Observability Analysis for AUV Range-only Localization and Mapping

Measures of Unobservability and Experimental Results

Mohammadreza Bayat and A. Pedro Aguiar

Institute for Systems and Robotics (ISR), Instituto Superior Técnico (IST), Lisbon, Portugal. {mbayat}{pedro}@isr.ist.utl.pt

Abstract: We investigate the observability problem of the Simultaneous Localization and Mapping (SLAM) process of an Autonomous Underwater Vehicle (AUV) equipped with inertial sensors, a depth sensor, and an acoustic ranging device that provides relative range measurements to stationary beacons. For trimming trajectories (that is, when the motion of the AUV is in steady-state with constant linear and angular velocities expressed in the body-frame), we provide conditions under which it is possible to reconstruct the initial state of the resulting SLAM system (and in particular the position of the AUV). We show that the unobservable subspace $\mathcal{UO}$ restricted to the assumption that the position of one of the beacons or the initial position of the AUV is known, contains only the zero vector with exception of a distinct case where the $\mathcal{UO}$ is composed by a finite set of isolated points. Another problem that we also address in this paper is to understand (independently of the observer scheme) how difficult it is to accurately estimate the state of the system from the pair input/output measured signals. To this end, we compute explicitly the unobservability index and the estimation condition number that provide measures of the *degree* of unobservability. Simulation and experimental results with the Medusa robotic vehicle are presented and discussed.

Keywords: Observability, SLAM, Range Measurements, AUV.

1. INTRODUCTION

Worldwide, there has been a growing interest in cooperative motion control of multiple autonomous marine robotic vehicles for search, surveillance, and survey operations. An important and key enabling element for the execution of such missions is the availability of advanced systems for localization. In the last two decades, a variety of different approaches has been developed that make use of acoustic signals. Ultra Short BaseLine (USBL), Long BaseLine (LBL), and GPS Intelligent Buoy (GIB) are examples of underwater navigation and positioning systems where all of them use the concept of beacons, transponders, and range measurements taken from relative/absolute time of flight of acoustic signals.

Another promising and interesting approach is to use only one beacon for localization. One of the first works on single beacon acoustic navigation was reported in (Larsen, 2000), where the author describes a Synthetic Long Base-Line (SLBL) navigation algorithm, which makes use of a single LBL in combination with a a high performance dead-reckoning navigation system. In (Casey and Hu, 2007) the authors design an extended Kalman filter (EKF) for localization of an AUV using a single beacon, and in (Saude and Aguiar, 2009) combining the dead-reckoning information with multiple range measurements taken at different instants of time from the vehicle to a single beacon, a robust estimator algorithm was proposed for vehicle localization in the presence of constant unknown ocean currents. In (Olson et al., 2004), the authors described a range only beacon localization, which assumes no prior knowledge of beacons' location. A pure range only sub-sea SLAM has been designed in (Newman and Leonard, 2003). Cooperative AUV navigation using a single maneuvering surface craft has been studied in (Fallon et al., 2009). Regarding observability studies in the single beacon AUV localization, one of the first results are described in (Gadre and Stilwell, 2005; Gadre, 2007), where the authors have investigated the observability of the linearized single beacon navigation system. Another study that reformulates the problem to a linear time varying (LTV) system is reported in (Batista et al., 2010). In (Arrichiello et al., 2011), the authors exploited the nonlinear observability concepts of a nonlinear inter-vehicle ranging system using observability rank conditions and the results obtained are validated experimentally in an equivalent single beacon navigation scenario.

This work addresses the single/multiple beacon observability analysis of the SLAM for AUV navigation using range measurements to stationary beacons. To this effect, we first apply a coordinate transformation similar to the one presented in (Aguiar and Hespanha, 2006) and then a time-scaling transformation to obtain a LTV system. Then, we investigate for the case that the motion of the AUV corresponds to constant linear and angular velocities

⋆ This work was supported in part by projects MORPH (EU FP7 under grant agreement No. 288704), CONAV/FCT-PT (PTDC/EEA-CRO/113820/2009), and the FCT [PEst-OE/EEI/LA0009/2011]. The first author benefited from a PhD scholarship of the Foundation for Science and Technology (FCT), Portugal.

expressed in the body-frame, (also known as trimming or steady-state trajectories), under which conditions it is possible to reconstruct the initial state of the resulting SLAM system (and in particular the position of the AUV). We show that the unobservable subspace $\mathcal{UO}$ restricted to the assumption that the position of one of the beacons or the initial position of the AUV is known, contains only the zero vector with exception of a particular case where the $\mathcal{UO}$ is composed by a finite set of isolated points. Another more critical issue that we also address is to understand (independently of the observer scheme) how difficult it is to accurately estimate the state of the system from the pair input/output measured signals. To this end, we compute explicitly the unobservability index and the estimation condition number that provide measures of the *degree* of unobservability. Simulation and experimental results with the Medusa robotic vehicle are presented and discussed.

2. PROCESS MODEL

This section describes the process model of the problem of computing in real-time an estimate of the position of an AUV while simultaneously constructing a map of its surrounding. The map whose building process is based on ranging measurements obtained from stationary acoustic modems (beacons) contains an estimate of the location of the beacons. To formulate the process model we consider two coordinate frames: fixed earth or inertial coordinate frame $\{\mathcal{I}\}$, and body fixed coordinate frame $\{\mathcal{B}\}$ that is attached to the AUV, which moves with respect to the coordinate frame $\{\mathcal{I}\}$. Let $({}^{\mathcal{I}}\boldsymbol{p}_{\mathcal{B}}, {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}) \in \mathrm{SE}(3)$ be the configuration of the frame $\{\mathcal{B}\}$ with respect to $\{\mathcal{I}\}$, where ${}^{\mathcal{I}}\boldsymbol{p}_{\mathcal{B}}$ indicates the position of the AUV in frame $\{\mathcal{I}\}$, and ${}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}$ its rotation matrix from $\{\mathcal{B}\}$ to $\{\mathcal{I}\}$. The equations of motion can be written as

$$ {}^{\mathcal{I}}\dot{\boldsymbol{p}}_{\mathcal{B}} = {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\,\boldsymbol{\nu} \tag{1} $$

$$ {}^{\mathcal{I}}_{\mathcal{B}}\dot{\mathcal{R}} = {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\,S(\boldsymbol{\omega}) \tag{2} $$

where the linear and angular velocities ($\boldsymbol{\nu}, \boldsymbol{\omega} : [0, \infty) \to \mathbb{R}^3$) are viewed as input signals to the system (1)-(2). Consider now n stationary beacons located at unknown positions ${}^{\mathcal{I}}\boldsymbol{q}_i$, that is,

$$ {}^{\mathcal{I}}\dot{\boldsymbol{q}}_i = 0 \tag{3} $$

For each $i \in \{1, 2, ..., n\}$, let $r_i(t)$ be an acoustic ranging measurement acquired at time t from the i^{th} beacon. In this case, the measurement or output model is given by

$$ r_i = \|{}^{\mathcal{I}}\boldsymbol{q}_i - {}^{\mathcal{I}}\boldsymbol{p}_{\mathcal{B}}\| \tag{4} $$

$$ z_i = [0, 0, 1]\,{}^{\mathcal{I}}\boldsymbol{q}_i \tag{5} $$

$$ z_0 = [0, 0, 1]\,{}^{\mathcal{I}}\boldsymbol{p}_{\mathcal{B}} \tag{6} $$

where z_0 is the depth of the AUV that is assumed to be available (we consider the practical situation that the AUV is equipped with a depth sensor). We also consider that the location of the beacons q_i are only unknown in the horizontal plane, that is, we assume that we know the depth z_i. This is a reasonable assumption if each beacon is attached to a buoy that is at the surface.

Equations (1)-(6) represent the process model of SLAM problem for the AUV. In this paper we address the observability problem. To this end, from the nonlinear system (1)-(6) we construct a new linear time varying system (LTV) and derive under what conditions the new system is *equivalent* to (1)-(6). Note that once we have a LTV system we can apply the tools of linear estimation theory for observability analysis. The strategy to obtain a LTV system does not follow the ones described in (Krener and Isidori, 1983; Plestan and Glumineau, 1997) but it is specific tailored for our application. The idea is to view the beacons $\boldsymbol{q}_i$ in body frame $\{\mathcal{B}\}$ and introduce a *virtual* beacon $\boldsymbol{q}_0$, located at the origin of $\{\mathcal{I}\}$. Following this strategy and resorting to some of the ideas in (Aguiar and Hespanha, 2006), we first express $\boldsymbol{q}_0$ in $\{\mathcal{B}\}$ as

$$ {}^{\mathcal{B}}\boldsymbol{q}_0 = {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}'\,{}^{\mathcal{I}}\boldsymbol{q}_0 - {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}'\,{}^{\mathcal{I}}\boldsymbol{p}_{\mathcal{B}} $$

whose its dynamical equation is given by

$$ \begin{aligned} {}^{\mathcal{B}}\dot{\boldsymbol{q}}_0 &= {}^{\mathcal{I}}_{\mathcal{B}}\dot{\mathcal{R}}'\,({}^{\mathcal{I}}\boldsymbol{q}_0 - {}^{\mathcal{I}}\boldsymbol{p}_{\mathcal{B}}) + {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}'\,{}^{\mathcal{I}}\dot{\boldsymbol{q}}_0 - {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}'\,{}^{\mathcal{I}}\dot{\boldsymbol{p}}_{\mathcal{B}} \\ &= -S(\boldsymbol{\omega})\,{}^{\mathcal{B}}\boldsymbol{q}_0 - \boldsymbol{\nu} \end{aligned} $$

where we have used (3). To obtain the dynamics of the position of the other beacons $\boldsymbol{q}_i$ in body frame, we introduce the vector $\boldsymbol{p}_i$ that connects the virtual beacon $\boldsymbol{q}_0$ to $\boldsymbol{q}_i$. Note that ${}^{\mathcal{I}}\boldsymbol{p}_i$ is a stationary vector, while ${}^{\mathcal{B}}\boldsymbol{p}_i$ is in general a time dependent vector (with same magnitude of ${}^{\mathcal{I}}\boldsymbol{p}_i$ but rotated by ${}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}'$). Therefore,

$$ \begin{aligned} {}^{\mathcal{B}}\boldsymbol{p}_i &= {}^{\mathcal{B}}\boldsymbol{q}_i - {}^{\mathcal{B}}\boldsymbol{q}_0 \\ {}^{\mathcal{B}}\dot{\boldsymbol{p}}_i &= -S(\boldsymbol{\omega})\,{}^{\mathcal{B}}\boldsymbol{p}_i \end{aligned} \tag{7} $$

From (4), (7), and using the fact that

$$ {}^{\mathcal{I}}\boldsymbol{q}_i = {}^{\mathcal{I}}\boldsymbol{p}_{\mathcal{B}} + {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\,{}^{\mathcal{B}}\boldsymbol{q}_i $$

the measurement model can be written as

$$ r_i = \|{}^{\mathcal{I}}\boldsymbol{q}_i - {}^{\mathcal{I}}\boldsymbol{p}_{\mathcal{B}}\| = \|{}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\,{}^{\mathcal{B}}\boldsymbol{q}_i\| = \|{}^{\mathcal{B}}\boldsymbol{p}_i + {}^{\mathcal{B}}\boldsymbol{q}_0\| $$

where we have used the fact that $\|\mathcal{R}\| = 1$. Introducing the scalar state variable $\chi_i = \|{}^{\mathcal{B}}\boldsymbol{p}_i + {}^{\mathcal{B}}\boldsymbol{q}_0\|$ the output equation (4) becomes $r_i = \chi_i$, where after some algebraic simplification χ_i satisfies

$$ \dot{\chi}_i = \frac{-\boldsymbol{\nu}'({}^{\mathcal{B}}\boldsymbol{p}_i + {}^{\mathcal{B}}\boldsymbol{q}_0)}{r_i} $$

Using the equalities

$$ {}^{\mathcal{I}}\boldsymbol{q}_i = {}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\,{}^{\mathcal{B}}\boldsymbol{p}_i, \quad {}^{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{B}} = -{}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\,{}^{\mathcal{B}}\boldsymbol{q}_0 $$

we can write the output equation (5) and (6) as

$$ \begin{aligned} z_i &= [0, 0, 1]\,{}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\,{}^{\mathcal{B}}\boldsymbol{p}_i \\ z_0 &= -[0, 0, 1]\,{}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}\,{}^{\mathcal{B}}\boldsymbol{q}_0 \end{aligned} $$

In summary we obtain a LTV system described by

$$ \begin{aligned} \dot{\boldsymbol{x}}(t) &= A\left(\boldsymbol{u}(t), \boldsymbol{y}(t)\right)\boldsymbol{x}(t) + \boldsymbol{b}\left(\boldsymbol{u}(t)\right) \\ \boldsymbol{y}(t) &= C\left(\boldsymbol{u}(t)\right)\boldsymbol{x}(t) \end{aligned} \tag{8} $$

where

$$ \begin{aligned} \boldsymbol{x} &:= \left[\, {}^{\mathcal{B}}\boldsymbol{q}_0', {}^{\mathcal{B}}\boldsymbol{p}_1' \, {}^{\mathcal{B}}\boldsymbol{p}_2' \ldots {}^{\mathcal{B}}\boldsymbol{p}_n', \chi_1\, \chi_2 \ldots \chi_n \,\right]' \\ \boldsymbol{y} &:= [r_1 \, \ldots \, r_n, \; z_0, \; z_1 \, \ldots \, z_n]' \\ \boldsymbol{u} &:= \left[\, \boldsymbol{\nu}' \;\, \boldsymbol{\omega}' \;\, \boldsymbol{\eta}' \,\right]' \\ \boldsymbol{s} &:= \left[\, {}^{1}\!/_{r_1} \;\, {}^{1}\!/_{r_2} \;\, \ldots \;\, {}^{1}\!/_{r_n} \,\right]' \end{aligned} $$

$$ A(\boldsymbol{u}, \boldsymbol{y}) := \begin{bmatrix} -S(\boldsymbol{\omega}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -I_n \otimes S(\boldsymbol{\omega}) & \mathbf{0} \\ -\boldsymbol{s} \otimes \boldsymbol{\nu}' & -I_n \otimes (\boldsymbol{s} \otimes \boldsymbol{\nu}') & \mathbf{0} \end{bmatrix} $$

$$ \boldsymbol{b}(\boldsymbol{u}) := \left[\, -\boldsymbol{\nu}' \;\, \mathbf{0} \;\, \mathbf{0} \,\right]' $$

$$ C(\boldsymbol{u}) := \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0}\, I_n \\ -[0, 0, 1]\,{}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}(\boldsymbol{\eta}) & \mathbf{0} & \mathbf{0}\,\mathbf{0} \\ \mathbf{0} & I_n \otimes \left([0, 0, 1]\,{}^{\mathcal{I}}_{\mathcal{B}}\mathcal{R}(\boldsymbol{\eta})\right) & \mathbf{0}\,\mathbf{0} \end{bmatrix} $$

We remark that (8) is not defined when $r_i = 0$, which corresponds to the particular case that the position of the AUV coincides with the location of the i^{th} beacon.

3. OBSERVABILITY ANALYSIS

SLAM is a technique to build an estimate of the environment map (within a complete unknown environment or with some a priori knowledge of the environment), while simultaneously compute an estimate of the position of the vehicle. Unless there is an *anchor* that relates the relative localization with the global (inertial) position, the SLAM process in (8) (where ${}^{\mathcal{B}}\boldsymbol{q}_0$ can be related to the vehicle position and the rest of the states to the beacons positions) is not observable. In fact the idea is to use a priori knowledge of one of the states and estimate the other unknown ones. For example, one assumption is to consider that the initial condition of the location of the AUV is known (which can be done in practice if the AUV starts at the surface and there is GPS). Then, since ${}^{\mathcal{B}}\boldsymbol{q}_0(t_0)$ and the input $\boldsymbol{u}(t)$ are known, it follows that ${}^{\mathcal{B}}\boldsymbol{q}_0(t)$ will also be known for all $t \geq 0$. In this case, we can remove ${}^{\mathcal{B}}\boldsymbol{q}_0$ from the state vector and instead consider it as an input signal. Using the above argument and considering at this stage only one beacon, $n = 1$, from (8) we obtain the single beacon system with squared range state χ_1^2 and output r_1^2, as follows

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \nu(t)\left(A\left(\boldsymbol{u}\left(t\right)\right)\boldsymbol{x}(t) + \boldsymbol{b}(\boldsymbol{u}(t))\right) \\ \boldsymbol{y}(t) &= C(\boldsymbol{u}(t))\boldsymbol{x}(t) \end{aligned} \tag{9}$$

where

$$\begin{aligned} \boldsymbol{x} &:= \left[\, {}^{\mathcal{B}}\boldsymbol{p}_1' \;\; \chi_1^2 \,\right]' = \left[\, {}^{\mathcal{B}}p_{1,x} \;\; {}^{\mathcal{B}}p_{1,y} \;\; {}^{\mathcal{B}}p_{1,z} \;\; \chi_1^2 \,\right]' \\ \boldsymbol{y} &:= \left[r_1^2,\; z_1\right]' \\ \boldsymbol{u} &:= \left[\, \boldsymbol{\nu}' \;\; \boldsymbol{\omega}' \;\; \boldsymbol{\eta}' \;\; {}^{\mathcal{B}}\boldsymbol{q}_0' \,\right]' \\ A(\boldsymbol{u}) &:= \begin{bmatrix} 0 & \omega_3(t)/\nu(t) & -\omega_2(t)/\nu(t) & 0 \\ -\omega_3(t)/\nu(t) & 0 & \omega_1(t)/\nu(t) & 0 \\ \omega_2(t)/\nu(t) & -\omega_1(t)/\nu(t) & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix} \\ \boldsymbol{b}(\boldsymbol{u}) &:= \left[\, 0 \;\; 0 \;\; 0 \;\; -2\,{}^{\mathcal{B}}q_{0,x}(t) \,\right]' \\ C(\boldsymbol{u}) &:= \begin{bmatrix} 0 & 0 & 0 & 1 \\ -s\theta(t) & s\phi(t)c\theta(t) & c\phi(t)c\theta(t) & 0 \end{bmatrix} \end{aligned}$$

with $\boldsymbol{\omega}(t) = [\omega_1(t), \omega_2(t), \omega_3(t)]'$, $\boldsymbol{\nu}(t) = [\nu(t), 0, 0]'$, and ${}^{\mathcal{B}}\boldsymbol{q}_0 = [{}^{\mathcal{B}}q_{0,x}, {}^{\mathcal{B}}q_{0,y}, {}^{\mathcal{B}}q_{0,z}]'$.

Remark 1. In (9) we have considered squared ranging measurement to simplify the observability analysis. This does not change the observability results. In fact if we conclude that (9) is observable with a given input $\boldsymbol{u}^*$ in the sense that for every pair of distinct initial conditions $(\boldsymbol{x}_0, \boldsymbol{z}_0)$ there exists a time interval $t \in [t^*, t_f]$, $t^* \geq t_0$ such that the corresponding squared range outputs are different,

$$y(t;\boldsymbol{u}^*,\boldsymbol{x}_0) := r_1^2(t;\boldsymbol{u}^*,\boldsymbol{x}_0) \neq y(t;\boldsymbol{u}^*,\boldsymbol{z}_0) := r_1^2(t;\boldsymbol{u}^*,\boldsymbol{z}_0)$$

then it also follows that $r_1(t;\boldsymbol{u}^*,\boldsymbol{x}_0) \neq r_1(t;\boldsymbol{u}^*,\boldsymbol{z}_0)$, which implies that the initial conditions $(\boldsymbol{x}_0, \boldsymbol{z}_0)$ for the original system (using ranges) will produce different outputs. □

Remark 2. For observability analysis we can consider without loss of generality that the linear velocity $\boldsymbol{\nu}$ does not include any non null term in y and z components. If this is not the case (e.g., there is nonzero sideslip in steady state), then the process model (9) is obtained by substituting the body fixed frame $\{\mathcal{B}\}$ by the flow frame $\{\mathcal{F}\}$, which its origin coincides with $\{\mathcal{B}\}$ but the orientation is such that the linear velocity expressed in $\{\mathcal{F}\}$ is $\boldsymbol{\nu}(t) = [\nu(t), 0, 0]'$. Note that in this case the orientation $\boldsymbol{\eta}$ is with respect to $\{\mathcal{F}\}$. □

To re-write (9) in a standard LTV system we further apply a time scale transformation with $\dot{\tau} = \nu(t)$ and assume that $\nu(t) \neq 0$. In this case, we obtain

$$\begin{aligned} \frac{d\boldsymbol{x}}{d\tau} &= A(\tau)\boldsymbol{x}(\tau) + b(\tau) \\ \boldsymbol{y}(\tau) &= C(\tau)\boldsymbol{x}(\tau) \end{aligned} \tag{10}$$

Since (10) is a time scaled version of (9), then both systems are equivalent in observability sense. Furthermore without loss of generality we can set $\nu(t) = 1$ in (9). We are now ready to study the observability of the single beacon case. We will consider that the vehicle is in steady state motion, that is, the linear and angular velocities are constant.

Theorem 1. Consider system (9) without depth measurement, that is $C = [0, 0, 0, 1]$, and suppose that

$$\omega_1 \neq 0 \text{ and } (\omega_2 \neq 0 \text{ or } \omega_3 \neq 0) \tag{11}$$

Then the system is observable.

Proof. Since we are assuming that the linear and angular velocities are constant, it turns out that system (9) without depth measurement is an LTI system. In this case the observability matrix is computed as

$$\mathcal{O}_4 = \left[\, C; CA; CA^2; CA^3 \,\right] = 2\begin{bmatrix} 0 & 0 & 0 & 0.5 \\ -1 & 0 & 0 & 0 \\ 0 & -\omega_3 & \omega_2 & 0 \\ \omega_2^2+\omega_3^2 & -\omega_1\omega_2 & -\omega_1\omega_3 & 0 \end{bmatrix}$$

If the conditions of theorem hold, it follows that $\mathcal{O}$ is a full rank matrix and therefor the system is observable. ■

We remark that the fact of introducing a new output to the system (in this case the depth measurement) will not change the observability results of Theorem 1 but as it will be seen in the following it is possible to obtain less conservative results.

Theorem 2. Consider system (9) and suppose that

$$\omega_3 \neq -\omega_2 \tan(\phi) \tag{13}$$

Then, the system is observable.

Proof. Using the observability rank condition we can conclude that if the following matrix

$$\mathcal{O}_3 = \begin{bmatrix} C \\ CA + \dot{C} \\ CA^2 + 2\dot{C}A + C\dot{A} + \ddot{C} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -s\theta & c\theta s\phi & c\phi c\theta & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2\omega_3 & 2\omega_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is full rank then the system (9) is observable. Computing the determinant (taking out the nonzero rows), it follows that $\mathcal{O}_3$ is full rank if (13) holds. ■

From the above results we can conclude that the condition

$$\omega_3 = -\omega_2 \tan(\phi) \tag{14}$$

is a particular case. In fact, in this case the Euler angle rates satisfy

$$\left[\, \dot{\phi} \;\; \dot{\theta} \;\; \dot{\psi} \,\right]' = \left[\, \omega_1 \;\; \frac{\omega_2}{\cos\phi} \;\; 0 \,\right]' \tag{15}$$

Notice also that from the assumption of $\boldsymbol{\omega}$ being constant, using (14) we can conclude that ϕ has to be constant, which implies that $\dot{\phi} = 0$ and consequently from (15) that $\omega_1 = 0$.

$$\mathcal{W}(0,t) = \begin{bmatrix} \frac{t\omega_2^3\sin^2\theta_0 - \sin(2t\omega_2) - 2t\omega_2}{\omega_2^3} & 0 & \frac{-t\omega_2^3\sin(2\theta_0) - 16\sin(\frac{t\omega_2}{2})^4}{2\omega_2^3} & \frac{2\cos(t\omega_2) - 2}{\omega_2^2} \\ 0 & 0 & 0 & 0 \\ \frac{-t\omega_2^3\sin(2\theta_0) - 16\sin(\frac{t\omega_2}{2})^4}{2\omega_2^3} & 0 & \frac{t\omega_2^3\cos^2(\theta_0) + \sin(2t\omega_2) - 8\sin(t\omega_2) + 6t\omega_2}{\omega_2^3} & \frac{-2\sin(t\omega_2) - 2t\omega_2}{\omega_2^2} \\ \frac{2\cos(t\omega_2) - 2}{\omega_2^2} & 0 & \frac{-2\sin(t\omega_2) - 2t\omega_2}{\omega_2^2} & t \end{bmatrix} \tag{12}$$

Theorem 3. Consider system (9) and suppose that (14) holds. Then, it is possible to reconstruct the initial condition of the state $\boldsymbol{x}(0)$ from the input and output signals on the interval $[0, t_f]$ with the exception that the pair $({}^{\mathcal{B}}p_{1,y}, {}^{\mathcal{B}}p_{1,z})(0)$ can have two possible solutions.

Proof. Suppose first that $\phi_0 = 0$. Since $\phi(t) = \phi_0 = 0$ for all $t \geq 0$ it is possible to compute explicitly the Gramian of observability

$$\mathcal{W}(0,t) = \int_0^t \Phi'(s,0)C'(s)C(s)\Phi(s,0)ds$$

which is given in (12). Note that the second row and column are zero, which means that it is not possible using the Gramian to reconstruct the initial condition $\boldsymbol{x}(0)$. Now consider a reduced order system with state $\boldsymbol{x}_r = [{}^{\mathcal{B}}p_{1,x}, {}^{\mathcal{B}}p_{1,z}, \chi_1^2]$. Clearly this system is observable because the observability Gramian matrix is full rank (rank 3). Furthermore, in this case it is possible to reconstruct the initial condition $\boldsymbol{x}_r(0)$. Using now the fact that $\chi_1^2 = {}^{\mathcal{B}}q_{1,x}^2 + {}^{\mathcal{B}}q_{1,y}^2 + {}^{\mathcal{B}}q_{1,z}^2$ it follows that we can obtain two possible solutions for ${}^{\mathcal{B}}p_{1,y}$.
Consider now the case $\phi_0 \neq 0$ and let $\boldsymbol{\eta}_0 = [\phi_0, 0, 0]'$. Note that $\phi(t) = \phi_0$ for all $t \geq 0$. By performing a change of coordinates

$$\begin{aligned} \bar{\boldsymbol{\eta}} &= \boldsymbol{\eta} - \boldsymbol{\eta}_0 \\ {}^{\mathcal{B}}\bar{\boldsymbol{p}}_1 &= {}^{\mathcal{I}}_{\mathcal{B}}R(-\boldsymbol{\eta}_0){}^{\mathcal{B}}\boldsymbol{p}_1 \\ {}^{\mathcal{B}}\bar{\boldsymbol{q}}_0 &= {}^{\mathcal{I}}_{\mathcal{B}}R(-\boldsymbol{\eta}_0){}^{\mathcal{B}}\boldsymbol{q}_0 \end{aligned}$$

we obtain an equivalent system to the previous one but with $\bar{\phi}_0 = 0$ and $\chi_1^2 = {}^{\mathcal{B}}\bar{q}_{1,x}^2 + {}^{\mathcal{B}}\bar{q}_{1,y}^2 + {}^{\mathcal{B}}\bar{q}_{1,z}^2$. Using the same reasoning as before, it follows that we can obtain two possible solutions for ${}^{\mathcal{B}}\bar{p}_{1,y}$. Rotating back the solution to the original coordinate, ${}^{\mathcal{B}}\boldsymbol{p}_1$, we can conclude that the pair $({}^{\mathcal{B}}p_{1,y}, {}^{\mathcal{B}}p_{1,z})$ can have two possible solutions. This concludes the result. ■

From Theorems 1-3 we can now characterize the Unobservable subspace ($\mathcal{UO}$) of (9) that corresponds to the subspace given by the kernel of $\mathcal{W}(t_0, t)$, $\forall t \in [0, t_f]$. Note that the importance of $\mathcal{UO}$ stems from the fact that if x_0 is a given initial condition of (9) for a compatible input/output pair, then the initial condition $z_0 = x_0 + x_{uo}$, $x_{uo} \in \mathcal{UO}$ is also compatible with the same input/output pair (Hespanha, 2009).

Corollary 1. The $\mathcal{UO}$ of system (9) is given by

$$\mathcal{UO} = \begin{cases} \{\alpha[0, c\phi_0, s\phi_0, 0]'\} & ,\omega_3 = -\omega_2\tan(\phi) \\ \{\boldsymbol{0}\} & ,\omega_3 \neq -\omega_2\tan(\phi) \end{cases}$$

where $\alpha \in \{0, -2(\bar{y}_1 c\phi_0 + \bar{z}_1 s\phi_0)\}$ with $[\bar{x}_1, \bar{y}_1, \bar{z}_1]' = {}^{\mathcal{B}}\boldsymbol{q}_0(0) + {}^{\mathcal{B}}\boldsymbol{p}_1(0)$.

Proof. In the case of $\omega_3 \neq -\omega_2\tan(\phi)$, system (9) is observable and the unobservable set $\mathcal{UO}$ contains only the origin. If instead condition $\omega_3 = -\omega_2\tan(\phi)$ holds, by computing the $\ker(\mathcal{W}(t_0, t))$ we get

$$\ker(\mathcal{W}(t_0, t)) = \{\alpha[0, 1, 0, 0]' : \alpha \in \mathbb{R}\}$$

where we have used the assumption that $\phi_0 = 0$. If this is not the case, we just have to perform a rotation of ${}^{\mathcal{I}}_{\mathcal{B}}R(\eta_0)$ to obtain

$$\ker(\mathcal{W}(t_0, t, \phi_0)) = \{\alpha[0, c\phi_0, s\phi_0, 0]' : \alpha \in \mathbb{R}\}$$

We now have to consider the effect of the constraint $\chi_1^2 = {}^{\mathcal{B}}q_{1,x}^2 + {}^{\mathcal{B}}q_{1,y}^2 + {}^{\mathcal{B}}q_{1,z}^2$. To this end, consider two initial conditions $\boldsymbol{x}_0$ and $\boldsymbol{z}_0 = \boldsymbol{x}_0 + \alpha[0, c\phi, s\phi, 0]'$, $\alpha \in \mathbb{R}$. Given the constraint $\chi_1^2 = {}^{\mathcal{B}}q_{1,x}^2 + {}^{\mathcal{B}}q_{1,y}^2 + {}^{\mathcal{B}}q_{1,z}^2$ and apply the constraint to obtain

$$\begin{aligned} &\bar{x}_1^2 + (\bar{y}_1 + \alpha c\phi_0)^2 + (\bar{z}_1 + \alpha s\phi_0)^2 = \chi_1(t_0)^2 \\ &\bar{x}_1^2 + \bar{y}_1^2 + \bar{z}_1^2 = \chi_1(t_0)^2 \end{aligned}$$

Solving the above equation yields

$$\alpha^2 = -2\alpha(\bar{y}_1 c\phi_0 + \bar{z}_1 s\phi_0)$$

which results into two possible solutions

$$\alpha = 0 \text{ or } \alpha = -2(\bar{y}_1 c\phi_0 + \bar{z}_1 s\phi_0)$$

This completes the proof. □

Until this point we have investigated the case where it is assumed that the initial location of the AUV ${}^{\mathcal{B}}q_0(t_0)$, is known and the position of the beacon ${}^{\mathcal{B}}p_1$, is unknown. We now consider the dual case, that is, ${}^{\mathcal{B}}p_1$ is known but not the position of the AUV. In this case we obtain the system

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \nu(t)\left(A(t)\boldsymbol{x}(t) + \boldsymbol{b}(t)\right) \\ \boldsymbol{y}(t) &= C(t)\boldsymbol{x}(t) \end{aligned} \tag{16}$$

but with

$$\begin{aligned} \boldsymbol{x} &:= \left[\, {}^{\mathcal{B}}\boldsymbol{q}_0' \;\; \chi_1^2 \,\right]' = \left[\, {}^{\mathcal{B}}q_{0,x} \;\; {}^{\mathcal{B}}q_{0,y} \;\; {}^{\mathcal{B}}q_{0,z} \;\; \chi_1^2 \,\right]' \\ \boldsymbol{y} &:= \left[r_1^2,\; z_0\right]' \\ \boldsymbol{u} &:= \left[\, \boldsymbol{\nu}' \;\; \boldsymbol{\omega}' \;\; \phi \;\; \theta \;\; \psi \;\; {}^{\mathcal{B}}\boldsymbol{p}_1' \,\right]' \\ A(\boldsymbol{u}) &:= \begin{bmatrix} 0 & \omega_3(t)/\nu(t) & -\omega_2(t)/\nu(t) & 0 \\ -\omega_3(t)/\nu(t) & 0 & \omega_1(t)/\nu(t) & 0 \\ \omega_2(t)/\nu(t) & -\omega_2(t)/\nu(t) & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix} \\ \boldsymbol{b}(\boldsymbol{u}) &:= \left[\, -1 \;\; 0 \;\; 0 \;\; -2{}^{\mathcal{B}}p_{1,x} \,\right]' \\ C(\boldsymbol{u}) &:= \begin{bmatrix} 0 & 0 & 0 & 1 \\ s\theta(t) & -s\phi(t)c\theta(t) & -c\phi(t)c\theta(t) & 0 \end{bmatrix} \end{aligned}$$

The following result holds.

Theorem 4. The dual system (16) has the same observability properties as the system (9), that is, Theorems 1-3 hold.

Proof. Comparing (16) with (9) it can be seen that the state matrix A is the same and the output matrix C differs by a negative sign in the second row. Clearly this makes no difference in the observability analysis of the dual system and the results obtained for system (9) hold. ■

We are now ready to state the main result for the SLAM system (8) which extends to more than one beacon.

Theorem 5. Consider the SLAM system (8) with constant linear velocity $\boldsymbol{\nu} \neq 0$ and angular velocity $\boldsymbol{\omega}$. Suppose that there is an anchor, that is, the initial condition ${}^{\mathcal{B}}\boldsymbol{q}_0(t_0)$ or the position of one of the beacons ${}^{\mathcal{B}}\boldsymbol{p}_i(t_0)$ is known. Then the initial condition of other states can be reconstructed from the observed input and output pair $\{\boldsymbol{u}(t), \boldsymbol{y}(t)\}$, $t \in [t_0, t_f)$, provided that (13) holds. If (13) does not hold then the initial condition of each unknown vector in $\{{}^{\mathcal{B}}\boldsymbol{q}_0, {}^{\mathcal{B}}\boldsymbol{p}_i; i = 1, 2, ..., n\}$ has two possible solutions.

Proof. For only one beacon the result follows from Theorems 1-4. Consider now more than one beacon. In this case it can be concluded from (8) that the dynamic equations of each pair $\{{}^{\mathcal{B}}\boldsymbol{p}_i, \chi_i\}$ does not depend on the other pairs. This means that the observability of the multiple beacon system can be investigated by analyzing the observability of each single beacon system and therefore the result follows from Theorems 1-4. ■

4. MEASURES OF UNOBSERVABILITY

Until this point we have analyzed the observability from a yes or no point of view. Although this is important, a more critical issue is to understand (independently of the observer scheme) how difficult it is to accurately estimate the state of the system from the pair input/output measured signals. In (Krener and Ide, 2009), the authors propose two measures: the *unobservability index* and the *estimation condition number*. The unobservability index defined as the reciprocal of the smallest local singular value of the Gramian of observability can be seen as a measure of how difficult it is to estimate the initial condition from the observed output signal. If this index is very large then the observation noise can have a large impact on the estimation error. The estimation condition number is the ratio of the largest local singular value to the smallest. If this is large then the estimation problem is ill conditioned, which means that for example the effect on the output due to a small change in the initial condition in one direction can be overwhelmed by the effect on the output of a change in another direction.

In general, these two measures of the degree of unobservability are difficult to compute. In our case, we could compute explicitly the Gramian of observability, which simplified considerable the computation of the degrees of unobservability and avoided the use of empirical measures. Figure 1 shows the two unobservability measures for the single-beacon case where we have considered that the AUV is moving with constant linear and angular velocity and such that $\phi(t) = 0$, and $\theta(t) = 0$ for all $t \geq 0$. As it can be seen, the degree of unobservability increases significantly when ω_3 is near zero, which corresponds to the condition (14) (straight-line motion).

5. SIMULATION AND EXPERIMENTAL RESULTS

To illustrate the observability results we have considered two scenarios:

- A simulated helix type trajectory given by a constant pitch $\theta = (\frac{-\pi}{10})$ and turning rate $\boldsymbol{\omega} = [\tan(\frac{-\pi}{10})\frac{\pi}{50}, 0, \frac{\pi}{50}]'$, which means that conditions (11) and (13) hold.

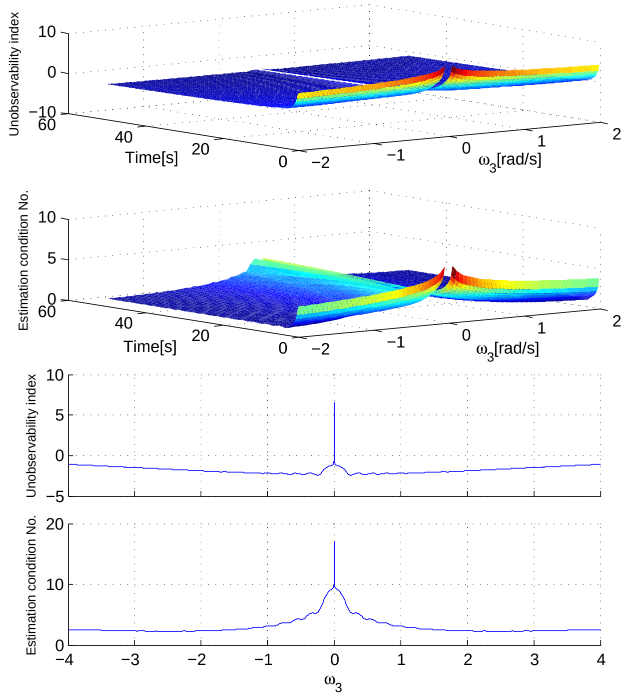

Fig. 1. Log of measures of unobservability versus time and ω_3 for constant pitch $\theta = 0$ with a cut at time $t = 30$.

- Experimental result with a lawn mowing type trajectory where the condition $\omega_3 = -\omega_2 \tan(\phi)$ holds except when the AUV is turning.

For the first scenario, we consider three stationary beacons as shown in Figure 2 with the symbol (o). The AUV is moving with a constant forward speed of $0.5[m/s]$ starting from the initial condition (□). To estimate the states of the model introduced in (8) we use a continuous time constrained estimator Aguiar and Hespanha (2003) combined with a multiple model approach. The models in the estimator are initialized with the same initial condition but with different up to a sign in initial condition ${}^{\mathcal{B}}p_{i,y}(t_0)$.

For the second scenario, experimental results were done with the three autonomous surface vehicles (Medusas) that were developed at the Laboratory of Robotics and Systems in Engineering and Science (LARSyS) of the Instituto Superior Técnico of Lisbon. Each vehicle has two side thrusters that can be independently controlled and is equipped with IMU, GPS, and compass. The communications with other devices are done via wifi or an underwater acoustic modem (Tritech Micron Modem). In this scenario, two of the Medusa vehicles have been in hold position mode to act only as stationary beacons.

Figure 3 shows the estimated and the true trajectory of the moving Medusa obtained by GPS, and also the location of the beacons (shown by (o). Starting from the initial position (□) the Medusa moves on a straight line. As soon as the vehicle turns, the observability condition (13) holds and the estimation errors converge to a small neighborhood of zero.

6. CONCLUSIONS

We addressed the observability problem of SLAM for an AUV equipped with inertial sensors, depth sensor, and

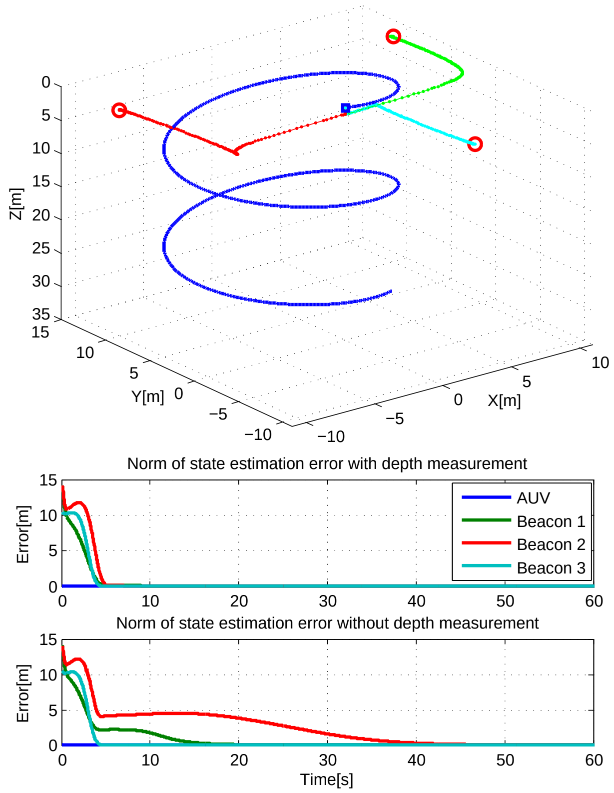

Fig. 2. Simulated trajectory, beacons configuration, and norm of state estimation error along time with and without depth measurement.

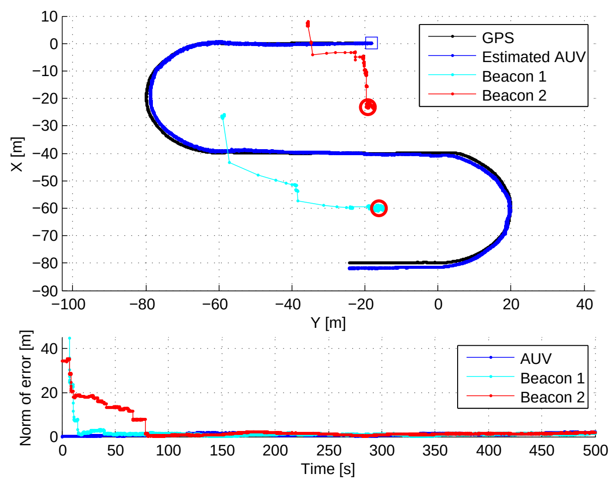

Fig. 3. Estimated trajectory of AUV, beacon location and corresponding norm of errors.

an acoustic ranging device to obtain relative range measurements to stationary beacons. We provided conditions under which it is possible to reconstruct the initial state of the resulting SLAM system (and in particular the position of the AUV). In addition, we computed the unobservability index and the estimation condition number to understand how difficult it is to accurately estimate the state of the system from the pair input/output measured signals. We could conclude that the degrees of unobservability are only significant when the system is very close to condition (14). Simulation and experimental results with the Medusa robotic vehicle validated the theoretical results.

REFERENCES

Aguiar, A.P. and Hespanha, J.a.P. (2003). Minimum-energy State Estimation for Systems with Perspective Outputs and State Constraints. In *Proc. 42nd IEEE Conference on Decision and Control*. Maui, Hawaii.

Aguiar, A.P. and Hespanha, J.a.P. (2006). Minimum-Energy State Estimation for Systems with Perspective Outputs. *IEEE Transactions on Automatic Control*, 51(2), 226–241.

Arrichiello, F., Antonelli, G., Aguiar, A.P., and Pascoal, A. (2011). Observability metric for the relative localization of auvs based on range and depth measurements: Theory and experiments. In *IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*, 3166 –3171. doi:10.1109/IROS.2011.6094466.

Batista, P., Silvestre, C., and Oliveira, P. (2010). Single Beacon Navigation: Observability Analysis and Filter Design. In *Proc. of American Control Conference, Marriott Waterfront, Baltimore, MD, USA*, 6191–6196.

Casey, T. and Hu, J. (2007). Underwater vehicle positioning based on time of arrival measurements from a single beacon. In *Proc. of the MTS/IEEE Oceans 2007, Vancouver, BC, Canada*, 1–8.

Fallon, M.F., Papadopoulos, G., and Leonard, J.J. (2009). Cooperative AUV Navigation using a Single Surface Craft. In *Proc. of Field and Service Robots*, 1–10. Cambridge MA.

Gadre, A.S. (2007). *Observability Analysis in Navigation Systems with an Underwater Vehicle Application.* Ph.D. thesis, Virginia Polytechnic Institute and State University.

Gadre, A.S. and Stilwell, D.J. (2005). Underwater navigation in the presence of unknown currents based on range measurements from a single location. In *Proceedings of the 2005, American Control Conference*, volume 2, 656–661. Portland, OR, USA.

Hespanha, J.P. (2009). *Linear Systems Theory.* Princeton Press, Princeton, New Jersey.

Krener, A. and Ide, K. (2009). Measures of unobservability. In *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, 6401 –6406. doi:10.1109/CDC.2009.5400067.

Krener, A.J. and Isidori, A. (1983). Linearization by output injection and nonlinear observers. *Systems & Control Letters*, 3, 47–52.

Larsen, M.B. (2000). Synthetic long baseline navigation of underwater vehicles. In *Proc. of MTS/IEEE Conference and Exhibition OCEANS.*

Newman, P. and Leonard, J. (2003). Pure range-only subsea SLAM. In *Robotics and Automation, 2003. Proceedings. ICRA '03. IEEE International Conference on*, 1921–1926.

Olson, E., Leonard, J.J., and Teller, S. (2004). Robust Range-Only Beacon Localization. In *Proc. of IEEE/OES conference on Autonomous Underwater Vehicles*, 66–75. doi:10.1109/JOE.2006.880386.

Plestan, F. and Glumineau, A. (1997). Linearization by generalized input-output injection. *Systems & Control Letters*, 31, 115–128.

Saude, J.a. and Aguiar, A.P. (2009). Single Beacon Acoustic Navigation for an AUV in the presence of unknown ocean currents. In *Proc. of MCMC'09 - 8th IFAC Conference on Manoeuvring and Control of Marine Craft.* Guarujá (SP), Brazil.