Exploration of Kinematic Optimal Control on the Lie Group $SO(3)$ ⋆

Alessandro Saccon * **John Hauser** ** **A. Pedro Aguiar** *

* *Institute for Systems and Robotics (ISR), Instituto Superior Técnico (IST), Lisboa. {asaccon, pedro}@isr.ist.utl.pt*
** *Dept. of Electrical, Computer, and Energy Engineering, University of Colorado, Boulder. hauser@colorado.edu*

Abstract: In this paper, we investigate a generalization of the infinite time horizon linear quadratic regulator (LQR) for systems evolving on the special orthogonal group SO(3). Using Pontryagin's Maximum Principle, we derive the necessary conditions for optimality and the associated Hamiltonian equations. For a special class of weighting matrices, we show that the optimal feedback can be computed explicitly and we prove that the non differentiable value function is the viscosity solution of an appropriate Hamiltn-Jacobi-Bellman equation on SO(3). For arbitrary positive definite weighting matrices, numerical simulations allow us to explore the relationship between the optimal trajectories and weighting matrices, and in particular to highlight nontrivial non differentiability properties of the value function.

Keywords: Optimal control, Special orthogonal group SO(3), Rotation matrices, Lie groups, Pontryagin's Maximum Principle, Hamilton-Jacobi-Bellman equation

1. INTRODUCTION

Given the linear system

$$\dot{x}(t) = Ax(t) + Bu(t)\,, \qquad x(0) = x_0 \in \mathbb{R}^n\,, \tag{1}$$

the Linear Quadratic Regulator (LQR) problem

$$\min_{u(\cdot)} \int_0^\infty x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau)\,d\tau\,, \tag{2}$$

is a standard method to obtain an asymptotic stabilizing controller. The weighting matrices Q and R affect the closed loop behavior of the system, and provide a penalty of the state and input of the system, respectively. In this paper we address a similar problem but for systems that evolve on the special orthogonal group SO(3). In this case, we would like to modify the quadratic incremental cost in (2) such that it makes intrinsic sense on SO(3) and resembles on this manifold a quadratic cost. While there is a rich literature on the optimal stabilization of the attitude of a rigid body controlled by means of independent input torques (we refer the reader to, e.g., Tsiotras (1996), Spindler (1998), Krstic and Tsiotras (1999), Bhat and Bernstein (1998) and reference therein) we could not find, as far as the authors know, a study of the problem we are introducing in this work. The stabilization of rigid body dynamics is often addressed employing a set of local coordinates (such as the Cayley-Rodrigues parameters) or a covering map (such as the 2:1 mapping from unit quaternions to rotational matrices) to describe rotational dynamics. As mentioned in Bhat and Bernstein (1998), a proof of global convergence in these sets of coordinates results either in a feedback law which is only defined locally (in the case that a set of local coordinates is chosen) or exhibits *unwinding phenomenon* (when a cover map is chosen), where the initial attitude may start arbitrarily close to the desired final attitude and yet rotate through large angles before coming to rest. In this paper, we are interested in working explicitly with the group of rotational matrices SO(3), studying the properties of the value function (e.g., its continuity and differentiability) as a mapping from SO(3) to $\mathbb{R}$. Our long term goal is to give a complete characterization of the optimal solutions and develop a set of numerical tools for the exploration of optimal control problem on finite dimensional Lie group. Due to space limitations, the proofs have been omitted. A preprint including proofs is available upon request. A journal and complete version of this paper is in preparation.

The paper is organized as follow. In Section 2, we present the notation we use throughout the paper. In Section 3 we introduce the optimal control problem that we are interested to solve and derive the necessary conditions for optimality. A discussion on the incremental cost we have chosen is also provided. In particular, we propose a way to define a "quadratic function" on SO(3) which replace the term $\|x\|^2$ in the incremental cost of (2). In Section 4, we derive the explicit solution of the optimal control problem defined in the previous section. For a special subset of weighting matrices, we show that the continuous (but non differentiable) value function we obtain is the viscosity solution of the Hamilton-Jacobi-Bellman equation on SO(3). In Section 5, we discuss some properties of the solution of the optimal control problem on SO(3) and outline the

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numerical method we used to compute an arbitrarily accurate solution to it. The theoretical results derived in the previous sections are illustrated through numerical simulations. Conclusions and future work are discussed in Section 6.

2. NOTATION AND DEFINITIONS

In the sequel, SO(3) denotes the *special orthogonal group* of dimension 3, defined as the set

$$\mathrm{SO}(3) = \{g \in \mathbb{R}^{3\times 3} : g^T g = I, det(g) = 1\}\,,$$

where the group operation is the standard matrix multiplication. SO(3) is a Lie group, i.e., a smooth manifold with a smooth group operation. We refer to, e.g., (Marsden and Ratiu, 1999, Chapter 9) for basic definitions and properties of Lie Groups. The group SO(3) of rotation matrices can be used, e.g., to describe the attitude of a rigid body with respect to a reference frame. The group identity is the identity matrix, usually indicated by e.

We denote by $T_g\mathrm{SO}(3)$ the *tangent space* and by $T_g^*\mathrm{SO}(3)$ the *cotangent space* of $SO(3)$ at g. The disjoint union of these spaces form the *tangent bundle* $T\mathrm{SO}(3)$ and *cotangent bundle* $T^*\mathrm{SO}(3)$ of the group. Given a vector space X and its dual X^*, the bilinear operator $\langle\cdot,\cdot\rangle : X^* \times X \to \mathbb{R}$, $(\alpha, v) \mapsto \langle \alpha, v\rangle := \alpha(v)$ is the *natural pairing* between the vector space X and its dual. Thus, given $p \in T_g^*\mathrm{SO}(3)$ and $v \in T_g\mathrm{SO}(3)$, the linear map $p(v)$ will be written as $\langle p, v\rangle$.

Given a differentiable mapping between two manifolds $f : M \to N$, $m \mapsto n = f(m)$, we will indicate its tangent map (i.e., its differential) by $T_m f : T_m M \to T_{f(m)}N$, $w_m \mapsto v_n = T_m f\, w_m$. When clear from the context, we will drop the subscript m indicating where the differential is computed.

The maps $L_h : g \mapsto hg$ and $R_h : g \mapsto gh$, $h, g \in \mathrm{SO}(3)$, are called, respectively, *left translation* and *right translation*. Given a vector $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathbb{R}^3$, we denote by $\hat{\xi}$ the skew-symmetric matrix

$$\hat{\xi} = \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix}.$$

The operator $\wedge$ (hat) defines a mapping between $\mathbb{R}^3$ and the space of 3×3 skew-symmetric matrices. Conversely, the operator $\vee$ (vee) defines the inverse mapping that given any 3×3 skew-symmetric matrix S returns the associated vector $\xi = S^\vee$ such that $\hat{\xi} = S$.

The *Lie algebra* of SO(3) and its dual are denoted respectively by $\mathfrak{so}(3)$ end $\mathfrak{so}^*(3)$. The Lie algebra is the set of tangent vectors at the identity $T_e\mathrm{SO}(3)$ (the set of skew-symmetric 3×3 matrices) with the Lie bracket being the matrix commutator

$$[\hat{\xi}, \hat{\eta}] = \hat{\xi}\hat{\eta} - \hat{\eta}\hat{\xi}\,.$$

We can identify the Lie algebra $\mathfrak{so}(3)$ with the Lie algebra $\mathbb{R}^3$ with the (cross product) bracket $[\xi, \eta] = \hat{\xi}\eta$. The maps $\wedge$ (hat) and its inverse $\vee$ (vee) are used to pass from $\mathbb{R}^3$ to $\mathfrak{so}(3)$ and vice versa. The *adjoint representation* of $\mathfrak{so}(3)$, denoted by ad, is defined as

$$\mathrm{ad}_{\hat{\xi}}\hat{\eta} = [\hat{\xi}, \hat{\eta}]$$

or, in the equivalent $\mathbb{R}^3$ notation, $\mathrm{ad}_\xi \eta = [\xi, \eta]$.

Given $\mu \in \mathfrak{so}^*(3)$ and $\xi \in \mathfrak{so}(3)$, the linear map $\mu(\xi)$ will be denoted by $\langle \mu, \xi\rangle$. Given a linear mapping $L : X \to Y$ between the vector spaces X and Y, its *adjoint* mapping $L^* : Y^* \to X^*$ is the unique linear map such that for every $x \in X$, $w \in Y^*$

$$\langle w, Lx\rangle_Y = \langle L^* w, x\rangle_X\,.$$

Given $p = (TR_{g^{-1}})^*\mu \in T_g^*\mathrm{SO}(3)$ and $v = TR_g\xi \in T_g\mathrm{SO}(3)$, we have

$$\begin{aligned}\langle p, v\rangle_{T_g\mathrm{SO}(3)} &= \left\langle (TR_{g^{-1}})^*\mu, TR_g\xi\right\rangle = \left\langle \mu, TR_{g^{-1}}TR_g\xi\right\rangle \\ &= \langle \mu, \xi\rangle_{\mathfrak{so}(3)}\end{aligned}$$

The set $\mathbb{S}^3$ denotes the set of unit quaternions. A unit quaternion in $\mathbb{S}^3 \subset \mathbb{R}^4$ is represented as (a, q), with $a \in \mathbb{R}$ (the scalar part) and $q \in \mathbb{R}^3$ (the vector part). By definition, $\|(a, q)\| = a^2 + q^T q = 1$. Finally, for any square matrix M, $\mathrm{tr}\, M$ is the *trace* of M, the sum of its diagonal elements.

3. OPTIMAL CONTROL ON $SO(3)$

In this section we derive the necessary conditions for optimality of the optimal control problem

$$\min_{\xi(\cdot)} \frac{1}{2}\int_0^\infty \mathrm{tr}\,(e - g(\tau)) + \xi(\tau)^T R\xi(\tau)\, d\tau\,,$$

subject to (3)

$$\dot{g}(t) = \hat{\xi}(t)g(t)\,, \quad g(0) = g_0 \in SO(3)\,,$$

with $g \in \mathrm{SO}(3)$, $\xi \in \mathbb{R}^3$, and $R = R^T > 0$. The main theoretical tool we use is the Pontryagin's Maximum Principle (PMP). From a geometric point of view, this is a statement about the possibility of lifting an optimal trajectory on a state manifold M to a trajectory of an associated vector field on the cotangent bundle T^*M. As the manifold of interest is not an arbitrary manifold but the Lie Group SO(3), we can specialize the result identifying the cotangent bundle $T^*\mathrm{SO}(3)$ with the direct product $\mathrm{SO}(3)\times\mathfrak{so}^*(3)$. Hamiltonian equations originating from the PMP must be modified as to be well defined in this new space. This has the advantage of writing the adjoint equation in the vector space $\mathfrak{so}^*(3)$ which can be identified with $\mathbb{R}^3$. Also, we show that for the special case $R = rI$, $r \in \mathbb{R}$, we can find an explicit expression for the optimal control and value function for the optimization problem (3). In Section 5, we report numerical results obtained for the case of an arbitrary positive definite matrix R.

The following subset of SO(3) plays an important role in our discussion.

Definition 3.1. Let $\Pi \subset \mathrm{SO}(3)$ be defined as

$$\Pi := \{g \in \mathrm{SO}(3) : g = \exp(\pi\hat{n}), n \in \mathbb{R}^3, \|n\| = 1\}\,.$$

The set Π is the set of all rotation matrices which define a rotation of π radians about some axis.

To provide an insight about the incremental cost

$$l(g, \xi) := 1/2\,\mathrm{tr}\,(e - g) + 1/2\,\xi^T R\xi$$

introduced in (3), we first discuss the properties of the function $1/2\,\mathrm{tr}\,(e - g)$.

Proposition 3.1. The function $1/2\,\mathrm{tr}\,(e - g)$ has a unique global minimum at $g = e$ and its minimum value is zero. Also, its maximum value is 2 and is attained at every point in Π.

We can now conclude from the previous proposition, together with the hypothesis that $R = R^T > 0$, that the incremental cost $l(g, \xi)$ of the optimal control problem (3) has a unique minimum for $g = e$ and $\xi = 0$ and that the minimum value is zero. Note also that for each $g \in \mathrm{SO}(3)$ a straightforward computation show that $2\mathrm{tr}\,(e-g) = \|e-g\|_F^2$, where $\|A\|_F$ denotes, for each matrix $A \in \mathbb{R}^{n\times n}$, the *Frobenius* matrix norm defined as $\sqrt{\mathrm{tr}\,(A^T A)}$.

The incremental cost $l(g, \xi)$ provides an appropriate generalization of the quadratic cost function $\|x\|^2/2 + \|u\|_R^2/2$ for the considered state-control manifold $T\mathrm{SO}(3)$. Since this "quadratic" cost function has its unique minimum at $(g, \xi) = (e, 0)$, it is clear that any trajectory $(g(t), \xi(t))$, $t \geq 0$, such that $\int_0^\infty l(g(\tau), \xi(\tau))\, d\tau$ is finite must have the property that $g(t) \to 0$ exponentially fast. Thus, it follows that every minimizing trajectory from a given initial g_0 will have this property. As we will see, there are points in Π such that there is more than one minimizing trajectory. At those points the value function is not differentiable.

3.1 Necessary conditions for optimality

For unconstrained optimal control problems, the PMP requires one to form the pre-Hamiltonian function

$$\hat{H}(g, \xi, p) = 1/2\,\mathrm{tr}\,(e-g) + 1/2\,\xi^T R \xi + \left\langle p, \hat{\xi} g \right\rangle \tag{4}$$

where $p \in T^*\mathrm{SO}(3)$ is the *adjoint* state and $\langle \cdot, \cdot \rangle$ denotes the natural pairing between the tangent and cotangent spaces of SO(3). Then, one defines the Hamiltonian

$$H(g, p) = \min_{\xi} \hat{H}(g, \xi, p)$$

with associated optimal control

$$\xi^*(g, p) = \arg\min_{\xi} \hat{H}(g, \xi, p)\,.$$

The PMP states that, for extremal trajectories, the state and adjoint variables must satisfy the Hamiltonian equations

$$\dot{g} = \frac{\partial H}{\partial p}\,, \qquad \dot{p} = -\frac{\partial H}{\partial g} \tag{5}$$

with appropriate boundary conditions (that we will specify further below). Note that an each element of the cotangent space of SO(3) at g can be uniquely written as $p = (TR_{g^{-1}})^* \mu = \hat{\mu} g$, with $\mu \in \mathfrak{so}^*(3) \approx \mathbb{R}^3$. Thus, we can conclude that the natural pairing between $T_g^*\mathrm{SO}(3)$ and $T_g\mathrm{SO}(3)$ appearing in (4) satisfies

$$\langle p, \dot{g} \rangle_{SO(3)} = \left\langle p, \hat{\xi} g \right\rangle_{SO(3)} = \langle \mu, \xi \rangle_{\mathfrak{so}(3)}\,. \tag{6}$$

Equation (6) allows one to see that the pre-Hamiltonian (4) is a convex quadratic function with respect to the input ξ for any $(g, p) \in T^*\mathrm{SO}(3)$ and therefore has a unique minimum. The minimum is given by

$$\xi^*(g, p) = -R^{-1}\mu|_{\mu = (pg^{-1})^\vee}\,.$$

Substituting this expression in the pre-Hamiltonian (4), we obtain the Hamiltonian

$$\begin{aligned} H(g, p) &= \min_{\xi \in \mathfrak{so}(3)} \hat{H}(p, \xi, g) = \hat{H}(p, \xi^*(g, p), g) \\ &= \frac{1}{2}\,\mathrm{tr}\,(e-g) - \frac{1}{2}\,\mu^T R^{-1} \mu|_{\mu=(pg^{-1})^\vee} \end{aligned} \tag{7}$$

and, as previously mentioned, the PMP states that the state and adjoint variables must satisfy the Hamiltonian equations (5) with appropriate boundary conditions. For a finite time horizon optimal control problem with fixed initial state, free final state, and no terminal cost, the boundary conditions are $g(0) = g_0$ and $p(T) = 0$. For an infinite time horizon optimal control problem, the last boundary condition becomes $\lim_{T\to\infty} p(T) = 0$. It is worth mentioning that the Hamiltonian equations (5) can be derived from the variational principle

$$\delta \int_0^T \langle p, \dot{g} \rangle - H(g, p)\, dt = 0\,, \tag{8}$$

with the variation δg fixed at initial point $\delta g(0) = 0$ and variation δp fixed at final point $\delta p(T) = 0$.

3.2 Right-trivialized Hamiltonian equations

Although formally correct, the Hamiltonian equations written on the cotangent bundle $T^*\mathrm{SO}(3)$ are difficult to manipulate. Equivalent necessary conditions for optimality can be obtained using a right-trivialized version of the pre-Hamiltonian (4) and Hamiltonian equations (5). This is possible only because the state space is a Lie group so that there is a diffeomorphism between $T^*\mathrm{SO}(3)$ and the direct product $\mathrm{SO}(3) \times \mathfrak{g}^*$. The right-trivialized pre-Hamiltonian is defined as

$$\hat{H}^+(g, \xi, \mu) := \hat{H}(g, \xi, p)|_{p=(TR_{g^{-1}})^*\mu}$$

which in our specific case becomes

$$\hat{H}^+(g, \xi, \mu) = 1/2\,\mathrm{tr}\,(e-g) + 1/2\,\xi^T R \xi + \langle \mu, \xi \rangle$$

where $\mu \in \mathfrak{so}^*(3)$ is the right-trivialized adjoint variable. Minimizing the pre-Hamiltonian $\hat{H}^+$ with respect to the input ξ, we obtain the (right-trivialized) Hamiltonian

$$\begin{aligned} H^+(g, \mu) &= \min_{\xi} \hat{H}^+(g, \xi, \mu) \\ &= \hat{H}^+(g, \xi^*(g, \mu), \mu) \\ &= 1/2\,\mathrm{tr}\,(e-g) - 1/2\,\mu^T R^{-1} \mu \end{aligned} \tag{9}$$

where the associated optimal control

$$\xi^*(g, \mu) = \arg\min_{\xi} H^+(g, \xi, \mu) = -R^{-1}\mu \tag{10}$$

minimizes the pre-Hamiltonian H^+ for each $(g, \mu) \in \mathrm{SO}(3) \times \mathfrak{g}^*$. The PMP requires the optimal state-adjoint trajectory to satisfy the right-trivialized Hamiltonian equations (see, e.g., (Jurdjevic, 1997, Theorem 1, Chapter 12))

$$\begin{aligned} \dot{g} g^{-1} &= \frac{\partial H^+}{\partial \mu} \\ \dot{\mu} &= -\mathrm{ad}^*_{\partial H^+/\partial \mu}\, \mu - (TR_g)^* \frac{\partial H^+}{\partial g} \end{aligned} \tag{11}$$

with boundary conditions $g(0) = g_0$ and $\mu(T) = 0$ for the finite time horizon optimal control problem and $\lim_{T\to\infty} \mu(T) = 0$ for the infinite time-horizon optimal control problem.

Note that for the right-trivialized Hamiltonian (9), for any $\eta \in \mathfrak{so}(3)$,

$$\left\langle (TR_g)^* \frac{\partial H^+}{\partial g}, \eta \right\rangle = -\frac{1}{2}\mathrm{tr}\,(\hat{\eta} g) =: w(g)^T \eta\,, \tag{12}$$

that is

$$(TR_g)^* \frac{\partial H^+}{\partial g} = w(g) = -\frac{1}{2}\begin{bmatrix} \mathrm{tr}\,(g\hat{e}_1) \\ \mathrm{tr}\,(g\hat{e}_2) \\ \mathrm{tr}\,(g\hat{e}_3) \end{bmatrix}. \tag{13}$$

Direct computation shows also that $\widehat{w}(g) = (g - g^T)/2$. Using (13) and noting that for the right-trivialized Hamiltonian (9)

$$\frac{\partial H^+}{\partial \mu} = -R^{-1}\mu\,,$$

we see that the right-trivialized Hamiltonian equations (11) specialize into

$$\begin{aligned} \dot{g}g^{-1} &= -(R^{-1}\mu)^\wedge \\ \dot{\mu} &= (-R^{-1}\mu)\times\mu - w(g)\,. \end{aligned} \qquad (14)$$

In the following, we emphasize an important property of this Hamiltonian system that it is key to understand the structure of the infinite time horizon optimal control.

Lemma 3.2. The point $(e, 0) \in \mathrm{SO}(3)\times\mathfrak{g}^*$ is an hyperbolic equilibrium point of (14).

4. SCALAR CONTROL WEIGHTING

In this section we show that, for the special case $R = rI$, $r > 0$, $r \in \mathbb{R}$, we can obtain explicit expressions for the value function and optimal feedback associated to the optimal control problem (3). Observe that in this case, the optimal feedback (10) must satisfy

$$\frac{\partial H^+}{\partial \mu} = -R^{-1}\mu = -\frac{1}{r}\mu\,.$$

Using the expression for $(TR_g)^*\,\partial H^+/\partial g$ derived in (13), it follows that the right-trivialized Hamiltonian equation (11) become

$$\begin{aligned} \dot{g}g^{-1} &= -1/r\,\hat{\mu} \\ \frac{d}{dt}\mu &= (-1/r\,\mu)\times\mu - w(g) = -w(g)\,. \end{aligned} \qquad (15)$$

We now show that we can explicitly compute the stable manifold associated to the equilibrium point $(e, 0) \in \mathrm{SO}(3)\times\mathfrak{so}(3)$.

Proposition 4.1. The set $\mathcal{N}_{\mu_s} = \{(g,\mu) \in \mathrm{SO}(3)\times\mathfrak{so}^*(3) : g \in SO(3)\backslash\Pi, \mu = \mu_s(g)\}$, where

$$\mu_s(g) := 2\sqrt{r}\frac{w(g)}{\sqrt{1+\mathrm{tr}\,(g)}}\,, \qquad (16)$$

defines the *invariant stable* manifold of the hyperbolic equilibrium point $(g,\mu) = (e,0) \in \mathrm{SO}(3)\times\mathfrak{so}^*(3)$ of the left-trivialized Hamiltonian equations (15).

Let us define

$$p_s(g) := (TR_{g^{-1}})^*\mu_s(g) \in T_g^*\mathrm{SO}(3)\,. \qquad (17)$$

Due to the diffeomorphism between $\mathrm{SO}(3)\times\mathfrak{so}^*(3)$ and $T^*\mathrm{SO}(3)$ provided by the mapping $(TR_{g^{-1}})^*$, it follows that the set $\mathcal{N}_{p_s} := \{(g,p) \in T^*\mathrm{SO}(3) : g \in \mathrm{SO}(3)\backslash\Pi, p = p_s(g)\}$ defines the stable manifold associated to the equilibrium point $(e, 0) \in T^*\mathrm{SO}(3)$ of the Hamiltonian equations (5). Since the dimension of $\mathcal{N}_{p_s}$ is half the dimension of $T^*\mathrm{SO}(3)$, this also implies that $\mathcal{N}_{p_s}$ is a Lagrangian submanifold of $T^*\mathrm{SO}(3)$ on which the canonical symplectic 2-form vanishes. This result can be found in (van der Schaft, 1991, Lemma 1). Note that the stable manifold $\mathcal{N}_{p_s}$ is the graph of the 1-form $p_s(g)$, defined on $\mathrm{SO}(3)\backslash\Pi$. The 1-form $p_s(g)$ is *closed* if and only if its graph is a Lagrangian submanifold, see, e.g., (Abraham and Marsden, 1987, Proposition 5.3.15). As $\mathcal{N}_{p_s}$ is Lagrangian, then $p_s(g)$ is *closed* and, since $\mathrm{SO}(3)\backslash\Pi$ is simply connected, $p_s(g)$ is also *exact*, that is there exists a unique function (modulus an additive constant) $V(g)$ such that

$$p_s(g) = \frac{\partial V}{\partial g}(g) \qquad (18)$$

on $\mathrm{SO}(3)\backslash\Pi$. If we choose such an additive constant so that $V(g)$ is zero at the identity, then $V(g)$ is the value function associated to the optimal control problem (3). Indeed, the next result provides an explicit expression for $V(g)$.

Proposition 4.2. The function

$$V(g) = 2\sqrt{r}(2 - \sqrt{1+\mathrm{tr}\,g})\,. \qquad (19)$$

is the value function of the optimal control problem (3) for the case $R = rI$, $r > 0$, $r \in \mathbb{R}$. Moreover, the optimal control $\xi^*(g)$ is given by the feedback law

$$\xi^*(g) = -\frac{1}{r}\mu_s(g) = -\frac{2}{\sqrt{r}}\frac{w(g)}{\sqrt{1+\mathrm{tr}\,(g)}}\,. \qquad (20)$$

The optimal control $\xi^*(g)$ is defined for all g in $\mathrm{SO}(3)\backslash\Pi$, the set of differentiability points of $V(g)$.

5. GENERAL CONTROL WEIGHTING

We could not find an explicit expression for the value function V when R is not a multiple of the identity matrix. However, we have solved the optimization problem (3) numerically in order to explore the relationship between the weighting matrix R and value function V. We can restrict our attention without loss of generality to *diagonal* positive definite weighting matrix R. Indeed, any positive definite $R \in \mathbb{R}^3$ can be always decomposed as $U^T\Lambda U$, where $\Lambda \in \mathbb{R}^{3\times3}$ is a diagonal positive definite matrix and $U \in \mathrm{SO}(3)$. Let $\eta \in \mathfrak{so}(3)$ and $\tilde{g} \in \mathrm{SO}(3)$ satisfy $\xi = U\eta$ and $g = U\tilde{g}U^T$. Substituting these expressions for ξ and g into (3), we obtain the optimal control problem (3) rewritten in the form

$$\min_{\eta(\cdot)} \frac{1}{2}\int_0^\infty \mathrm{tr}\,(e - \tilde{g}(\tau)) + \eta(\tau)^T\Lambda\eta(\tau)\,d\tau\,, \qquad (21)$$

subject to

$$\dot{\tilde{g}}(t) = \hat{\eta}(t)\tilde{g}(t)\,, \quad \tilde{g}(0) = \tilde{g}_0 \in SO(3)\,,$$

where $\tilde{g}_0$ satisfies $g_0 = U\tilde{g}_0U^T$. In Section 4, for the special case $R = rI$, we concluded that the set of non-differentiable points for the value function is Π. According to numerical evidence, we *claim* that this is also true for an arbitrary positive definite diagonal weighting matrix Λ. More precisely, we claim the following.

Claim 1. For any diagonal positive definite matrix R, the value function V associated to the optimal control problem (3) is everywhere differentiable with the exception of the set Π.

Remark. If one accepts this claim, then one can prove that Π is the set of non-differentiable points of the value function V associated to *any* positive definite matrix R. This result easily follows from the equivalence between the optimal control problems (3) and (21) under the state-input transformation $\xi = U\eta$ and $g = U\tilde{g}U^T$ satisfying $R = U\lambda U^T \in SO(3)$ and from the equality $U\Pi U^T = \Pi$ valid for each $U \in SO(3)$. ■

5.1 How to solve the optimal control problem numerically

The *dynamic programming principle* (see, e.g., Bardi and Capuzzo-Dolcetta (1997)) states that for any $T \geq 0$, the

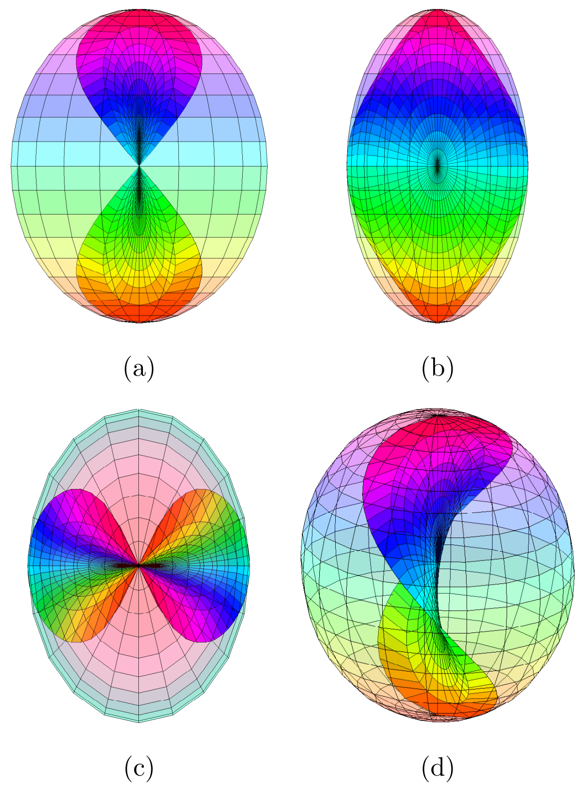

Fig. 1. Image of the x-z disk of radius one through the mapping $\mu_s(\cdot) : \mathrm{SO}(3)\backslash\Pi \approx B^{\mathbb{R}^3}_{[0,1)} \to \mathfrak{so}^*(3) \approx \mathbb{R}^3$ for $R = diag(1,2,3)$: Part (a) plots the $y - z$ section Part (b) plots the $x - z$ section; Part (c) plots the $x - y$ section; Plot (d) has been chosen to highlight the twisting of the surface

value function $V(\cdot)$ of the infinite time horizon optimal control problem (3) satisfies

$$
\begin{aligned}
V(g(0)) &:= \min_{\xi(\cdot)} \frac{1}{2}\int_0^\infty l(g(\tau),\xi(\tau))\,d\tau = \\
&= \min_{\xi(\cdot)}\left\{\frac{1}{2}\int_0^T l(g(\tau),\xi(\tau))\,d\tau + V(g(T))\right\} \quad (22)
\end{aligned}
$$

where $\dot g(t) = \hat\xi(t)g(t)$, $g(0) = g_0$. Note that for each initial condition g_0, any optimal state trajectory $g(t)$ starting at g_0 must converge to the group identity e. Following the approach detailed in Jadbabaie et al. (2001) and reference therein, we approximate the infinite time horizon problem (3) with the finite horizon optimal control problem with *terminal cost*

$$
\min_{\xi(\cdot)} \int_0^T l(g(\tau),\xi(\tau))\,d\tau + W(g(T)) \quad (23)
$$

subject to

$$
\dot g(t) = \hat\xi(t)g(t)\,, \quad g(0) = g_0 \in SO(3)\,,
$$

where the terminal cost W is a sufficiently accurate approximation around the identify of the infinite horizon optimal control problem (3). The time horizon T must be chosen large enough to guarantee that for any initial condition g_0 the optimal trajectory $g(\cdot)$ of (23) at time T is sufficiently close to the identity so that $W(g(T))$ is equal to the infinite time horizon value function $V(g(T))$ within numerical accuracy. It is straightforward to verify that each $g \in \mathrm{SO}(3)\backslash\Pi$ can be identified with the two unit quaternions (a,q) and $-(a,q)$ that satisfy

$$
a = \sqrt{1 - x^T x}\,, \qquad q = x\,, \quad (24)
$$

with $x = w(g)/\sqrt{1 + \mathrm{tr}\, g}$. Equation (24) suggests that we can use the open ball of radius one in $\mathbb{R}^3$ (that we write $B^{\mathbb{R}^3}_{[0,1)}(0)$) as a set of local coordinates for $\mathrm{SO}(3)\backslash\Pi$. Moreover, we can approximate the optimal control problem (3) on $\mathrm{SO}(3)\backslash\Pi$ as

$$
\min_{\xi(\cdot)} \frac{1}{2}\int_0^T x(\tau)^T Q x(\tau) + \xi(\tau)^T R \xi(\tau)\,d\tau + \frac{1}{2}x(T)^T P x(T)
$$

subject to

$$
\dot x = \frac{1}{2}(\sqrt{1 - x^T x} I - \hat x)\xi\,, \quad x(0) = x_0 \in B^{\mathbb{R}^3}_{[0,1)}(0)\,, \quad (25)
$$

where $Q = 4I$, $R = R^T > 0$, and P is the solution of the algebraic Riccati equation $PR^{-1}P + Q = 0$. Note that $x^T Q x = 2q^T 2q = \mathrm{tr}\,(e - g)$ (see proof of Proposition 3.1). The quadratic function $1/2\, x^T P x$ is the approximation of value function of the infinite time horizon version of (25). This approximation is obtained by solving the infinite time horizon LQR problem with quadratic cost

$$
\frac{1}{2}\int_0^\infty x(\tau)^T Q x(\tau) + \xi(\tau)^T R \xi(\tau)\,d\tau
$$

and dynamics $\dot x = 1/2\xi$ obtained by linearizing the nonlinear dynamics of (25) around the equilibrium state-control trajectory $(g(t),\xi(t)) \equiv (e,0)$. We solved the optimal control problem iteratively (25) using the *Projection Operator based Optimization Strategy* described in Hauser (2002) and Hauser and Saccon (2006). Remarkably, we found that the numerical algorithm converges (first order variation less than 10^{-7}) even when the initial condition is chosen quite close to Π ($\|x\| \approx 0.9999999$), which is the border of the domain of definition of the coordinate map (24). In the following, we discuss the numerical results we obtained.

Remark. One of the first test that we performed was to check that the numerical values computed with our method agree, for $R = rI$, with those predicted by the theory. With a time horizon $T \approx 10$ s, numerical solution agreed within an error of 10^{-12} with the explicit solution. Due to space limitation this is not reported here. ■

5.2 Numerical results

For a general weighting matrix R, the adjoint variable satisfies the following property.

Proposition 5.1. For any optimal trajectory, the associated (right trivialized) adjoint variable $\mu(t) = TR^*_{g^{-1}}p(t)$ satisfies for a.e. $t \geq 0$

$$
\|\mu(t)\|_R^2 \leq 4 \quad (26)
$$

that is, it is contained in the *ellipsoid* implicitly defined by the equation $\mu^T R \mu = 4$.

Proof: The state-adjoint trajectory associated to an optimal trajectory of (3) satisfies for a.e. $t \geq 0$

$$
H^+(g(t),\mu(t)) = \frac{1}{2}\big(\mathrm{tr}\,(e - g(t)) - \|\mu(t)\|_R^2\big) = 0\,.
$$

Since $0 \leq 1/2\,\mathrm{tr}\,(e-g) \leq 2$, $\forall g \in \mathrm{SO}(3)$, it follows (26). □

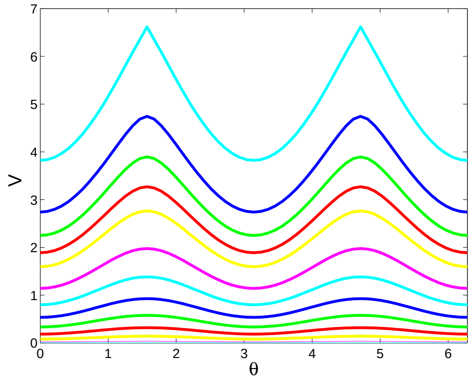

Fig. 2. Representation of value function for $R = \operatorname{diag}(1, 1, 3)$. The different lines represent the value function for different value of radial distance ρ.

As previously mentioned, without loss of generality, we restrict our attention to a positive definite diagonal weighting matrix $R = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$. Numerical results for $(\lambda_1, \lambda_2, \lambda_3) = (1, 2, 3)$ are presented in Figure 1. The figure provides a graphical representation of the mapping μ_s that goes from $\mathrm{SO}(3)\backslash\Pi \approx B^{\mathbb{R}^3}_{[0,1)}$ to $\mathfrak{so}(3)^* \approx \mathbb{R}^3$ and that was introduced, only for the case $R = rI$, in Proposition 4.1. This should allow the reader to get some intuition on how the stable manifold of the Hamiltonian equations looks like for a general weighting matrix R. The mapping $\mu_s(\cdot)$ is a mapping from a three dimensional space to a three dimensional space. For visualization, we found convenient to show only the image of a two dimensional subset of the domain of this mapping. The optimal control problem (25) has been solved on a disk, lying on the x-z plane, centered at the origin of $B^{\mathbb{R}^3}_{[0,1)}$ and with radius strictly less than one. More specifically, we solved (25) numerically on a finite set of points defining a grid the x-z disk. The points on this grid are parameterized by the the *radial distance* $\rho \in [0, 1]$ and *angular displacement* $\theta \in [0, 2\pi]$ so that

$$x_0(\rho, \theta) := [\rho \cos\theta \;\; 0 \;\; \rho \sin\theta]^T \; . \tag{27}$$

Figure 1(d) depicts the image of the x-z disk through the mapping μ_s. The image of x-z disk is a twisted surface which, as predicted by proposition (5.1), is contained in the ellipsoid $\mu_1/1+\mu_2^2/2+\mu_3^2/3 = 4$, which is also displayed. To help the reader to visualize such a surface, we also provide three sections of this surface in parts (a), (b), and (c) of the same figure. Very similar pictures can be produced "slicing" the domain $B^{\mathbb{R}^3}_{[0,1)}$ along a plane containing at least one of the x, y, or z axes. The image of those disk-shaped "slices" is always a twisted surface quite similar to that presented in Figure 1.

5.3 Kinks along the ridge

A very interesting phenomenon, which has not an explanation yet, has being noted when the weighting matrix R has two equal elements. A representation of the value function for the case $R = (1, 1, 3)$ is given in Figure 2. Figure 2 represents the value function for different value of the radial distance $\rho \in 0.999 \times \{10^{-3}\, .1\, .2\, .3\, .4\, .5\, .6\, .7\, .8\, .85\, .9\, .95\, 1\}$ and for $\theta \in [0, 2\pi]$. The value function appears to have two quite interesting kinks in correspondence of the points $\exp(\pi\hat{e}_3)$ and $\exp(-\pi\hat{e}_3)$ in Π, which are represented in polar coordinates (ρ, θ) by $(1, \pi/2)$ and $(1, 3\pi/2)$, respectively. The value function appear to have a ridge not only as we approach Π (as we claim it always does, according to Claim 1), but a kink also appears as we consider the value of V on a series of concentric spheres whose radius (ρ) tends to one.

6. CONCLUSIONS AND FURTHER WORK

We have presented an optimal stabilizing controller for the driftless dynamics $\dot{g}(t) = \hat{\xi}(t)g(t)$, $g(0) = g_0$, showing that a closed form solution exist for the special case $R = rI$ and studied the nature of the optimal solution by means of numerical optimization for a general weight R. We are interested to further investigate the optimal solution for an arbitrary weighting matrix R and introduce a general weighting matrix Q for the rotational matrix g. Moreover, in another publication, we will show that the value function we have detailed for the special case $R = rI$ can be used to obtain a inverse optimal controller for the stabilization of the dynamics of a rigid body with arbitrary inertia matrix. This will be done working directly on $T\mathrm{SO}(3)$, without the need of using a set of local coordinates as done in, e.g., Krstic and Tsiotras (1999).

REFERENCES

Abraham, R. and Marsden, J.E. (1987). *Foundations of Mechanics*. Addison-Wesley, 2nd edition.

Bardi, M. and Capuzzo-Dolcetta, I. (1997). *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*. Systems & Control: Foundations & Applications. Birkhuser.

Bhat, S.P. and Bernstein, D.S. (1998). A topological obstruction to global asymptotic stabilization of rotational motion and the unwinding phenomenon. In *American Control Conference*. Philadelphia, Pennsylvania, USA.

Hauser, J. (2002). A projection operator approach to the optimization of trajectory functionals. In *Proceedings of the 15th IFAC World Congress*. Barcelona, Spain.

Hauser, J. and Saccon, A. (2006). A barrier function method for the optimization of trajectory functionals with constraints. In *45th IEEE Conference on Decision and Control (CDC)*. San Diego, CA, USA.

Jadbabaie, A., Yu, J., and Hauser, J. (2001). Unconstrained receding-horizon control of nonlinear systems. *IEEE Transactions on Automatic Control*, 46(5), 776–783.

Jurdjevic, V. (1997). *Geometric control theory*. Cambridge University Press.

Krstic, M. and Tsiotras, P. (1999). Inverse optimal stabilization of a rigid spacecraft. *IEEE Transactions on Automatic Control*, 44(5), 1042–1049.

Marsden, J.E. and Ratiu, T.S. (1999). *Introduction to Mechanics and Symmetry*. Springer, 2nd edition.

Spindler, K. (1998). Optimal control on lie groups with applications to attituted control. *Math. Control Signal Systems*, 11, 197–219.

Tsiotras, P. (1996). Stabilization and optimality results for the attitute control problem. *Journal of Guidance, Control, and Dynamics*, 19(4), 772–779.

van der Schaft, A. (1991). On a state space approach to nonlinear H_∞ control. *Systems Control Letters*, 1, 1–8.