State estimation for systems on SE(3) with implicit outputs: An application to visual servoing ⋆

Sérgio S. Rodrigues * **Naveena Crasta** ** **António Pedro Aguiar** ** **Fátima Silva Leite** ***

* *University of de Cergy-Pontoise, Department of Mathematics, UMR CNRS 8088, F-95000 Cergy-Pontoise, France (e-mail: sergio.rodrigues@u-cergy.fr)*
** *Institute for Systems and Robotics, Instituto Superior Técnico, Technical University of Lisbon, Portugal (e-mail: {ncrasta, pedro}@isr.ist.utl.pt)*
*** *Department of Mathematics and Institute for Systems and Robotics, University of Coimbra, Portugal (e-mail: fleite@mat.uc.pt)*

Abstract: Motivated by applications in visual servoing, we consider the state estimation problem for a class of systems described by implicit outputs and whose state lives in the special Euclidean group SE(3). We propose an observer in the group of motion SE(3) that preserves invariance and therefore takes explicitly into consideration the geometry of the problem. We discuss conditions under which the linearized state estimation error converges exponentially fast. Furthermore, we analyze the problem when the system is subject to disturbances and noises and show that the estimate converges to a neighborhood of the real solution. The size of the neighborhood increases/decreases gracefully with the bound of the disturbance and noise. We apply and illustrate these results through an application of position and attitude estimation of a rigid body using measurements from a camera attached to the rigid body.

Keywords: Observers, Lie groups, Visual servoing, Robotics.

1. INTRODUCTION

During the last few decades there has been an extensive study on the design of observers for nonlinear systems. In simple terms, an observer or estimator can be defined as a process that provides in real time the estimate of the state (or some function of it) of the plant from partial and possibly noisy measurements of the inputs and outputs, and inexact knowledge of the initial condition. For linear systems evolving on n-dimensional vector spaces, state observer and filter designs employ the traditional Kalman filter (Kalman, 1960) and Luenberger type observer (Luenberger, 1964). In fact, it is well-known that the Kalman filter is the optimal state estimation algorithm in a well defined sense (see eg. Anderson and Moore (1979)).

For nonlinear systems, the extended Kalman filter is a widely used method for estimating the state. It is obtained by linearizing the nonlinear dynamics and the observation along the trajectory of the estimate. However, if there are substantial nonlinearities or the state lives in some special manifold, there are no guarantees that the state estimate will evolve in the same manifold and even that the estimate will converge to a neighborhood of the true one. These problems are particularly relevant because they arise in many modern day applications such as the motion control of unmanned aerial vehicles, underwater vehicles, and autonomous robots. See e.g. (Bullo, 2000), (Al-Hiddabi and McClamroch, 2002), (Skjetne et al., 2004), (Aguiar and Hespanha, 2007). Other engineering applications that were studied in (Bonnabel et al., 2008b) are exothermic chemical reactor, a nonholonomic car, and a velocity-aided inertial navigation (Bonnabel et al., 2006). Typically, these applications require the design of robust nonlinear observers for systems evolving on Lie groups. In (Bonnabel et al., 2009a), (Bonnabel and Rouchon, 2005), (Lageman et al., 2009a), (Lageman et al., 2008b), (Lageman et al., 2009b), (Lageman et al., 2008a) a geometrical framework for the design of symmetry preserving observers on finite-dimensional Lie groups is described. In (Bonnabel et al., 2008a), it is shown that when the output map associated with a left-invariant dynamics on an arbitrary Lie group is right-left equivariant, then it is possible to build non-linear observers such that the error equation is autonomous. See also the recent work in (Bonnabel et al., 2009b) where an invariant extended Kalman filter is proposed.

In this paper, we consider left-invariant dynamical systems with implicit outputs, for which the results mentioned above do not apply. Systems of this kind typically arise in mobile robotic applications using dynamic vision such as the estimation of a motion of a camera from a sequence of images. In particular, in (Aguiar and Hespanha, 2006) and (Aguiar and Hespanha, 2009), the problem of

⋆ This work was supported in part by projects DENO / FCT-PT (PTDC/EEA-ACR/67020/2006), Co3-AUVs (EU FP7 under grant agreement No. 231378), FCT (ISR/IST plurianual funding) through the PIDDAC Program funds, and the CMU-Portugal program.

estimating the position and orientation of a controlled rigid body using measurements from a monocular charged-coupled-device (CCD) camera attached to the vehicle was addressed. The reader is referred to (Ghosh et al., 1994), (Ghosh and Loucks, 1995), (Takahashi and Ghosh, 2001) for several other examples of implicit output systems in the context of motion and shape estimation. In this work, we propose an observer in the group of motion SE(3) and discuss conditions under which the linearized state estimation error converges exponentially fast. The proposed observer takes explicitly into consideration the geometry of the problem, namely, the observer is constructed to also satisfy the left invariance property that the process dynamics possess. We show that when the system is subject to disturbances and noises, the estimate converges to a neighborhood of the real solution. The size of the neighborhood increases/decreases gracefully with the bound of the disturbance and noise. An application to the estimation of the position and attitude of a rigid body using measurements from a monocular camera on-board illustrates the results.

The outline of the paper is as follows. Section 2 introduces the mathematical preliminaries and Section 3 formulates the state estimation problem. In Section 4 we propose a left-invariant dynamic observer for estimating the state of systems on SE(3) with implicit outputs, and determine under what conditions the state estimate converges exponentially to the true state. In Section 5 we analyze the robustness of the proposed observer in the presence of disturbance and noise. An example in visual control illustrates the results in Section 6. Concluding remarks are given in Section 7.

Due to space limitations, some of the proofs are omitted. These can be found in (Rodrigues et al., 2009).

2. MATHEMATICAL PRELIMINARIES

In this section we introduce notations and definitions used through out the paper. We denote the Euclidean norm in $\mathbb{R}^n$ by $\|\cdot\|$, and the identity matrix of size n by I_n. Given $A \in \mathbb{R}^{n\times n}$, we let $\det(A)$ and $\mathrm{Tr}(A)$ denote the determinant and the trace of the matrix A, respectively. We consider the scalar product of $A, B \in \mathbb{R}^{n\times n}$ as being defined by $\langle A, B\rangle \overset{\text{def}}{=} \mathrm{Tr}(A^{\mathrm{T}}B)$. The corresponding norm $\|A\| = \sqrt{\langle A, A\rangle}$ is the so-called Frobenius norm. Further, if $A \in \mathbb{R}^{n\times n}$ is time dependent and invertible, from the identity $A^{-1}A = I_n$, one may deduce

$$\dot{A}^{-1}A + A^{-1}\dot{A} = 0. \tag{1}$$

The cross product of vectors $u, v \in \mathbb{R}^3$ is denoted by $u \times v$. For every $u \in \mathbb{R}^3$, $(u\times) = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$ denotes the matrix representation of the linear map $v \mapsto u \times v$, $v \in \mathbb{R}^3$. It can be easily shown that, for every $u, v \in \mathbb{R}^3$, $\mathrm{Tr}((u\times)^{\mathrm{T}}(v\times)) = 2u^{\mathrm{T}}v$. Given a vector $u \in \mathbb{R}^3$, we denote by $\bar{u} \in \mathbb{R}^4$ its homogeneous coordinates, that is, $\bar{u} = (u,\, 1)'$(Ma et al., 2005).

The *special orthogonal* group in three-dimensions is denoted by SO(3) and its Lie algebra is denoted by so(3). We denote the *special Euclidean* group and its Lie algebra by SE(3) and se(3), respectively. We follow the standard notations of (Ma et al., 2005). Elements $g \in \mathrm{SE}(3)$ and $\Omega \in \mathrm{se}(3)$ are represented respectively by matrices with the following structures:

$$g = \begin{bmatrix} g_R & g_T \\ 0 & 1 \end{bmatrix}, \Omega = \begin{bmatrix} \Omega_R & \Omega_T \\ 0 & 0 \end{bmatrix}, \tag{2}$$

where $g_R \in \mathrm{SO}(3)$, $\Omega_R \in \mathrm{so}(3)$, and $g_T, \Omega_T \in \mathbb{R}^3$. If $g \in \mathrm{SE}(3)$ is time dependent, we may use (1) to write $\dot{g} = g(-\dot{g}^{-1}g)$. Using (2) it is easy to check that $-\dot{g}^{-1}g \in \mathrm{se}(3)$. So, the equation $\dot{g}(t) = g(t)\Omega(t)$, which will be used in the next section, makes sense.

We next present a result that will be useful later in the paper.

Lemma 1. Consider $\xi = \begin{bmatrix} \xi_R & \xi_T \\ 0 & 0 \end{bmatrix} \in \mathrm{se}(3)$, where $\xi_R = (\bar{\xi}\times)$ and $\bar{\xi}, \xi_T \in \mathbb{R}^3$. Then $\|\xi\|^2 = 2\|\bar{\xi}\|^2 + \|\xi_T\|^2$.

The Lie bracket of two matrices $A, B \in \mathbb{R}^{n\times n}$ is denoted by $[A, B]$ or, equivalently, $\mathrm{ad}_A B$, and is defined as the commutator $[A, B] = AB - BA$. Given $A, B \in \mathbb{R}^{n\times n}$, we denote $\mathrm{ad}_A^1 B = \mathrm{ad}_A B$ and $\mathrm{ad}_A^{k+1} B = \mathrm{ad}_A \mathrm{ad}_A^k B$ for every $k \in \mathbb{N}$.

3. PROBLEM STATEMENT

Consider a left-invariant dynamical system evolving on SE(3), described by

$$\dot{g}(t) = g(t)\Omega(t), \quad g(0) = g_0, \tag{3}$$

where Ω takes values in se(3) and is assumed to be known for all $t \geq 0$.

Consider a set of given points $p_1, \ldots, p_N \in \mathbb{R}^3$, and let $y_j = [y_{j_1} \;\; y_{j_2} \;\; 1]^{\mathrm{T}} \in \mathbb{R}^3$, $j \in \mathcal{J}$ be the outputs (measurement signals) of the dynamical system (3) described implicitly by

$$\alpha_j(t)y_j(t) = F(t)\Pi_0 g(t)\bar{p}_j, \tag{4}$$

where $\mathcal{J} \subseteq \{1, 2, \ldots, N\}$ is an index set that may depend on time, $\bar{p}_j \in \mathbb{R}^4$ is the homogeneous representation of p_j, the α_j's are unknown scalar continuous function of time satisfying $\alpha_j(t) > 0$ for every $t \geq 0$, $F(t) \in \mathbb{R}^{3\times 3}$ is a known nonsingular matrix, and $\Pi_0 = [I_3 \; 0] \in \mathbb{R}^{3\times 4}$ is often referred to as the standard (or canonical) projection matrix (Ma et al., 2005). We assume that the right-hand-side of (4) and $\Pi_0 g(t)\bar{p}_j$ are both bounded below and above, that is, there exist m and M, $0 < m \leq M$, such that for all $t \geq 0$,

$$m \leq \|F(t)\Pi_0 g(t)\bar{p}_j\|, \|\Pi_0 g(t)\bar{p}_j\| \leq M. \tag{5}$$

The problem addressed in this paper can be stated as follows. *Consider the continuous-time left-invariant dynamical system described by* (3)-(4). *Let* $\hat{g} \in \mathrm{SE}(3)$ *be the estimate of the state* g *with a given initial estimate* $\hat{g}(0) = \hat{g}_0$. *Design a state observer for* (3)-(4) *that accepts as inputs the measured input* $\Omega(\tau)$ *and the output of the process* $y_j(\tau)$ *for every* $\tau \in [0, t)$, $j \in \mathcal{J}$, *and returns* $\hat{g}(t)$ *at time* t, *for every* $t \geq 0$. *The observer should satisfy some desired performance and robustness properties that will be mentioned later in the paper.*

Remark 3.1. System (3)–(4) arises for example when one needs to estimate the position and orientation of a robotic vehicle using measurements from an on-board monocular charged-coupled-device (CCD) camera. In that case, adopting the frontal pinhole camera model (Ma et al., 2005), the scalar α_j captures the unknown depth of a point p_j, and F is a matrix transformation that depends on the parameters of the camera such as the focal length, the scaling factors, and the center offsets. The assumption in (5) is very reasonable and only means that the image points are well defined in the sense that they live in some compact set. Notice that if for some point that assumption does not hold, then this only implies to take it out from the index set $\mathcal{J}$.

4. OBSERVER DESIGN AND CONVERGENCE ANALYSIS

Consider the continuous-time left-invariant dynamical system (3)-(4). We propose the nonlinear observer

$$\dot{\hat{g}}(t) = \hat{g}(t)\Omega(t) + \zeta\,\Theta(\hat{g}(t), y(t))\,\hat{g}(t), \quad \hat{g}(0) = \hat{g}_0, \quad (6)$$

where $\hat{g} \in \mathrm{SE}(3)$ is the estimate of the state g, and $\Theta(\hat{g}, y) \in \mathrm{se}(3)$ is given by

$$\Theta(\hat{g}, y) \overset{\text{def}}{=} \begin{bmatrix} \Theta_R(\hat{g}, y) & \Theta_T(\hat{g}, y) \\ 0 & 0 \end{bmatrix}, \quad (7)$$

with

$$\Theta_R(\hat{g}, y) = \sum_{j\in\mathcal{J}} \frac{((((\tilde{y}_j \times \Pi_0\hat{g}\bar{p}_j) \times \Pi_0\hat{g}\bar{p}_j) \times \Pi_0\hat{g}\bar{p}_j)\times)}{D(\hat{g}\bar{p}_j)}, \quad (8)$$

$$\Theta_T(\hat{g}, y) = \sum_{j\in\mathcal{J}} \frac{1}{D(\hat{g}\bar{p}_j)}((-2\tilde{y}_j \times \Pi_0\hat{g}\bar{p}_j) \times \Pi_0\hat{g}\bar{p}_j), \quad (9)$$

$$\tilde{y}_j = F^{-1}\frac{y_j}{\|y_j\|}, \quad (10)$$

$$D(\hat{g}\bar{p}_j) \overset{\text{def}}{=} (\#\mathcal{J})\|\Pi_0\hat{g}\bar{p}_j\|^2(1 + \|\Pi_0\hat{g}\bar{p}_j\|). \quad (11)$$

The symbol $\#\mathcal{J}$ means the number of elements of $\mathcal{J}$, and the parameter $\zeta > 0$ is a tuning constant.

At this point it is convenient to stress that the proposed observer is a nonlinear dynamic system with inputs Ω and y_j and whose state evolves on SE(3). Also, defining $\hat{\Theta} \overset{\text{def}}{=} \hat{g}^{-1}\Theta\hat{g}$, system (6) can be rewritten as $\dot{\hat{g}} = \hat{g}(\Omega+\zeta\hat{\Theta})$, and, by a direct computation we can show that $\hat{\Theta} \in \mathrm{se}(3)$. Thus, like the dynamics of g in (3), also the dynamics of $\hat{g}$ is *left-invariant*. Moreover, if $\hat{g}(0) = g(0)$, then $\hat{\Theta} = 0$ for every $t \geq 0$, which means that the observer dynamics in that case is exactly the same as the original system.This last fact, can be easily checked from the error dynamics equation defined in the next section.

4.1 The error dynamics

We now proceed with the derivation of the error dynamics. Since $\alpha_j > 0$, the expressions (4) and (10) imply that

$$\tilde{y}_j = \frac{\Pi_0 g\bar{p}_j}{\|F\Pi_0 g\bar{p}_j\|}. \quad (12)$$

Using the Lagrange identity for the cross product of vectors together with (12), we have that (8) and (9) can be simplified to

$$\Theta_R(\hat{g}, y) = \sum_{j\in\mathcal{J}} \frac{\|\Pi_0\hat{g}\bar{p}_j\|^2}{D(\hat{g}\bar{p}_j)} \frac{((\Pi_0\hat{g}\bar{p}_j \times \Pi_0 g\bar{p}_j)\times)}{\|F\Pi_0 g\bar{p}_j\|}, \quad (13)$$

$$\Theta_T(\hat{g}, y) = \sum_{j\in\mathcal{J}} \frac{-2}{D(\hat{g}\bar{p}_j)} \frac{((\Pi_0 g\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j) \times \Pi_0\hat{g}\bar{p}_j)}{\|F\Pi_0 g\bar{p}_j\|}. \quad (14)$$

Remark 4.1. Note that from (5) and (11) if $\mathcal{J} \neq 0$ then a lower bound for $D(\hat{g}\bar{p}_j)\|F\Pi_0 g\bar{p}_j\|$ is given by $m^2(m+1)m$, which implies that the observer is well defined.

Define now the error $\eta(t) \overset{\text{def}}{=} \hat{g}(t)g^{-1}(t)$. Using (1), one can write

$$\dot{\eta} = \dot{\hat{g}}g^{-1} + \hat{g}\dot{g}^{-1} = \zeta\,\Theta(\hat{g}, y)\eta, \quad \eta(0) = \hat{g}_0 g_0^{-1}, \quad (15)$$

where taking into account that $g = \eta^{-1}\hat{g}$, it follows that $\Theta(\hat{g}, y)$ can be rewritten as $\Theta(\hat{g}, y) = \Theta(\eta) = \begin{bmatrix} \Theta_R(\eta) & \Theta_T(\eta) \\ 0 & 0 \end{bmatrix}$, with

$$\Theta_R(\eta) = \sum_{j\in\mathcal{J}} \frac{\|\Pi_0\hat{g}\bar{p}_j\|^2}{D(\hat{g}\bar{p}_j)} \frac{((\Pi_0\hat{g}\bar{p}_j \times \Pi_0\eta^{-1}\hat{g}\bar{p}_j)\times)}{\|F\Pi_0 g\bar{p}_j\|}, \quad (16)$$

$$\Theta_T(\eta) = \sum_{j\in\mathcal{J}} \frac{-2}{D(\hat{g}\bar{p}_j)} \frac{((\Pi_0\eta^{-1}\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j) \times \Pi_0\hat{g}\bar{p}_j)}{\|F\Pi_0 g\bar{p}_j\|}. \quad (17)$$

Since a Lie group is a complex geometric object, it is a standard procedure to estimate results on a matrix Lie group G from results in the vector space which is its Lie algebra, here denoted by $\mathcal{L}$. We will adopt this procedure to analyze the error η and, later, prove convergence results. The Lie algebra $\mathcal{L}$ is the best linear approximation of G in the neighborhood of the identity I, and the exponential map exp, which sends elements in $\mathcal{L}$ to elements in G plays a crucial role in transferring data and results from one structure to the other. The exponential mapping is known to be bijective from a small neighborhood of $0 \in \mathcal{L}$ to a small neighborhood of the identity in G, and its inverse is denoted by log.

If η is sufficiently close to the identity, there is a representation $\eta = \exp(\epsilon\xi)$, where $\epsilon > 0$ and $\xi \in \mathrm{se}(3)$ satisfies $\|\xi\| = 1$. Since, $\exp(\epsilon\xi) = I_4 + \epsilon\xi + O(\epsilon^2)$, where $O(\epsilon^2)$ represents the terms containing ϵ^k, for $k \geq 2$, for small ϵ, $I_4 + \epsilon\xi$ is a good approximation for η. In the rest of the paper, and for the sake of simplicity, we may use the alternative notation e^A instead of $\exp(A)$. We henceforth make the following assumption.

Assumption 4.1. The error η is close enough to I_4, that is, $\eta \in \mathcal{N}_\epsilon \overset{\text{def}}{=} \{v = \exp(\epsilon\xi)\colon \xi \in \mathrm{se}(3) \text{ and } \|\xi\| = 1\}$, where $0 \leq \epsilon < 1$.

Remark 4.2. We may, without loss of generality assume that η is close to the identity. This is due to the fact that $x\mathcal{L} \sim \mathcal{L}$, for $x \in G$, is the best linear approximation of G in the neighborhood of x. So, if η is in the neighborhood of $x \in G$, then ηx^{-1} is close to the identity.

Using Lemma 1.7.3 of (Machado, 2006), which can be deduced from Lemma 3.4 in (Sattinger and Weaver, 1980), we have $\frac{\mathrm{d}}{\mathrm{d}t}(\epsilon\xi) = \left.\frac{u}{e^u-1}\right|_{u\,=\,\mathrm{ad}_{\epsilon\xi}}(\dot{\eta}\eta^{-1})$, where $\frac{u}{e^u-1} = \sum_{m=0}^{+\infty}\frac{(-1)^m}{m+1}(e^u - 1)^m$. Using (15), we have $\dot{\eta}\eta^{-1} = \zeta\Theta(\hat{g}, y)$ and hence $\frac{\mathrm{d}}{\mathrm{d}t}(\epsilon\xi) = \left.\frac{u}{e^u-1}\right|_{u\,=\,\mathrm{ad}_{\epsilon\xi}}(\zeta\Theta)$ or, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\epsilon\xi) = \zeta\Theta - \frac{1}{2}\mathrm{ad}_{\epsilon\xi}\zeta\Theta - \frac{1}{2}\sum_{k=2}^{\infty}\frac{1}{k!}\mathrm{ad}_{\epsilon\xi}^k\zeta\Theta + \left.\sum_{m=2}^{+\infty}\frac{(-1)^m}{m+1}(e^u - 1)^m\right|_{u\,=\,\mathrm{ad}_{\epsilon\xi}}(\zeta\Theta). \quad (18)$$

On the other hand $\exp(\epsilon\xi) = I_4+\epsilon\xi+O(\epsilon^2)$, $\exp(-\epsilon\xi) = I_4-\epsilon\xi+O(\epsilon^2)$ and, using the fact that $\Theta(g, y) = \Theta(I_4) = 0$ and noticing that Θ is defined in the linear space of 4×4 real matrices, containing both SE(3) and se(3), we have

$$\Theta_R(\eta) = \sum_{j\in\mathcal{J}} \frac{-\|\Pi_0\hat{g}\bar{p}_j\|^2}{D(\hat{g}\bar{p}_j)} \frac{((\Pi_0\hat{g}\bar{p}_j \times \Pi_0\epsilon\xi\hat{g}\bar{p}_j)\times)}{\|F\Pi_0 g\bar{p}_j\|} + O(\epsilon^2), \qquad (19)$$

$$\Theta_T(\eta) = \sum_{j\in\mathcal{J}} \frac{2}{D(\hat{g}\bar{p}_j)} \frac{((\Pi_0\epsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j) \times \Pi_0\hat{g}\bar{p}_j)}{\|F\Pi_0 g\bar{p}_j\|} + O(\epsilon^2). \quad (20)$$

From (16)-(18) with $\Theta(I_4) = 0$, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\epsilon\xi) = \zeta\bar{\Theta}(\epsilon\xi) = \zeta\begin{bmatrix}\bar{\Theta}_R(\epsilon\xi) & \bar{\Theta}_T(\epsilon\xi)\\ 0 & 0\end{bmatrix} + O(\epsilon^2),$$

where

$$\bar{\Theta}_R(\epsilon\xi) = \sum_{j\in\mathcal{J}} \frac{\|\Pi_0\hat{g}\bar{p}_j\|^2}{D(\hat{g}\bar{p}_j)} \frac{((\Pi_0\epsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j)\times)}{\|F\Pi_0 g\bar{p}_j\|}, \qquad (21)$$

$$\bar{\Theta}_T(\epsilon\xi) = \sum_{j\in\mathcal{J}} \frac{-2}{D(\hat{g}\bar{p}_j)} \frac{((\Pi_0\hat{g}\bar{p}_j \times \Pi_0\epsilon\xi\hat{g}\bar{p}_j) \times \Pi_0\hat{g}\bar{p}_j)}{\|F\Pi_0 g\bar{p}_j\|}. \qquad (22)$$

Up to an approximation of the order ϵ^2, we obtain that $\epsilon\xi$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t}(\epsilon\xi) = \zeta\begin{bmatrix}\bar{\Theta}_R(\epsilon\xi) & \bar{\Theta}_T(\epsilon\xi)\\ 0 & 0\end{bmatrix}$. We have the following result.

Proposition 2. Up to an approximation of the order ϵ^3, the following result holds.

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\epsilon\xi\|^2 = -4\zeta\sum_{j\in\mathcal{J}} \frac{1}{D(\hat{g}\bar{p}_j)} \frac{\|\Pi_0\epsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2}{\|F\Pi_0 g\bar{p}_j\|}. \qquad (23)$$

4.2 Exponential convergence

In this section we show that under suitable persistency of excitation (PE) like condition, the estimation error converges exponentially to zero as $t \to \infty$. Let M be an upper bound for $\|F\Pi_0 g\bar{p}_j\|$ for all j.

Our next result is as follows.

Theorem 3. Let $\bar{T} \in [0, +\infty]$ and $\lambda > 0$ be such that

$$\sum_{j\in\mathcal{J}} \frac{\|\Pi_0\epsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2}{(\#\mathcal{J})\|\Pi_0\hat{g}\bar{p}_j\|^2(1+\|\Pi_0\hat{g}\bar{p}_j\|)} \geq \lambda\|\epsilon\xi\|^2 \qquad (24)$$

on the time interval $[0, \bar{T})$. Then, for every $t \in [0, \bar{T})$,

$$\|\epsilon\xi(t)\|^2 \leq \|\epsilon\xi(0)\|^2 e^{-4\zeta\lambda M^{-1}t}.$$

In particular, if $\bar{T} = +\infty$, then $\|\epsilon\xi(t)\|^2$ converges exponentially fast to zero as $t \to \infty$.

From (23), one can conclude that to guarantee exponential convergence of the estimation error with a strictly positive rate we will need some kind of PE condition, like for example (24). Note that the rate of convergence can be improved by tuning $\zeta > 0$, that is, the rate of convergence increases with ζ. Next we state the following result for a less restrictive PE like condition.

Theorem 4. Let $\bar{T} \in [0, +\infty]$. Suppose there exists $T > 0$ such that, for every $t \geq 0$, with $t + T \leq \bar{T}$,

$$\frac{1}{T}\int_t^{t+T} \sum_{j\in\mathcal{J}} \frac{\|\Pi_0\epsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2}{(\#\mathcal{J})\|\Pi_0\hat{g}\bar{p}_j\|^2(1+\|\Pi_0\hat{g}\bar{p}_j\|)} \frac{1}{\|\epsilon\xi\|^2}\mathrm{d}\tau \geq \lambda. \qquad (25)$$

Then, for $n \in \mathbb{N}$ with $t \geq 0$, $t + nT \leq \bar{T}$,

$$\|\epsilon\xi(t+nT)\|^2 \leq \|\epsilon\xi(t)\|^2 e^{-4\zeta\lambda M^{-1}nT}.$$

In particular, if $\bar{T} = +\infty$, then $\|\epsilon\xi(t)\|^2$ converges exponentially fast to zero as $t \to \infty$.

Remark 4.3. The PE condition (25) only requires that (24) holds in an integral sense, not pointwise in time. Theorem 3 may be seen as the limit of Theorem 4 when T goes to 0.

5. ROBUSTNESS ANALYSIS OF THE OBSERVER

In this section, we investigate the effect of disturbance and noise on the estimation error. Consider the process model (3)-(4) subjected to disturbances and noise

$$\dot{g}(t) = g(t)\left(\Omega(t) + w(t)\right), \quad g(0) = g_0, \qquad (26)$$

$$y_j(t) = \tilde{y}_j(t) + v_j(t), \qquad (27)$$

where $w \in \mathrm{se}(3)$ is the disturbance, $\tilde{y}_j = [\tilde{y}_{j1}\ \tilde{y}_{j2}\ \ 1]^{\mathrm{T}} \in \mathbb{R}^3$ is the output defined implicitly by $\alpha_j\tilde{y}_j = FHg\bar{p}_j$ with $0 < \kappa \leq \alpha_j$, $y_j = [y_{j1}\ y_{j2}\ \ 1]^{\mathrm{T}} \in \mathbb{R}^3$ is the measured output with noise $v_j = [v_{j1}\ v_{j2}\ \ 0]^{\mathrm{T}} \in \mathbb{R}^3$. Further, the disturbance and noise signals are assumed to be deterministic but unknown. Note that (27) is equivalent to $y_j = \alpha_j^{-1}(F\Pi_0 g\bar{p}_j + \alpha_j v_j)$. Define $M_p \stackrel{\text{def}}{=} \sup_{t\in[0,\,t_1],\,j\in\mathcal{J}} \|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|$. Let $|F^{-1}|$ denotes a bound for the functional norm of $F^{-1}(t)$ defined by $|F^{-1}(t)| \stackrel{\text{def}}{=} \sup\{F^{-1}(t)u\colon u \in \mathbb{R}^3 \text{and } \|u\| = 1\}$, that is, we assume $F^{-1}(t)$ is bounded in the time interval $[0,\, t_1]$ where the estimator is defined. Define $M_v \stackrel{\text{def}}{=} \sup_{t\in[0,\,t_1],\,j\in\mathcal{J}} \|v_j(t)\|$ and $M_w \stackrel{\text{def}}{=} \sup_{t\in[0,\,t_1]} \|w(t)\|$, that is, M_v and M_w denote the upper bounds for the noise $\|v_j\|$ and disturbance $\|w\|$, respectively.

We consider the same observer described in (6), which can be rewritten as

$$\dot{\hat{g}}(t) = \hat{g}(t)\Omega(t) + \zeta\Theta\left(\hat{g}(t), y(t)\right)\hat{g}(t), \quad \hat{g}(0) = \hat{g}_0. \quad (28)$$

Remark 5.1. Note that $\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|$ is equal to $\alpha_j\|\alpha_j^{-1}F\Pi_0 g\bar{p}_j + v_j\| = \alpha_j\|y_j\| \geq \alpha_j \geq \kappa$ and from (5), it follows that if $\mathcal{J} \neq 0$ then $m^2(m+1)\kappa$ is a lower bound for $D(\hat{g}\bar{p}_j)\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|$. From this we conclude that the observer is well defined.

As in Section 4, in a small neighbourhood of I_4 we have $\eta = \exp(\epsilon\xi) = I_4 + \epsilon\xi + O(\epsilon^2)$. After some algebraic computations, we have that up to an approximation of the order ϵ^3

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\epsilon\xi\|^2 = 2\langle\zeta\bar{\Theta}(\epsilon\xi),\,\epsilon\xi\rangle + 2\langle\zeta\tilde{\Theta},\,\epsilon\xi\rangle - 2\langle\hat{g}w\hat{g}^{-1},\,\epsilon\xi\rangle - \langle[\epsilon\xi,\,\zeta\tilde{\Theta} - \hat{g}w\hat{g}^{-1}],\,\epsilon\xi\rangle, \qquad (29)$$

from which we derive the following result.

Proposition 5. Up to an approximation of the order ϵ^3 , the following result holds.

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\epsilon\xi\|^2 \leq \sum_{j\in\mathcal{J}} \frac{-4\zeta}{D(\hat{g}\bar{p}_j)} \frac{\|\Pi_0\epsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2}{\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|} + 6\zeta|F^{-1}|M_v + 3M_w.$$

By (5), $\|F\Pi_0 g\bar{p}_j\|$ is bounded above by M and, by the definition $\alpha_j \leq \|F\Pi_0 g\bar{p}_j\|$. Then each $\|F\Pi_0 g\bar{p}_j + \alpha_j v_j\|$ is bounded above by $M_p = M(1 + M_v)$. Next, we derive the noisy versions of theorems 3 and 4, respectively.

Theorem 6. Define $\beta \overset{\text{def}}{=} 4\zeta\lambda M_p^{-1}$. Let $\bar{T} \in [0, +\infty]$ and $\lambda > 0$ be such that

i) $\|\epsilon\xi(0)\| < 1$,

ii) $\displaystyle\sum_{j\in\mathcal{J}} \frac{\|\Pi_0\epsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2}{(\#\mathcal{J})\|\Pi_0\hat{g}\bar{p}_j\|^2(1+\|\Pi_0\hat{g}\bar{p}_j\|)} \geq \lambda\|\epsilon\xi\|^2$, and

iii) $\beta^{-1}(6\zeta|F^{-1}|M_v + 3M_w) < 1$,

are satisfied on the time interval $[0, \bar{T})$. Then, for every $t \in [0, \bar{T})$,

$$\|\epsilon\xi(t)\|^2 \leq \|\epsilon\xi(0)\|^2 e^{-\beta t} + \beta^{-1}(6\zeta|F^{-1}|M_v + 3M_w)$$

In particular, if $\bar{T} = +\infty$, then for every constant $\rho > 0$ there exists $t_\rho \geq 0$ such that for all $t \geq t_\rho$

$$\|\epsilon\xi(t)\|^2 < \beta^{-1}(6\zeta|F^{-1}|M_v + 3M_w) + \rho.$$

Theorem 7. Define $\beta \overset{\text{def}}{=} 4\zeta\lambda M_p^{-1}$. Let $\bar{T} \in]0, +\infty]$. Suppose there exist positive constants T, λ such that,

i) $\|\epsilon\xi(0)\|^2 \leq (1 - e^{-\beta T})^{-1}(6\zeta\,|F^{-1}|\,M_v + 3M_w)T$,

ii) $\displaystyle\frac{1}{T}\int_t^T \sum_{j\in\mathcal{J}} \frac{\|\Pi_0\epsilon\xi\hat{g}\bar{p}_j \times \Pi_0\hat{g}\bar{p}_j\|^2}{(\#\mathcal{J})\|\Pi_0\hat{g}\bar{p}_j\|^2(1+\|\Pi_0\hat{g}\bar{p}_j\|)}\,\mathrm{d}\tau \geq \lambda$ for every $0 \leq t,\ t + T < \bar{T}$, and

iii) $(1 - e^{-\beta T})^{-1}(6\zeta\,|F^{-1}|\,M_v + 3M_w)T(2 - e^{-\beta T}) < 1$,

are satisfied. Then for all $s \in [0, \bar{T})$,

$$\|\epsilon\xi(s)\|^2 \leq (1 - e^{-\beta T})^{-1}(6\zeta\,|F^{-1}|\,M_v + 3M_w)T(2 - e^{-\beta T}).$$

6. ILLUSTRATIVE EXAMPLE

In this section, we apply the results developed in the previous sections to the problem of estimating the position and attitude of a rigid body with respect to an inertial coordinate frame defined by visual landmarks. The measurements are provided by a monocular charged-coupled-device (CCD) camera mounted on-board (see Fig. 1). We consider right-handed orthogonal body-fixed frame {b} attached to a rigid body that is rotating relative to a right-handed orthogonal inertial-frame {i}. The rigid body is equipped with a camera, and let {c} denotes the camera-fixed frame where the third vector of its base is the pointing sense of the camera and the observes a set of fixed points in an inertial frame {i}. We assume that the camera cannot do translation motion relative to the body-fixed frame, so that we may consider the origins of the frames {b} and {c} to be coincident.

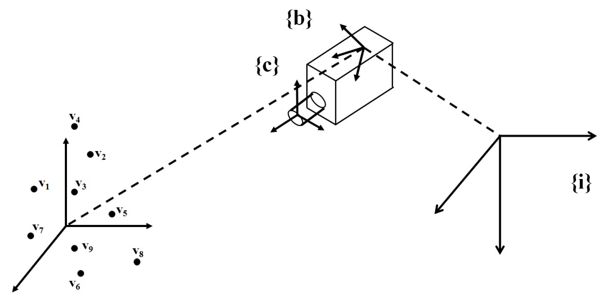

Fig. 1. Visual servoing application

The dynamics and output equation are given by

$$\dot{g}_{\mathrm{bi}} = g_{\mathrm{bi}}\Omega,\ \alpha_j y_j = \tilde{F}\Pi_0 g_{\mathrm{cb}} g_{\mathrm{bi}}\bar{v}_{\mathrm{i}}^j \tag{30}$$

where $g_{\mathrm{bi}} \in \mathrm{SE}(3)$ be the instantaneous state of the frame {b} relative to the frame {i}, $\Omega \in \mathrm{se}(3)$ is known, $v_{\mathrm{i}}^1, \ldots, v_{\mathrm{i}}^N \in \mathbb{R}^3$, $N > 0$, are a set of fixed points whose inertial components are known, $\alpha_j > 0$, $F \in \mathbb{R}^{3\times3}$ is a nonsingular upper-triangular matrix given by $F = \begin{bmatrix} f_{sx} & f_{s\theta} & o_x \\ 0 & f_{sy} & o_y \\ 0 & 0 & 1 \end{bmatrix}$, $\Pi_0 = [I_3 \ \ 0] \in \mathbb{R}^{3\times4}$, and $g_{\mathrm{cb}} = \begin{bmatrix} h_R & 0 \\ 0 & 1 \end{bmatrix}$. It now follows that $\alpha_j y_j = \tilde{F} h_R \Pi_0 g_{\mathrm{bi}} \bar{v}_{\mathrm{i}}^j$. Notice that by taking $F \overset{\text{def}}{=} \tilde{F} h_R$ and $g \overset{\text{def}}{=} g_{\mathrm{bi}}$ we are in the context of Section 5. For the purpose of illustration, we assume that the rigid body starts at the initial position $g_{\mathrm{bi}T}(0) = [0\ 0\ 4]^{\mathrm{T}}$ with initial orientation $g_{\mathrm{bi}R}(0) = I_3$ and follows a circular path with a camera looking up at four non-coplanar points $v_{\mathrm{i}}^1 = [1\ \ 0\ \ -1]^{\mathrm{T}}, v_{\mathrm{i}}^2 = [3\ \ -1\ \ 0]^{\mathrm{T}}, v_{\mathrm{i}}^3 = [4\ \ 0\ \ 0]^{\mathrm{T}}$, and $v_{\mathrm{i}}^4 = [1\ 3\ 2]^{\mathrm{T}}$. The linear velocity is $[0\ 0\ 1]^{\mathrm{T}}$ m/s and the angular velocity is $[0\ 0.2\ 0]^{\mathrm{T}}$ rad/s. The tuning constant $\zeta = 300$. The camera parameters are given by $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.1 \\ 0 & 0 & 1 \end{bmatrix}$ and $h_R = I_3$.

Figure 2 displays the time evolution of the estimation errors without noise and disturbances.

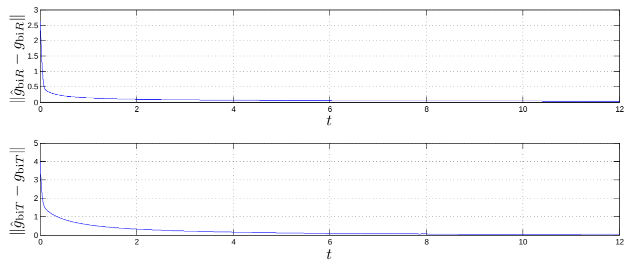

Fig. 2. Time evolution of $\|\hat{g}_{\mathrm{bi}R} - g_{\mathrm{bi}R}\|$ and $\|\hat{g}_{\mathrm{bi}T} - g_{\mathrm{bi}T}\|$

Figure 3 and 4 show the time evolution of the estimation errors with disturbance and the measurements corrupted with additive Gaussian noise. It can be seen that the estimated pose without noise converges to zero and in the presence of noise tends to a small neighborhood of the true value.

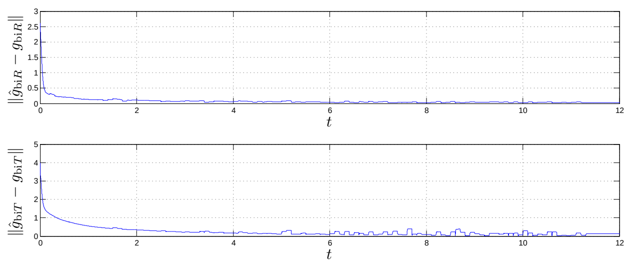

Fig. 3. Time evolution of $\|\hat{g}_{\mathrm{bi}R} - g_{\mathrm{bi}R}\|$ and $\|\hat{g}_{\mathrm{bi}T} - g_{\mathrm{bi}T}\|$

7. CONCLUSION

We proposed a nonlinear observer design for a left-invariant dynamical system evolving on the SE(3) with measurements given by implicit functions. The observer

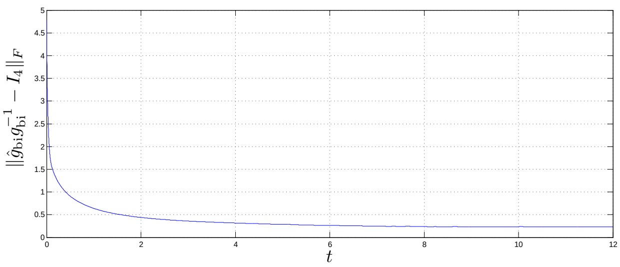

Fig. 4. Time evolution of $\|\hat{g}_{\mathrm{bi}} g_{\mathrm{bi}}^{-1} - I_4\|_F$

dynamics preserves the left-invariance property. Under suitable assumptions, we show that the linearized state estimation error converges exponentially fast to the true state. Furthermore, we show that if the dynamical system is subject to disturbance and noise, the estimated state converges to an open neighborhood of the true value. The size of the neighborhood increases/decreases gracefully with the bound of the disturbance and noise. An visual servoing application illustrate the results.

REFERENCES

Aguiar, A.P. and Hespanha, J.P. (2007). Trajectory-tracking and path-following of underactuated autonomous vehicles with parametric modeling uncertainty. *IEEE Transactions on Automatic Control*, 52(8), 1362–1379.

Aguiar, A.P. and Hespanha, J.P. (2009). Robust filtering for deterministic systems with implicit outputs. *Systems & Control Letters*, 58(4), 263–270.

Aguiar, A.P. and Hespanha, J.P. (2006). Minimum-energy state estimation for systems with perspective outputs. *IEEE Transactions on Automatic Control*, 51(2), 226–241.

Al-Hiddabi, S. and McClamroch, N. (2002). Tracking and maneuver regulation control for nonlinear nonminimum phase systems: Application to flight control. *IEEE Transactions on Control Systems Technology*, 10(6), 780–792.

Anderson, B.D.O. and Moore, J.B. (1979). *Optimal filtering*. Prentice-Hall, New Jersey, USA.

Bonnabel, S. and Rouchon, P. (2005). *Control and observer design for nonlinear finite and infinite dimensional systems*. Springer-Verlag LNCIS 322, Berlin Germany.

Bonnabel, S., Martin, P., and Rouchon, P. (2006). A nonlinear symmetry-preserving observer for velocity-aided inertial navigation. In *Proceedings of the 2006 American Control Conference*, 2910–2914.

Bonnabel, S., Martin, P., and Rouchon, P. (2008a). Nonlinear observer on Lie groups for left-invariant dynamics with right-left equivalent output. In *17th World Congress The International Federation of Automatic Control*, 8594–8598.

Bonnabel, S., Martin, P., and Rouchon, P. (2008b). Symmetry-preserving observers. *IEEE Transactions on Automatic Control*, 53, 2514–2526.

Bonnabel, S., Martin, P., and Rouchon, P. (2009a). Nonlinear symmetry-preserving observers on Lie groups. *IEEE Transactions on Automatic Control*, 5(7), 1709–1713.

Bonnabel, S., Martin, P., and Salaun, E. (2009b). Invariant extended Kalman filter: theory and application to a velocity-aided attitude estimation problem. In *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, 1297–1304.

Bullo, F. (2000). Stabilization of relative equilibria for underactuated systems on Riemannian manifolds. *Automatica*, 36, 1819–1834.

Ghosh, B.K., Jankovic, M., and Wu, Y.T. (1994). Perspective problems in system theory and its application in machine vision. *Journal of Mathematical Systems and Estimation Control*, 4(1), 3–38.

Ghosh, B.K. and Loucks, E.P. (1995). A perspective theory for motion and shape estimation in machine vision. *SIAM Journal of Control and Optimization*, 3(5), 1530–1559.

Kalman, R. (1960). A new approach in linear filtering and prediction problems. *Transactions of the American Society of Mechanical Engineers, Journal of Basic Engineering*, 82D, 35–45.

Lageman, C., Trumpf, J., and Mahony, R. (2008a). Observer design for invariant systems with homogeneous observations. *IEEE Transactions on Automatic Control*.

Lageman, C., Trumpf, J., and Mahony, R. (2008b). State observers for invariant dynamics on a Lie group. In *18th International Symposium on Mathematical Theory of Networks and Systems*.

Lageman, C., Trumpf, J., and Mahony, R. (2009a). Gradient-like observers for invariant dynamics on a Lie group. *IEEE Transactions on Automatic Control*.

Lageman, C., Trumpf, J., and Mahony, R. (2009b). Observers for systems with invariant outputs. In *European Control Conference 2009, Budapest, Hungary*, 4587–4592.

Luenberger, D. (1964). Observing the state of a linear system with observers of low dynamic order. *IEEE Transactions on Military Electronics*, 74–80.

Ma, Y., Soatto, S., Kosecka, J., and Sastry, S.S. (2005). *An Invitation to 3-D Vision*. Springer.

Machado, L. (2006). *Least squares problems on Riemannian manifolds*. Ph.D. thesis, University of Coimbra, Portugal.

Rodrigues, S.S., Crasta, N., Aguiar, A.P., and Leite, F.S. (2009). An exponential observer for systems on SE(3) with implicit outputs. Technical report, Institute for Systems and Robotics, IST, Portugal.

Sattinger, D.H. and Weaver, O.L. (1980). *Lie groups and algebras with applications to physics, geometry, and mechanics*. Springer-Verlag.

Skjetne, R., Fossen, T.I., and Kokotovic, P. (2004). Robust output maneuvering for a class of nonlinear systems. *Automatica*, 40(3), 373–383.

Takahashi, S. and Ghosh, B.K. (2001). Motion and shape parameters identification with vision and range. In *2001 American Control Conference*, volume 6, 4626–4631.