OPTIMAL DESIGN OF DISTRIBUTED SENSOR NETWORKS FOR FIELD RECONSTRUCTION

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ABSTRACT

The paper introduces a method by which to design the topology of a distributed sensor network that is minimal with respect to a communication cost function. In the scenario considered, sensor nodes communicate with each other within a graph structure to update their data according to linear dynamics using neighbor node data. A subset of sensors can also report their state to a central location. One physical interpretation of this situation would be a set of spatially distributed wireless sensors which can communicate with other sensors within range to update data and can possibly connect to a network backbone. The costs would then be related to transmission energy. The objective is to recover the vector of initial sensor measurements from the backbone outputs over time, which requires that the dynamics of the overall networked system be observable. The topology of the network is then determined by the nonzero elements of the optimal observable dynamics. The following text contributes an efficient algorithm for designing the optimal observable dynamics and the network topology for a given set of sensors and cost function, providing proof of correctness and example implementation.

Index Terms— distributed sensor network, structural observability, optimal design, field reconstruction, output cacti

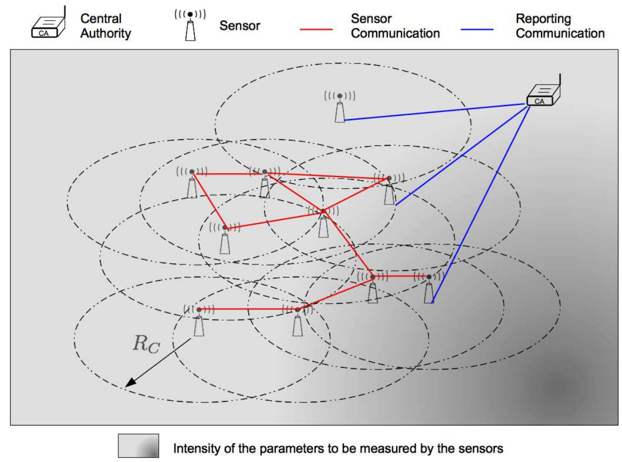

Fig. 1. The figure depicts a set of sensors deployed over a geographic area in which each sensor is equipped with a short range communication device (operating within a radius R_C) for local sensor-to-sensor communication. Some of the sensors are also equipped with long range communication devices to report their respective states to a central authority.

1. INTRODUCTION

Numerous applications including field surveillance, environmental study, and geo-scientific exploration [1] employ distributed sensor networks for wide area monitoring, leading to interesting topology and network configuration design problems that involve power, bandwidth, and range constraints [2]. These networks typically consist of a set of sensor nodes, each of which has a microprocessor that allows simple information processing and a transceiver that enables short distance communication. This paper examines situations in which some sensors can also potentially be equipped with long range transmission capability in order to communicate with a central reporting authority over the sensor network.

Consider sensor nodes communicating with each other within a graph structure to update their data according to linear dynamics using neighbor node data. Each sensor maintains a state x_n^i, where n is the iteration index and the initial state x_0^i corresponds to the field measurement collected by the ith sensor. A subset of sensors can also report their state to a central location or backbone node. Denote by $\mathbf{x}_n$ the vector of sensor states and $\mathbf{y}_n$ the reported outputs at iteration n. Thus, network communication is described by a dynamic system of the form

$$\mathbf{x}_{n+1} = A\mathbf{x}_n, \quad (1)$$
$$\mathbf{y}_n = C\mathbf{x}_n, \quad (2)$$

where A describes the state update dynamics, which respects the communication graph structure, and C encodes the backbone reporting infrastructure.

The network objective is to determine the vector of initial sensor states $\mathbf{x}_0$, the field measurements, at the central reporting authority based on the outputs $(\mathbf{y}_0, \cdots, \mathbf{y}_{N-1})$ obtained over time, where N is the number of sensors in the network. In order for this to be achieved, observable dynamics (A, C) from (1)-(2) are desired. Such a network is said to be an observable distributed sensor network.

The notion of an observable distributed sensor network has already been introduced in [3], where necessary conditions to ensure observability of the network are given in terms

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of graph partitions, though without providing or justifying the complexity of verifying such conditions. In contrast, this paper investigates the problem of optimal network topology design for attaining observable dynamics that incur minimum cost with respect to a generic communication cost function precisely defined in section 3. Also note that in contrast to the typical problems in consensus or gossip algorithms literature where the goal is to asymptotically recover the average or a specific linear function of the initial sensor states, as in [4], the observable dynamics discussed can be used to recover the entire collection of the initial sensor states in finite time.

The paper proposes an algorithm to efficiently compute the optimal observable dynamic structure, proving the correctness of the algorithm and demonstrating the algorithm in implementation. Section 2 introduces some preliminary concepts and terminology. Section 3 presents the problem formulation in greater detail, and section 4 follows with the proof of the main results. Section 5 illustrates concepts with simulation results. Finally, section 6 concludes the paper.

2. BACKGROUND CONCEPTS

This section introduces elementary concepts in graph theory, and structural systems [5]. The paper requires the following standard graph theoretic terminology [6].

A graph $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{E})$ in which $\mathcal{V}$ denotes a set of vertices and $\mathcal{E}$ represents a set of edges, such that the unordered pair (v_i, v_j) of vertices $v_i, v_j \in \mathcal{V}$ represents an undirected link between vertex v_i and vertex v_j. Any graph $\mathcal{G}_s(\mathcal{V}_s, \mathcal{E}_s)$ with $\mathcal{V}_s \subset \mathcal{V}$ and $\mathcal{E}_s \subset \mathcal{E}$ is called a subgraph of $\mathcal{G}(\mathcal{V}, \mathcal{E})$. If $\mathcal{V}_s = \mathcal{V}$, $\mathcal{G}_s$ is said to span $\mathcal{G}$. A sequence of edges $\{(v_1, v_2), (v_2, v_3), \cdots, (v_{k-1}, v_k)\}$, in which all the vertices are distinct, is called an elementary path from v_1 to v_k. When v_k coincides with v_1, the sequence is called a cycle. A graph is said to be connected if there exists an elementary path between any two vertices. A tree is a connected graph with no cycles. Finally, a weight graph $\mathcal{G}_w$ is a triple $(\mathcal{V}, \mathcal{E}, w)$ consisting of an undirected graph $(\mathcal{V}, \mathcal{E})$ and an edge weight function $w : \mathcal{E} \mapsto \mathbb{R}$ that assigns a weight $w(e)$ to an edge $e \in \mathcal{E}$. For brevity, the notation $(\mathcal{V}_w, \mathcal{E}_w)$ will be used often to denote the weighted graph $(\mathcal{V}, \mathcal{E}, w)$. Directed graphs, also known as digraphs, may be defined in a similar fashion. A directed graph $\mathcal{D}$ is a pair $(\mathcal{V}, \mathcal{E})$ in which $\mathcal{V}$ denotes the set of vertices and $\mathcal{E}$ represents the set of directed edges, such that the ordered pair $(v_i, v_j) \in \mathcal{E}$ of vertices $v_i, v_j \in \mathcal{V}$ represents a directed link from v_i to v_j.

2.1. Minimum Weight Spanning Tree

Key to the algorithm presented in section 4 is the concept of a minimum weight spanning tree. Let a connected, undirected, weight graph $\mathcal{G}_w = (\mathcal{V}_w, \mathcal{E}_w)$ be given, where the weight function $w : \mathcal{E}_w \to \mathbb{R}$ assigns a weight $w(e)$ to each edge $e \in \mathcal{E}$. The minimum weight spanning tree problem consists of finding a spanning connected subgraph of $\mathcal{G}_w$ with $|\mathcal{V}| - 1$ edges, a tree $\mathcal{T} = (\mathcal{V}_w, \mathcal{E}_\mathcal{T})$, such that the following function is minimized:

$$w(\mathcal{T}) = \sum_{e \in \mathcal{E}_\mathcal{T}} w(e).$$

The solution to the minimum weight spanning tree problem can be found by resorting to any of several efficient algorithms, such as those of Kruskal, Prim, or Boruvka [6]. Prim's algorithm is used in this paper, and it has complexity $\mathcal{O}(|\mathcal{V}_w|^2)$ or $\mathcal{O}(|\mathcal{E}_w| \log(|\mathcal{V}_w|))$ depending on the choice of the implementation.

2.2. Structural Dynamic Systems

Consider the system in (1)-(2) and denote the state variables and output variables by $\mathcal{X} = \{x^1, \cdots, x^N\}$ and $\mathcal{Y} = \{y^1, \cdots, y^M\}$, corresponding to the set of state vertices and output vertices, respectively. Let $\mathcal{E}_{\mathcal{X},\mathcal{X}} = \{(x^i, x^j) | \, [A]_{ji} \neq 0\}$ and $\mathcal{E}_{\mathcal{X},\mathcal{Y}} = \{(x^i, y^j) | \, [C]_{ji} \neq 0\}$ and define the directed graph $\mathcal{D}(A, C) = (\mathcal{X} \cup \mathcal{Y}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{X},\mathcal{Y}})$.

Given a digraph $\mathcal{D}(A, C)$, define the following special subgraphs:

- *State Stem* - An elementary path exclusively composed of state vertices, or a single state vertex.
- *Output Stem* - An elementary path composed of a state stem with an output vertex (the output stem tip) linked from the tip of the state stem.
- *State Cactus* - Defined recursively as follows: A state stem is a state cactus. A state cactus connected to a cycle from any point other than the tip is also a state cactus.
- *Output Cactus* - Defined recursively as follows: An output stem with at least one state vertex is an output cactus. An output cactus connected from a cycle to any point other than the root of the state stem is also an output cactus. The root and the tip of the output stem are also the root and tip of the associated cactus.
- *Output Cactus Patch* - A disjoint union of output cacti.

Note that the definition allows an output cactus to have an output vertex linked from several state vertices. That is, the output vertex must connect from the tip of a state stem but could be linked from one or more other states in state cycles. Furthermore, consider the following definition and result:

Definition 1 (Structural Observability) *A system defined by* (1)-(2) *is said to be structurally observable if there are observable matrices* (A_1, C_1) *with identical zero-nonzero sparsity structure as the matrices* (A, C).

A pair (A, C) is said to be structurally observable if there exists a pair (A', C') with the same structure as (A, C), i.e., same locations of zeros and non-zeros, such that (A', C') is observable. By density arguments, it may be shown that if a pair (A, C) is structurally observable then, in the Lebesgue measure theoretic sense, almost all pairs with the same structure as (A, C) are observable. In essence, structural observability is a property of the structure of the pair (A, C) and not the specific numerical values.

Theorem 1 (Structural Observability and Cacti) *For a linear time invariant system described by* (1)-(2)*, the following statements are equivalent [5]:*

(i) The corresponding structured linear system (A, C) *is structurally observable.*

(ii) The digraph $\mathcal{D}(A, C)$ *is spanned by an output cactus patch.*

3. PROBLEM FORMULATION

Consider a set $\mathcal{V}$ of N indexed sensor nodes with distance D_{ij} between each pair of sensor nodes $v_i, v_j \in \mathcal{V}$. Suppose that each node $v_i \in \mathcal{V}$ can communicate with any node $v_j \in \mathcal{V}$ of distance less than a maximum communication radius R_C with transmission energy related cost $(E_{CC})^2_{ij} = \alpha D^2_{ij}$ proportional to the square of communication distance[1]. That is,

$$(E_{CC})_{ij} = \begin{cases} \alpha D^2_{ij} & D_{ij} \leq R_C \\ \infty & D_{ij} > R_C \end{cases} . \tag{3}$$

Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be the resulting communication graph where

$$\mathcal{E} = \{(v_i, v_j) \in \mathcal{V}^2 | (E_{CC})_{ij} < \infty\} \tag{4}$$

and $(E_{CC})_{ij}$ is the weight of edge (v_i, v_j). Note that the graph $\mathcal{G}$ is not necessarily connected, and allows self loops of weight 0.

Sensors within range are allowed to communicate with each other and update their data as a linear function of data from their neighbors defined by a matrix $A \in \mathbb{R}^{N\times N}$. If the coefficient corresponding to a neighbor is 0, then no communication need take place, incurring no cost. It is clear that for A to be a feasible matrix, it must satisfy $A_{ij} = 0$ if $(v_i, v_j) \notin \mathcal{E}$. Let $\mathbf{x}_0$ be the vector of sensor readings and $\mathbf{x}_n$ be the vector of data at the nodes at update time step n. Thus, the node update dynamics are described by (1).

Furthermore, suppose that a subset $\mathcal{U} \subseteq \mathcal{V}$ of the nodes can be connected to the sensor network backbone node b with cost given by

$$(E_{BC})_{ii} = \beta D^2_{bi}, \tag{5}$$

for node $v_i \in \mathcal{U}$ where D_{bi} is the distance between v_i and b. For notational convenience, let $(E_{BC})_{ij} = \infty$ when $i \neq j$ or when $v_i \notin \mathcal{U}$. Define an output matrix $C \in \mathbb{R}^{N\times N}$ satisfying $C_{ij} \neq 0$ only if $i = j$ and $v_i \in U$. Let $\mathbf{y}_n$ be the output vector of reported states at time n. Then the backbone reporting function is described by (2).

Thus, total cost associated with a pair (A, C) is given by

$$f(A, C) = \mathbf{1}^T(\tilde{A} \circ E_{CC})\mathbf{1} + \mathbf{1}^T(\tilde{C} \circ E_{BC})\mathbf{1} \tag{6}$$

where

$$\tilde{A}_{ij} = \begin{cases} 0 & A_{ij} = 0 \\ 1 & A_{ij} \neq 0 \end{cases}, \quad \tilde{C}_{ij} = \begin{cases} 0 & C_{ij} = 0 \\ 1 & C_{ij} \neq 0 \end{cases},$$

and $\circ$ is the Hadamard (entrywise) product of matrices.

Now, consider the problem of choosing a node update and backbone reporting dynamics (A, C) incurring minimum cost as calculated in (6) that is feasible with respect to the communication graph and such that the initial sensor measurement $\mathbf{x}_0$ can be inferred from N backbone output readings $(\mathbf{y}_0, ..., \mathbf{y}_{N-1})$. This means the constraint that (A, C) is observable must be imposed.

Note that the cost does not depend on the values of the dynamic matrices (A, C) but only on their sparsity structure. From Theorem 1 it is known that if $(\tilde{A}, \tilde{C})$ is structurally observable, then there exist matrices (A, C) with the same sparsity structure as $(\tilde{A}, \tilde{C})$ that are observable. Thus, replacement of the constraint that (A, C) must be observable with the constraint that $(\tilde{A}, \tilde{C})$ be structurally observable allows the minimization to be performed without considering suitable values for the dynamics, which are guaranteed to exist (see Definition 1) and can be determined later. Hence, the optimal solution will specify the network topology of minimum cost such that observable dynamics exist. For this paper, the problem is solved for arbitrary communication cost functions under the following mild assumptions:

A.1 **(Symmetric Communication Cost):** The sensor to sensor communication cost matrix E_{CC} is symmetric but otherwise arbitrary. The sensor to backbone communication cost matrix E_{BC} has arbitrary diagonal entries for the ith diagonal element if $v_i \in U$. It has infinite off-diagonal entries and infinite diagonal entries for the ith diagonal element if $v_i \notin U$.

A.2 **(Connectedness):** Consider the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ already defined in (4), and construct the augmented undirected graph $\mathcal{G}'(\mathcal{V}', \mathcal{E}')$ comprised of all possible communication links between the sensors nodes and the backbone node. Hence, $\mathcal{V}' = \mathcal{V} \cup \{b\}$ and $\mathcal{E}' = \mathcal{E} \cup \{(v, b) | v \in \mathcal{U}\}$. The augmented graph $\mathcal{G}'$ must be connected.

Note that A.2 requires that there exists a path between each node in $\mathcal{V}$ and the backbone b. Thus, A.2 does not require the sensor to sensor communication graph, $\mathcal{G}$ itself, to be connected. The problem is transformed to the following final form: given arbitrary communication costs that satisfy the assumptions in A.1 and A.2, find the arguments $(\tilde{A}, \tilde{C})$ that produce the optimal solution to

$$\min_{\tilde{A},\tilde{C} \in \{0,1\}^{N\times N}} \mathbf{1}^T(\tilde{A} \circ E_{CC})\mathbf{1} + \mathbf{1}^T(\tilde{C} \circ E_{BC})\mathbf{1} \tag{7}$$

subject to the constraint that $(\tilde{A}, \tilde{C})$ be structurally observable.

4. MAIN RESULTS

This section presents the main result of the paper, which consists of an algorithm for solving the optimization problem in (7) with $O(N^2)$ computational complexity, where N is the number of sensor nodes[2]. In particular, it provides proof of the correctness of the algorithm in Theorem 2 and its complexity in Theorem 3.

Algorithm 1 is composed of five steps, the first three of which construct the minimum spanning tree of the augmented graph $\mathcal{G}'$ and assign orientations to the tree edges, such that each edge points toward the central reporting authority. The final two steps replace the single backbone node with several output nodes and turn the resulting directed output graph into an output cactus patch. From this spanning output cactus patch, the structure $(\tilde{A}^*, \tilde{C}^*)$ is then inferred.

Algorithm 1

Input *Sensor nodes* $\mathcal{V}$*, reporting node subset* $\mathcal{U}$*, central node* b*, possible communication links* $\mathcal{E}$*, sensor to sensor communication costs* E_{CC}*, and reporting costs* E_{BC}

[1]Note that, the transmission energy related cost structure is used for illustration purposes only. Our framework allows for more general cost structures as noted in A.1.

[2]Note that naively approaching the optimization problem in (7) using an exhaustive search over all possible network topologies would lead to exponential computational complexity.

Step 1 *Form the undirected weighted graph $\mathcal{G}'(\mathcal{V}_{G'}, \mathcal{E}_{G'})$ defined in A.2 with $(E_{CC})_{ij} + (E_{CC})_{ji} = 2(E_{CC})_{ij}$ as the weight of (v_i, v_j) and $2(E_{BC})_{ii}$ as the weight of (v_i, b).*

Step 2 *Find the minimum weight spanning tree $\mathcal{T}'(\mathcal{V}_{G'}, \mathcal{E}_{\mathcal{T}'})$ of $\mathcal{G}'$.*[3]

Step 3 *Identify the minimal weight directed graph $\mathcal{T}''(\mathcal{V}_{G'}, \mathcal{E}_{\mathcal{T}''})$ from $\mathcal{T}'$ such that for every $v \in V$ there is a directed elementary path in $\mathcal{T}''$ from v to b. In other words, $\mathcal{T}''$ is an orientation of T' that satisfies the above condition with $(E_{CC})_{ij}$ as the weight of (v_i, v_j) and $(E_{BC})_{ii}$ as the weight of (v_i, b).*

Step 4 *Form the directed spanning output forest $\mathcal{F}(\mathcal{V}_{\mathcal{F}}, \mathcal{E}_{\mathcal{F}})$ where an introduced output node r_i represents the backbone output capability of sensor node v_i for each $v_i \in U$,*

$$\mathcal{V}_{\mathcal{F}} = \mathcal{V} \cup \{r_i | (v_i, b) \in \mathcal{E}_{\mathcal{T}''}\},$$
$$\mathcal{E}_{\mathcal{F}} = \{(v_i, v_j) \in E_{\mathcal{T}''} | v_i, v_j \in \mathcal{V}\} \cup \{(v_i, r_i) | (v_i, b) \in \mathcal{E}_{\mathcal{T}''}\}$$

with $(E_{CC})_{ij}$ as the weight of (v_i, v_j) and $(E_{BC})_{ii}$ as the weight of (v_i, b).

Step 5 *Construct a spanning output cactus patch $\mathcal{P}(\mathcal{V}_{\mathcal{P}}, \mathcal{E}_{\mathcal{P}}) \simeq \mathcal{D}(\tilde{A}^*, \tilde{C}^*)$ from $\mathcal{F}$ (ensuring structural observability) as follows. Let*

$$\mathcal{V}_{\mathcal{P}} = \mathcal{V}_{\mathcal{F}},$$
$$\mathcal{E}_{\mathcal{P}} = \mathcal{E}_{\mathcal{F}} \cup \{(v_i, v_i) | v_i \in \mathcal{V}, r_i \notin \mathcal{V}_{\mathcal{F}}\}.$$

That is, add a self loop to every node that is not connected to an output. Then set

$$\tilde{A}^*_{ij} = \begin{cases} 1 & (v_i, v_j) \in \mathcal{E}_{\mathcal{P}} \\ 0 & (v_i, v_j) \notin \mathcal{E}_{\mathcal{P}} \end{cases},$$

$$\tilde{C}^*_{ij} = \begin{cases} 1 & i = j, (v_i, r_i) \in \mathcal{E}_{\mathcal{P}} \\ 0 & \text{otherwise} \end{cases}.$$

Output *Optimal matrix structure $(\tilde{A}^*, \tilde{C}^*)$*

The steps of Algorithm 1 are depicted in Figure 2. Theorem 2 now proves the optimality of the matrix structure $(\tilde{A}^*, \tilde{C}^*)$ output by this algorithm.

Theorem 2 (Correctness) *The pair $(\tilde{A}^*, \tilde{C}^*)$ output by Algorithm 1 is the optimal structurally observable solution to the optimization problem in* (7).

Proof First, it will be shown that $(\tilde{A}^*, \tilde{C}^*)$ is structurally observable and, thus, a feasible solution. From Theorem 1, $(\tilde{A}^*, \tilde{C}^*)$ is structurally observable if and only if $\mathcal{D}(\tilde{A}^*, \tilde{C}^*)$ is spanned by an output cactus patch. Step 5 of the algorithm defined $(\tilde{A}^*, \tilde{C}^*)$ such that $\mathcal{D}(\tilde{A}^*, \tilde{C}^*)$ is isomorphic to $\mathcal{P}$, where the isomorphism consists of a relabeling from $v_i \in \mathcal{V}_{\mathcal{P}}$ to $x_i \in \mathcal{X}$ and from $r_i \in \mathcal{V}_{\mathcal{P}}$ to $y_i \in \mathcal{Y}$. Since $\mathcal{F}$ is constructed as an output forest and $\mathcal{P}$ is formed from $\mathcal{F}$ by the addition of self loops to all nodes that do not have backbone outputs, $\mathcal{P}$ fits the recursive definition of an output cactus patch as it is composed of output stems of length one and self-loops attached to other self-loops or an output stem. Hence, $\mathcal{P} \simeq \mathcal{D}(\tilde{A}^*, \tilde{C}^*)$ is an output cactus, so $(\tilde{A}^*, \tilde{C}^*)$ is structurally observable.

[3]The minimum weight spanning tree exists by the assumption in A.2.

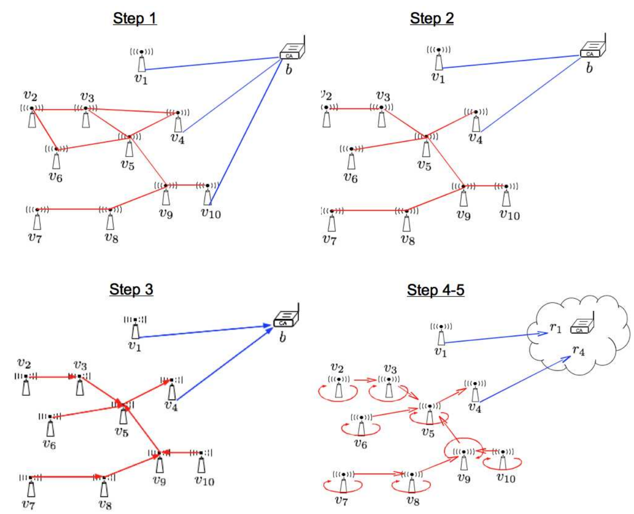

Fig. 2. Schematic representation of the steps in Algorithm 1.

Next, it is shown that no other feasible solution has lesser cost. Let $w(\mathcal{H})$ be the undirected edge weight sum of a graph $\mathcal{H}$. Note that all steps from 2 to 5 of the algorithm preserve edge weights except for step 3, which cuts it in half. Thus, $w(\mathcal{T}') = 2w(\mathcal{P})$. Assume by way of contradiction that $(\tilde{A}^*, \tilde{C}^*)$ is not a minimum solution to optimization problem (7). Then there is a feasible matrix pair $(\tilde{A}_1, \tilde{C}_1)$ which is a minimum solution and has lesser cost with respect to the objective function.

Because $(\tilde{A}_1, \tilde{C}_1)$ is a minimum feasible solution, $\mathcal{P}_1 \simeq \mathcal{D}(\tilde{A}_1, \tilde{C}_1)$ must be spanned by an output cactus patch. More precisely, it must be a minimum spanning output cactus patch, with the possible addition of extra zero cost self loops and with $w(\mathcal{P}_1) < w(\mathcal{P})$, since an additional directed edge would increase the cost. Because the minimum spanning cactus patch for $\mathcal{G}$ can have no cycles other than self loops (otherwise, breaking the cycle by removing an edge and giving all member nodes self loops would improve the cost), $\mathcal{P}_1$ has no cycles other than self loops. Now, consider the graph $\mathcal{T}_1'$ obtained by removing all self loops from $\mathcal{P}_1$, connecting all nodes in $\mathcal{P}_1$ that connect to a reporting node to b, and making all edges bidirectional. By this construction, $\mathcal{T}_1'$ must be a spanning tree for $\mathcal{G}'$ since $\mathcal{P}_1$ had no cycles other than self loops, which were removed, and since each connected component of $\mathcal{P}_1$ has only one reporting node. Then $w(\mathcal{T}_1') = 2w(\mathcal{P}_1) < 2w(\mathcal{P}) = w(\mathcal{T}')$.

Hence, the weigh of $\mathcal{T}_1'$ is less than the weight of $\mathcal{T}'$, contradicting the hypothesis that $\mathcal{T}'$ is a minimum spanning tree for $\mathcal{G}'$. Thus, $(\tilde{A}^*, \tilde{C}^*)$ is the minimum feasible solution to the problem in (7). ■

Theorem 2 stated that Algorithm 1 achieves the global minimum of the function (7) but made no comment concerning uniqueness. Note that since self loops have zero cost, there exist several alternatives that consist of disregarding particular combinations of self loops. Also, the minimum spanning tree $\mathcal{T}'$ might not be unique, leading to alternative solutions. To conclude this section, Theorem 3 shows that Algorithm 1 is computationally efficient.

Theorem 3 (Computational Complexity) *Algorithm 1 has computational complexity given by* $\mathcal{O}(|\mathcal{V}|^2)$.

Proof First, remark that Step 1 creates an undirected graph $\mathcal{G}'$ that can be generated linearly in the number of vertices and edges. In other words, the computational complexity is at most $\mathcal{O}(|\mathcal{V}|^2)$. Step 2 generates the minimum weight spanning tree $\mathcal{T}'$ of $\mathcal{G}'$, which requires computational complexity $\mathcal{O}(|\mathcal{E}|\log(|\mathcal{V}|))$ or $\mathcal{O}(|\mathcal{V}|^2)$ depending on the implementation of Prim's algorithm. This paper chooses the $\mathcal{O}(|\mathcal{V}|^2)$ implementation. Hence, Step 2 has complexity $\mathcal{O}(|\mathcal{V}|^2)$. Step 3 and 4 can be implemented with linear computational complexity $\mathcal{O}(|\mathcal{V}|)$ since it orients the tree edges, of which there are only N, and splits the backbone node into at most $|\mathcal{V}|$ output nodes. Finally, Step 5 adds self loops to the nodes, a $\mathcal{O}(|\mathcal{V}|)$ process, and constructs the matrices $(\tilde{A}^*, \tilde{C}^*)$, a linear operation in the number of the edges of the digraph generated in Step 4 (which is linear due to the aforementioned reason). Hence, the overall complexity is $\mathcal{O}(|\mathcal{V}|^2)$. ■

5. SIMULATION

In order to demonstrate the application of Algorithm 1, this section shows the results when an implementation of the algorithm is applied to $N = 50$ sensor nodes and a central authority node all randomly deployed in a 10×10 square region. All nodes have a sufficiently large communication radius to communicate with any other node, and all nodes have reporting potential. Sensor to backbone communication over a given distance is equal in cost to sensor to sensor communication over the same distance ($\alpha = \beta = 1$). This scenario can be visualized in Figure 2. The outcome of the simulation is depicted in Figure 3. Actual observable matrix values (A, C) can be derived from $(\tilde{A}^*, \tilde{C}^*)$ by sampling random matrices with the same sparsity structure as $(\tilde{A}^*, \tilde{C}^*)$ and verifying observability by evaluating the rank of the observability Gramian. Since the set of unobservable matrix pairs with the same sparsity structure as $(\tilde{A}^*, \tilde{C}^*)$ has zero measure, this observability test is passed on the first iteration with probability 1. The dynamic matrices obtained could then be used to simulate communication and reporting in the network and then estimate the sensor reading data $\mathbf{x}_0$ from the reported outputs. See, for instance, Theorem 15.4 in [7].

6. CONCLUSION

This paper posed the problem of finding the optimal network structure and linear update dynamics with respect to a communication cost function for a set of sensor nodes for which data must be communicated to a network backbone in order to reconstruct the set of initial sensor measurements. To address this problem, a solution algorithm was developed and shown to be formally correct and computationally efficient, the proofs of which are rooted in theorems from the field of structural dynamic systems and comprise the main result of this paper. Additionally, use of the algorithm was demonstrated in a simple simulation.

While the paper noted one approach by which observable dynamics can be found from the network structure produced by the algorithm, it does not guarantee good numerical properties of the observability Gramian matrix, such as a good ratio of maximum to minimum eigenvalues, that would be necessary to produce useful observers. Future work that produced well conditioned matrices from a given sparsity structure would greatly increase the usefulness of the algorithm as a network design tool, as would identification of practical sensor network applications in which a linear dynamic update scheme provides advantages.

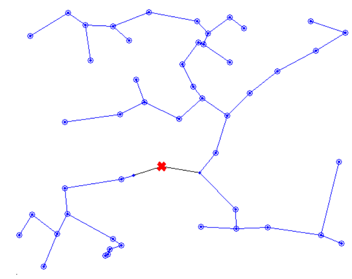

Fig. 3. A visualization of the communication graph corresponding to the optimal structurally observable network dynamics sparsity structure is shown. It is a minimum weight spanning tree of a randomly generated sensor network with added self loops (blue circles) at some nodes (blue points). All edges are directed towards the central node (red x).

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