mathrm{d}z^4 \neq 0$ (cf. Remark 2.1). We may set for example $z^3 = x^1x^3$ and $z^4 = x^1x^4$. We can now confirm that applying this coordinate transformation, the original system can be rewritten in the target form. A quick check of this fact can be done, using (6), by noticing that

$$\partial/\partial x^1 = \partial/\partial z^1 + \partial/\partial z^2 + \frac{z^3}{z^1}\partial/\partial z^3 + \frac{z^4}{z^1}\partial/\partial z^4,$$

$$\partial/\partial x^2 = \partial/\partial z^2, \quad \partial/\partial x^3 = z^1\partial/\partial z^3, \text{ and } \partial/\partial x^4 = z^1\partial/\partial z^4;$$

the vector field

$$\begin{aligned} f_u = {} & x^1\left(u + x^3\right)\partial/\partial x^1 + \left(-x^4(u+x^3) + x^3\right)\partial/\partial x^4 \\ & + \left(-x^1(u+x^3) + x^1x^4 + 1 + {x^2}/{x^1}\right)\partial/\partial x^2 \\ & + \left(-x^3(u+x^3) + 1 + {(x^2+u)}/{x^1}\right)\partial/\partial x^3 \end{aligned}$$

can be rewritten as

$$\begin{aligned} f_u = {} & x^1\left(u + x^3\right)\partial/\partial z^1 \\ & + \left(x^1(u+x^3){z^4}/{z^1} + (-x^4(u+x^3) + x^3)z^1\right)\partial/\partial z^4 \\ & + \left(x^1(u+x^3) - x^1(u+x^3) + x^1x^4 + 1 + {x^2}/{x^1}\right)\partial/\partial z^2 \\ & + \Big(x^1(u+x^3){z^3}/{z^1} + (-x^3(u+x^3) \\ & \qquad + 1 + {(x^2+u)}/{x^1})z^1\Big)\partial/\partial z^3 \end{aligned}$$

that is,

$$
\begin{aligned}
f_u &= \left(z^3 + uz^1\right) \partial/\partial z^1 + \left(z^4 + {}^{(x^2+x^1)}/_{x^1}\right) \partial/\partial z^2 \\
&\quad + \left((u + x^3)z^4 - x^4(u + x^3)z^1 + x^3 z^1\right) \partial/\partial z^4 \\
&\quad + \left((u + x^3)(z^3 - x^3 z^1) + z^1(1 + {}^{(x^2+u)}/_{x^1})\right)\partial/\partial z^3 \\
&= (z^3 + uz^1)\partial/\partial z^1 + (z^4 + y)\partial/\partial z^2 \\
&\quad + (z^2 + u)\partial/\partial z^3 + z^3\partial/\partial z^4.
\end{aligned}
$$

Therefore, we obtain the following system, in target form:

$$
\dot{z} = \begin{bmatrix} u & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ y \\ u \\ 0 \end{bmatrix}, \quad 0 = \begin{bmatrix} 0 & -1 \end{bmatrix} P^2 z + (P^2 z)^\top \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right] y.
$$

B. A system that is not suitable for the particular algorithm

Consider the following system with state $x = [x^1\ x^2\ x^3]^\top$, output $y = [y^1\, y^2]^\top$ and input $u \in \mathbb{R}$,

$$
\dot{x} = f_u(x) = \begin{bmatrix} \mathrm{e}^{-x^1}(x^3 + u) \\ x^3 + (x^1 + x^2 - \mathrm{e}^{x^1})^2 - \mathrm{e}^{-x^1}(x^3 + u) \\ x^3 + \sin(x^1 + x^2 - \mathrm{e}^{x^1}) \end{bmatrix}
$$

$$
y^1 = x^1 + x^2 - \mathrm{e}^{x^1}, \quad y^2 = \mathrm{e}^{-x^1}(x^1 + x^2).
$$

Our aim is to write this system in the form (3) in a neighborhood $\mathcal{N}$ of the point $x_0 = [0\ 0\ 0]^\top$. Rewriting the output equations as $\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} g + g^\top \mathrm{col}_{\mathcal{M}}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) y + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} y = 0$, with $g = \begin{bmatrix} g^1 & g^2 \end{bmatrix}^\top = \begin{bmatrix} \mathrm{e}^{x^1} & x^1 + x^2 \end{bmatrix}^\top$, we cannot apply the particular algorithm described in Section 3.2 because, as we can see, there is no sequence satisfying both the conditions in Lemma 3.1 and (12). We must apply the general algorithm: following its steps, as in [20], we will end up with a set of equations that characterize the pairs $X = (X_1,\, X_2)$ that lead to a desired change of coordinates, namely

$$
\begin{aligned}
&X_1 = \mathrm{e}^{-x^1}\partial/\partial x^1 - \mathrm{e}^{-x^1}\partial/\partial x^2 + X_1^3 \partial/\partial x^3, \\
&X_2 = \partial/\partial x^2 + X_2^3 \partial/\partial x^3,
\end{aligned} \tag{14a}
$$

$$
\partial X_2^3/\partial x^1 + (1 - \mathrm{e}^{x^1})\partial X_1^3/\partial x^2 = 0, \tag{14b}
$$

$$
\mathrm{d}(X_1^3 + X_2^3) = 0, \text{ and } \partial X_1^3/\partial x^3 = \partial X_2^3/\partial x^3 = 0. \tag{14c}
$$

We can see that, in this case, the solution exists and is not unique. Now we can set one of the solutions and built up the desired new coordinates. Following [20], if we choose the solution $X_1 = \mathrm{e}^{-x^1}\partial/\partial x^1 - \mathrm{e}^{-x^1}\partial/\partial x^2$, $X_2 = \partial/\partial x^2$, we can find the coordinates $(z^1,\, z^2,\, z^3) := (\mathrm{e}^{x^1},\, x^1 + x^2,\, x^3)$, in which the system takes the target form.

Remark 4.1: The equations in (14c) imply that $X_2^3 = \phi(x^1,\, x^2) = -X_1^3 + c$, with $c \in \mathbb{R}$; then (14b) just means that $\mathrm{d}\phi \wedge \mathrm{d}y^1 = 0$, which implies that $\phi = \phi(y^1)$ in a neighborhood of x_0. Thus, the solutions X are defined by the conditions: (14a), $X_2^3 = \phi(y^1) = -X_1^3 + c$, $c \in \mathbb{R}$ and $\phi \in C^\infty(I)$, for some neighborhood $I \subseteq \mathbb{R}$ of $-1 = y^1|_{x_0}$.

V. Concluding Remarks

We have proposed an algorithm to find a local change of coordinates that transforms a suitable nonlinear system into a time-varying linear system up to output and input injection. The procedure is illustrated with an example. It is important to underline that for any system written in the target form (3), there exists an observer (Kalman-like) that exhibits in the new coordinate system linear error dynamics.

Like the general algorithm proposed in [20], the new one propose here also aims to find a suitable s-tuple of vector fields from which we can obtain the desired change of coordinates. The major difference is that while the algorithm in [20] leads to a set of equations that the suitable s-tuples must satisfy, the new one gives us one of those s-tuples directly.

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