Moving Horizon Estimation with Decimated Observations ⋆

Rui F. Barreiro* A. Pedro Aguiar* João M. Lemos**

* *Institute for Systems and Robotics (ISR) Instituto Superior Tecnico (IST), Lisbon, Portugal*
** *INESC-ID/IST, Portugal (e-mail: jlml@inesc-id.pt).*

Abstract: This paper addresses the problem of moving horizon (MH) state estimation of discrete lumped nonlinear systems. It is assumed that the measurements of the observed variables are not available at every sampling instant (decimated observations). An estimation algorithm is provided for that purpose, together with results on its convergence. It is shown that, under convenient assumptions, the estimation error is bounded, with a bound that grows with the number of samples between consecutive observations. The algorithm features are illustrated by simulations concerning the application to state estimation in a model of the HIV-1 infection. The simulations show that the MH estimator exhibits superior performance over the extended Kalman filter. This difference of performance increases with the growth of the time interval between consecutive measurements.

Keywords: Moving Horizon, Estimation, Nonlinear, HIV-1 infection, stability, decimated observations.

1. INTRODUCTION

Although the original idea is old (Y.A. Thomas (1975)), moving horizon (MH) estimation has received an increased attention since the past 15 years (Robertson *et al.* (1996)) up to the present (Haverbeke *et al.* (2009)) (see also the references in Alessandri *et al.* (2008)). Besides its intrinsic robustness properties, that makes MH adequate to solve estimation problems in the presence of un-modeled dynamics, a major advantage is the capacity to incorporate constraints.

In addition to the above features, the solution of estimation problems in bio-medical applications require the ability to tackle estimation problems when the measurements of the observed variables are not available at every sampling instant. This is a situation referred hereafter as "decimated observations" which is not treated in the available literature. An example is provided by the control of HIV-1 infection (Perelson and Nelson (1999)).

The main contribution of the present work consists in an MH estimation algorithm for nonlinear discrete systems that copes with decimated observations. It is shown that, under convenient assumptions, the estimation error is bounded by a bound that grows with the number of samples between consecutive observations. The algorithm features are illustrated by simulations concerning the application to state estimation in a nonlinear model of the HIV-1 infection. The simulations show that the MH estimator exhibits superior performance over the extended Kalman filter. This difference of performance increases with the grow of the time interval between consecutive measurements.

This paper is organized as follows: Section 2 formulates the state estimation problem and proposes a MH estimator to deal with decimated observations. Section 3 presents stability properties. Section 4 provides an interpretation for the selection of the cost function and Section 5 includes simulation results that compare the performance with the extended Kalman filter (EKF) estimator in relation to a model of the HIV-1 infection. Finally, section 6 draws conclusions.

Due to space limitations some of the proofs are omitted. These can be found in Barreiro *et al.* (2010).

2. MH ESTIMATION WITH DECIMATED OBSERVATIONS

This section formulates the state estimation problem and introduces a MH estimator for a discrete (or sampled data representation) of a nonlinear system whose measurements are not available at every sampling instant.

2.1 Process Model

Let M be the set of time instants (indexes) where measurements are available, and $\sigma_k(i) : \mathbb{N} \to M$ the index time i of the kth measurement. We consider a dynamic system described by the discrete time equations

$$\begin{aligned} x_{i+1} &= \phi_i(x_i, u_i, \omega_i) \\ y_{\sigma_k} &= h_k(x_{\sigma_k}) + v_k \end{aligned} \tag{1}$$

where $x_i \in \mathbb{X}_i$ is the state vector of the system, u_i its control, y_{σ_k} denote the measurements, and $\omega_i \in \Theta_i$ and $v_k \in \mathbb{V}_k$ represent input disturbances and measurement

⋆ Research supported in part by projects DENO/FCT-PT (PTDC/EEA-ACR/67020/2006), HIVCONTROL/FCT-PT (PTDC/EEA-CRO/100128/2008), Co3-AUVs (EU FP7 no. 231378), and FCT-ISR/IST plurianual funding program through the PIDDAC program funds.

noise, respectively. The sets $\mathbb{X}_i$, Θ_i and $\mathbb{V}_k$ are subsets (with appropriate dimension) of the Euclidean space that incorporate the constraints associated to (1). The initial condition x_0 and the signals ω_i, v_k are assumed to be unknown. The function σ_k represents a renumbering of the time index. It stands for the fact that observations are not available at every time instant, but only in a subset of them. A frequent case is when an observation is only available every n_s samples. Then, $\sigma_k := n_s\, k$.

2.2 Decimated Observations

Before we describe the MH estimator algorithm, we first introduce two operators that allow us to work with a more convenient representation of (1).

The first operator, denoted by Σ, will permit to write in an appropriate way the recursive composition of a function and is defined as

$$\Sigma\left[\{\phi\}_a^{b-1}, z, \{\omega\}_a^{b-1}, a, b\right] := \begin{cases} z, & \text{if } a = b \\ \phi_{b-1}\Big(\Sigma\left[\{\phi\}_a^{b-2}, z, \{\omega\}_a^{b-2}, a, b-1\right], u_{b-1}, \omega_{b-1}\Big), & \text{otherwise} \end{cases}$$

where $\{\phi\}_a^b$ denotes a sequence of functions $\{\phi_a(\cdot), \phi_{a+1}(\cdot), ..., \phi_b(\cdot)\}$, z is the input and $\{\omega\}_a^b$ a sequence of input disturbances. Note that the state solution of system (1) at time $i+1$ with initial condition x_0 can be written as $x_{i+1} = \Sigma[\{\phi\}_0^i, x_0, \{\omega\}_0^i, 0, i+1]$.

The second operator, the accumulated noise χ, is defined as the difference between the evolution of state x with input disturbance and the state x with zero input disturbance, that is,

$$\chi\left[\{\phi\}_a^{b-1}, z, \{w\}_a^{b-1}, a, b\right] := \Sigma\left[\{\phi\}_a^{b-1}, z, \{w\}_a^{b-1}, a, b\right] - \Sigma\left[\{\phi\}_a^{b-1}, z, \{0\}_a^{b-1}, a, b\right]$$

For simplicity of notation the following abbreviations are used

$$\Sigma\left[\phi, z, \omega, a, b\right] := \Sigma\left[\{\phi\}_a^{b-1}, z, \{\omega\}_a^{b-1}, a, b\right]$$
$$\chi\left[\phi, z, \omega, a, b\right] := \chi\left[\{\phi\}_a^{b-1}, z, \{\omega\}_a^{b-1}, a, b\right].$$

We are now ready to introduce another representation of system (1) that will play an important role on the developments that follow.
Consider the system

$$x_{k+1} = f_k(x_k) + w_k \tag{2a}$$
$$y_k = h_k(x_k) + v_k \tag{2b}$$

where $f_k(x) = \Sigma\left[\phi, x, 0, \sigma_k, \sigma_{k+1}\right]$ and $w_k = \chi\left[\phi, x_{\sigma_k}, \omega, \sigma_k, \sigma_{k+1}\right]$. Since $f_k(x_k) + w_k$ is equal to $\Sigma\left[\phi, x, \omega, \sigma_k, \sigma_{k+1}\right]$, it is straightforward to conclude that x_k in (2) is equal to x_{σ_k} in (1). Thus, system (2) describes how the state in (1) is transfered from a point where a measurement is available to the next one (where a measurement occurs again). Note however that in (2) w_k might depend on x_k.

Hereafter we use the index k for solutions of system (2) and i for solutions of system (1).

To obtain x_i from (2), we have to perform the following computation

$$\begin{aligned} k &= \max\{k : \sigma_k \le i\} \\ x_i &= \Sigma\left[\phi, x_k, \omega, \sigma_k, i\right]. \end{aligned} \tag{3}$$

2.3 Moving Horizon Estimator

Using the notation in Rao *et al.* (2003), we denote by $x(k; z, l, \{w_j\})$ the solution of system (2) at time k when the initial state is z at time l and the disturbance sequence is $\{w_j\}_{j=l}^k$. When $w_j = 0$ we will write $x(k; z, l)$. Also $y(k; z, l, \{w_j\}) := h_k(y(k; z, l, \{w_j\}))$ and $y(k; z, l) := h_k(x(k; z, l))$.

The objective is to find the state sequence $\{\hat{x}_i\}$ that is most likely to be in some sense close to the real state $\{x_i\}$, given the sequence of observations $\{y_{\sigma_k}\}$, the inputs $\{u_i\}$ and the model with constraints described in (1).

To this effect, we consider the following objective function defined in the equivalent system (2),

$$\Phi_T(x_0, \{w_k\}) := \sum_{k=0}^{T-1} L_k(w_k, v_k) + \Gamma(x_0),$$

where $T > 0$ is the estimation horizon, $v_k = y_k - y(k, x_0, 0, \{w_j\}$, $L_k : \mathbb{W}_k \times \mathbb{V}_k \to \mathbb{R}_{\ge 0}\ \forall_{k \ge 0}$ is the running cost and $\Gamma : \mathbb{X}_0 \to \mathbb{R}_{\ge 0}$ represents a penalty on the initial condition. It is assumed that some prior information of the initial state is known, and this one is captured by $\Gamma(\cdot)$, that satisfies the following property

$$\Gamma(\hat{x}_0) = 0,\ \Gamma(x) > 0 \quad \forall_{x \ne \hat{x}_0}$$

where $\hat{x}_0 \in \mathbb{X}_0$ is the (a priori) must likely value of x_0. In Section 4, we provide a better insight of how to choose $L_k(\cdot)$ and $\Gamma(\cdot)$, but the main idea is that Φ_T will penalize large w_k and v_k, and x_0 far from an initial guess $\hat{x}_0$.

The optimization problem can now be stated as follows: Find the pair $(\hat{x}_0, \{\hat{w}_k\}_{k=0}^{T-1})$ that minimizes $\Phi_T(x_0, \{w_k\})$ subjected to $(x_0, \{w_k\}) \in \Omega_T$. The constraint set Ω_T is given by

$$\Omega_T := \left\{ (x_0, \{w_k\}) : \begin{array}{rl} x(k; x_0, 0, \{w_j\}) \in \mathbb{X}_{\sigma_k}, & k = 0, ..., T \\ w_k \in \mathbb{W}_k, & k = 0, ..., (T-1) \\ v_k = y_k - y(k; x_0, 0, \{w_j\}) \in \mathbb{V}_k, & k = 0, ..., (T-1) \end{array} \right\}$$

and it arises from the restrictions $\mathbb{X}_{\sigma_k}, \mathbb{W}_k$ and $\mathbb{V}_k$, where $\mathbb{W}_k$ inherits restrictions from Θ_i. The computation of the set $\mathbb{W}_k$ can be involved. However, for the particular case that Θ_i is a bounded set, Lemma 3 provides a bound for $\mathbb{W}_k$, although a smaller one might exist.

In general this optimization cannot be applied online because the computational complexity grows unbounded with increasing horizon T. To account for this problem and enforce a fixed dimension optimal control problem, a possible strategy is to explore the ideas of dynamical programing by breaking the summation in Φ_T as follows

$$\Phi_T(x_0, \{w_k\}) := \sum_{k=T-N}^{T-1} L_k(w_k, v_k) + \sum_{k=0}^{T-N-1} L_k(w_k, v_k) + \Gamma(x_0)$$

Since the first term of the right hand side depends only on the state x_{T-N} and on the sequences $\{w_k\}_{k=T-N}^{T-1}$ and $\{v_k\}_{k=T-N}^{T-1}$, the optimization problem can be reformulated as

$$\Phi^*_T(x_0,\{w_k\}) = \min_{z,\{w_k\}_{k=T-N}^{T-1}} \{ \sum_{k=T-N}^{T-1} L_k(w_k,v_k) + \mathcal{Z}_{T-N}(z) : (z,\{w_k\}) \in \Omega_T^N \}$$

where

$$\mathcal{Z}_\tau(z) = \min_{x_0,\{w_k\}_{k=0}^{\tau-1}} \{\Phi_T(x_0,\{w_k\}) : (x_0,\{w_k\}) \in \Omega_T, x(\tau,x_0,0,\{w_j\}) = z\},$$

and

$$\Omega_T^N := \left\{ (z,\{w_k\}) : \begin{array}{ll} x(k;z,T-N,\{w_j\}) \in \mathbb{X}_{\sigma_k}, & k=(T-N),...,T \\ w_k \in \mathbb{W}_k, & k=(T-N),...,(T-1) \\ v_k = y_k - y(k;z,T-N,\{w_j\}) \in \mathbb{V}_k, & k=(T-N),...,(T-1) \end{array} \right\}.$$

Here $\mathcal{Z}_\tau(z)$ is usually called the arrival cost, cost to come or cost to arrive.

The idea is now to summarize the past information given by the arrival cost $\mathcal{Z}_\tau(z)$ by an approximation of it $\hat{\mathcal{Z}}_\tau(z)$, and apply a moving horizon strategy by considering rather the following optimization (for $T > N$)

$$\hat{\Phi}_T(z,\{w_k\}) = \min_{z,\{w_k\}_{k=T-N}^{T-1}} \{ \sum_{k=T-N}^{T-1} L_k(w_k,v_k) + \hat{\mathcal{Z}}_{T-N}(z) : (z,\{w_k\}) \in \Omega_T^N \} \quad (4)$$

From this optimization, we obtain the pair

$$(z^*, \{\hat{w}_{k|T-1}\}_{k=T-N}^{T-1})$$

that allows to compute the sequence

$$\{\hat{x}_{T-N|T-1}, \hat{x}_{T-N+1|T-1}, ..., \hat{x}_{T|T-1}\}$$

by using (2a) with initial condition $x_{T-N} = z^*$.

To obtain the estimate of x_i of the original system (1) we use

$$\begin{aligned} k &= \max\{k : \sigma_k \le i\} \\ \hat{x}_i &= \Sigma\left[\phi, \hat{x}_k, 0, \sigma_k, i\right]. \end{aligned} \quad (5)$$

3. STABILITY RESULTS

In this section we provide conditions under which the state estimate computed in (5) converges to zero in absence of disturbances and noise, or to a small neighborhood of the true values in the presence of bounded disturbances and noise. To this effect, we first recall the following definitions that will be used in the sequel.

Definition 1. A function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a $\mathbf{K_\infty}$**-function** if it is continuous, strictly monotone increasing, $\alpha(x) > 0$ for all $x \neq 0$, $\alpha(0) = 0$ and $\lim_{x\to\infty} \alpha(x) = \infty$.

We will freely use basic results on K_∞ functions, such as, the inverse of $\alpha(\cdot) \in K_\infty$ exists and is a K_∞ function, also if $\alpha(\cdot) \in K_\infty$ then $a \le b \implies \alpha(a) \le \alpha(b)$.

Definition 2. System (1) is **uniformly observable** if there exist a positive integer N_o and a K_∞ function $\varphi(\cdot)$ such that for any two states x_1 and x_2,

$$\varphi(\|x_1 - x_2\|) \le \sum_{j=0}^{N_o-1} \|y(\sigma_{k+j}; x_1, \sigma_k) - y(\sigma_{k+j}; x_2, \sigma_k)\|,$$

where $y(\sigma_k; z, \sigma_l) = h_k(x(\sigma_k, z, \sigma_l))$ with $x(\sigma_k, z, \sigma_l)$ being the solution of (1) without disturbances at time σ_k when the state starts at time σ_l with value z .

The above definition takes into consideration the fact that system (1) may not have measurements at each sampling instant of time. From this definition, we can conclude that if system (1) is uniformly observable then system (2) is also uniformly observable, meaning that

$$\varphi(\|x_1 - x_2\|) \le \sum_{j=0}^{N_o-1} \|y(k+j; x_1, k) - y(k+j; x_2, k)\|.$$

The stability results presented in this section make use of the following assumptions:

A0) The vector fields $\phi_i(\cdot)$ and h_k satisfy the following growth conditions:
a)

$$\|\phi_i(z_1,u_k,\omega_1) - \phi_i(z_2,u_k,\omega_2)\| \le c_\phi \|(z_1,\omega_1) - (z_2,\omega_2)\|$$

b)

$$\|h_k(z_1) - h_k(z_2)\| \le c_h \|z_1 - z_2\|$$

for any $z_1, z_2 \in \mathbb{X}_i$, $u_k \in U_k$, $w_1, w_2 \in W_k$ and some positive numbers c_ϕ and c_h.

A1) $L_k(\cdot)$ and $\Gamma(\cdot)$ are left continuous in their arguments for all $k \ge 0$.

A2) There exist K_∞-functions $\eta(\cdot)$ and $\gamma(\cdot)$ such that

$$\eta(\|(w,v)\|) \le L_k(w,v) \le \gamma(\|(w,v)\|)$$
$$\eta(\|x - \hat{x}_0\|) \le \Gamma(x) \le \gamma(\|x - \hat{x}_0\|)$$

for all $(w,v) \in (\mathbb{W}_k \times \mathbb{V}_k)$, $x, x_0 \in \mathbb{X}_0$, and $k \ge 0$.

A3) There exists an initial condition x_0, disturbance sequence $\{w_k\}_{k=0}^\infty$ such that, for all $k \ge 0$, $(x_0, \{w_k\}k = 0^\infty) \in \Omega_k$.

A4) The interval of time between two consecutive measurements is finite, i.e. $\sigma_k - \sigma_{k-1} < n_{max}$ for some n_{max}.

A5) There exist K_∞-function $\bar{\gamma}(\cdot)$ such that

$$0 \le \hat{\mathcal{Z}}_k(z) - \hat{\Phi}_k \le \bar{\gamma}(\|z - \hat{x}_k\|)$$

for all $z \in \mathbb{X}_k$.

A6) Let

$$\mathcal{R}_\tau^N = \{x(\tau; z, \tau - N, \{w_k\}) : (z,\{w_k\}) \in \Omega_\tau^N\}$$

where $\mathcal{R}_\tau^N = \mathcal{R}_\tau$ for $\tau \le N$. For a horizon length N, any time $\tau > N$, and any $p \in \mathcal{R}_\tau^N$, the approximate arrival cost $\hat{\mathcal{Z}}_\tau(\cdot)$ satisfies the inequality

$$\hat{\mathcal{Z}}_\tau(p) \le \min_{z,\{w_k\}_{k=\tau-N}^{\tau-1}} \{ \sum_{k=\tau-N}^{\tau-1} L_k(w_k,v_k) + \hat{\mathcal{Z}}_{\tau-N}(z) : \begin{array}{c} (z,\{w_k\}) \in \Omega_\tau^N, \\ x(\tau; z, \tau-N, \{w_j\} = p \end{array} \}$$

subjected to initial condition $\hat{\mathcal{Z}}_0(\cdot) = \Gamma(\cdot)$. For $\tau \le N$, the approximate arrival cost $\hat{\mathcal{Z}}_\tau(\cdot)$ satisfies instead the inequality $\hat{\mathcal{Z}}_\tau(\cdot) \le \mathcal{Z}_\tau(\cdot)$.

With exception of **A4)** all the assumptions stated above were considered in Rao *et al.* (2003). Assumption **A6)** loosely speaking means that the approximate arrival cost should not add "information" that is not present in the data. See details and strategies to choose $\hat{\mathcal{Z}}$ in Rao *et al.* (2003).

With this framework adopted, the following preliminary technical results can be derived.

Lemma 3. If there exist positive constants δ and $\mathbf{d}$ such that for any sequence $\{\omega_i\}_a^{b-1}$, with $b > a$

$$\|\omega_i\| \leq \delta, \quad \|z_1 - z_2\| \leq \mathbf{d}$$

and Assumption **A0)** a) holds, then

$$\|\Sigma[\phi, z_1, w, a, b] - \Sigma[\phi, z_2, 0, a, b]\| \leq \left(\sum_{i=1}^{b-a} c_\phi^i\right)\delta + c_\phi^{b-a}\mathbf{d}$$

Lemma 3 provides a bound for the difference between two solutions of system (1) that depends on the difference between the initial condition of each solution and on the bounded disturbance at every step. This is an important Lemma that will be used often. Notice that in particular, this Lemma gives a bound for w_k.

Proposition 4. **A0)** and **A4** imply that
A0*)
a)

$$\|f_k(z_1) - f_k(z_2)\| \leq c_f \|z_1 - z_2\|$$

b)

$$\|h_k(z_1) - h_k(z_2\| \leq c_h \|z_1 - z_2\|$$

holds for system (2) for some $c_f, c_h > 0$.

Proof. This is a direct consequence of Lemma 3 and the definition of f_k.

The following Lemma establishes that, under reasonable assumptions, bounded noises and bounded estimates of the noises imply bounded estimation error.

Lemma 5. If system (2) is uniformly observable, $N > N_o$, **A0*)** holds and $\forall_{k:j\leq k\leq j+N-1} \|w_{k+j}\|, \|v_{k+j}\|, \|\hat{w}_{k+j}\|$ and $\|\hat{v}_{k+j}\|$ are all bounded by b; then there exists a K_∞ function $\zeta(\cdot)$ such that $\forall_{k:j\leq k\leq j+N} \|x_k - \hat{x}_k\| \leq \zeta(b)$.

The next two propositions taken from Rao *et al.* (2003) provide conditions for the existence of the solution to the optimization (4), and convergence of the estimation error $\hat{x}_k - x_k$, respectively.

Proposition 6. (Rao *et al.* (2003)). If assumptions **A0)-A3)** and **A5)** hold, system (2) is uniformly observable, and $N \geq N_o$, then a solution exists to optimization (4) for all $\hat{x}_0 \in \mathbb{X}_0$ and $T \geq 0$.

Proof. The proof is given in Rao *et al.* (2003) (Proposition 3.3).

Proposition 7. (Rao *et al.* (2003)). If assumptions **A0***, **A1-A3**, **A5** and **A6** hold, system (2) is uniformly observable, $N \geq N_o$ and $w_k, v_k = 0$, then for all $\hat{x}_0 \in \mathbb{X}_0$, $\|\hat{x}_k - x_k\| \to 0$ as $k \to \infty$.

Proof. The proof is given in Rao *et al.* (2003) (Proposition 3.4).

We are now ready to state the main results of this section. The first one states that if there are no disturbances or noise, then the estimation error converges to zero.

Corollary 8. If assumptions **A1-A6** hold, system (1) is uniformly observable, $N \geq N_o$ and $w_k, v_k = 0$, then for all $\hat{x}_0 \in \mathbb{X}_0$, $\|\hat{x}_i - x_i\| \to 0$ as $i \to \infty$.

Proof. By Proposition 4, **A0*** holds. Then by Proposition 7, $\|\hat{x}_k - x_k\| \to 0$ as $k \to \infty$. Thus, by Lemma 3 we can conclude that $\|x_i - \hat{x}_i\|$ with $\hat{x}_i$ obtained from (5) also converges to 0 as $i \to \infty$.

We now show under the following assumption that the estimate $\hat{x}_i$ converges to a neighborhood of the true value.

A7) There exists positive constants δ_w and δ_v such that $\Theta_k \subseteq \bar{B}_{\delta_\omega}$ and $\mathbb{V}_k \subseteq \bar{B}_{\delta_v}$ for all k, where $\bar{B}_\varepsilon = \{x : \|x\| \leq \varepsilon\}$.

Proposition 9. Suppose that **A0)**, **A4)** and **A7)** hold, a solution exists to (4) for all $\hat{x}_0 \in \mathbb{X}_0$, $N \geq N_o$, and system (1) is uniformly observable. Then the estimation error $\|\hat{x}_i - x_i\|$ for $i \geq \sigma_{N_o}$ are bounded by $\beta(\|\delta_\omega + \delta_v\|)$ where $\beta(\cdot)$ is a K_∞ function.

Proof. By Proposition 4, **A0*)** holds. Then by **A7)** and Lemma 5 it follows that the error $\|x_k - \hat{x}_k\|$ is bounded. Using Lemma 3 we can then conclude that $\|x_i - \hat{x}_i\|$ is also bounded.

4. IMPLEMENTATION

This section provides an interpretation of the minimization described in Section 2.3 and proposes a scheme to select the running cost $L_k(\cdot)$. We consider the class of systems where the process noise is only additive (like system (2)). In this case $w_k = x_{k+1} - f_k(x_k)$. Also, instead of computing $\hat{x}_{T|T-1}$ (as in section 2.3) we would like to compute $\hat{x}_{T|T}$, i.e. we use y_T to estimate x_T. We suppose that the input disturbances w_k and measurement noises v_k are stationary zero mean white Gaussian sequences of random variables, mutually independent with covariances Q_K and R_k, respectively. The initial priori information of the initial condition is also assumed to be a Gaussian random variable with covariance Π_0.

In this setup, as done in Goodwin *et al.* (2005), we would like to find the estimate that maximizes the probability density $p(x_0, x_1, ..., x_T | y_0, y_1, ..., y_T)$ given the observations, that is,

$$\{\hat{x}_k\}_{k=0}^T = \arg \max_{x_0, x_1, ..., x_T} \{p(x_0, x_1, ..., x_T | y_0, y_1, ..., y_T)\}.$$

Performing some straightforward computations and applying the Bayes Theorem we obtain

$$\{\hat{x}_k\}_{k=0}^T = \arg \max_{x_0, x_1, \ldots, x_T} \{\prod_{k=0}^{T} p_{V_k}(y_k - h_k(x_k)) p_{X_0}(x_0) \times \prod_{k=0}^{T-1} p_{W_k}(x_{k+1} - f_k(x_k))\}$$

where p_{V_k}, p_{W_k} and p_{X_0} denote the probability density functions of v_k, w_k and x_0, respectively. Applying logarithm and using the normal probability density function we get

$$\{\hat{x}_k\}_{k=0}^T = \arg \min_{x_0, x_1, \ldots, x_T} \{\sum_{k=0}^{T} \|y_k - h_k(x_k)\|^2_{R_k^{-1}} + \sum_{k=0}^{T-1} \|x_{k+1} - f_k(x_k, u_k)\|^2_{Q_k^{-1}} + \|x_0 - \bar{x}_0\|^2_{\Pi_0^{-1}}\}.$$

where $\|z\|_A^2 = z^{\mathsf{T}}Az$. We can now conclude that this optimization is the same as the one described in Section 2.3 if $L_k(\cdot)$ is chosen to be

$$L_k(w,v) = w'Q_k^{-1}w + v'R_k^{-1}v$$

and

$$\Gamma(x) = (x_0 - \bar{x}_0)'\Pi_0^{-1}(x_0 - \bar{x}_0)$$

plus the term $v_T'R_T^{-1}v_T$ that arises from taking account y_T to estimate x_T.

To compute the arrival cost, one strategy is to approximate it by employing a first order Taylor series approximation of the model around the estimated trajectory $\hat{x}_k$. This strategy yields the Extended Kalman Filter (EKF) covariance update formula. Thus,

$$\begin{aligned}\{\hat{x}_k\}_{k=T-N}^T = \arg \min_{x_{T-N}, x_{T-N+1}, \dots, x_T} \{ &\sum_{k=T-N}^{T} \|y_k - h_k(x_k)\|^2_{R_k^{-1}} \\ &+ \sum_{k=T-N}^{T-1} \|x_{k+1} - f_k(x_k,u_k)\|^2_{Q_k^{-1}} \\ &+ \alpha\, \|x_{T-N} - f_{T-N-1}(\hat{x}_{T-N-1})\|^2_{\Pi^{-1}_{T-N|T-N-1}} \},\end{aligned} \tag{6}$$

where α is a forgetting factor and Π is computed using the EKF formula.

Note however that since the original process model is (1) (with decimated observations) and not (2), an extra step is needed. Assuming that ω_i in system (1) is Gaussian with covariance Σ_ω^i, then we will approximate w_k of system (2) to be Gaussian with covariance Q_k, as it is described in the following pseudo code.

Pseudo Code 1

```
Initialization:
   x̄ = x̂_k;
   Σ = Σ_ω^k;

Use EKF to compute Q_k:
   for i = σ_k + 1, ..., σ_{k+1} − 1
       [x̄, Σ] = ekfUpdateTime(x̄, Σ, Σ_ω^i)
   end for

Return:
   Q_k = Σ;
```

where ekfUpdateTime($\bar{x}$, Σ, Σ_ω^i) returns one step ahead of the predicted mean and covariance using the standard EKF formulas.

The procedure to obtain the estimates is described in the following pseudo code.

Pseudo Code 2

```
Initialization:
   for i = 0, 1, ..., N − 1
       compute x̂_k using formula (6),
       but with T − N replaced by 0 everywhere,
       and the last term replaced by Γ(x_0);
   end for
   Σ = Π_0;
   x̄ = x̂_0;
Update estimates:
   for k = N, N + 1, ... and j = 1, 2, ...
       Σ = ekfUpdateMeasurement(Σ, x̄, y_{j−1}, R_{j−1});
       x̄ = x̂_{j−1};
       for l = σ_{j−1}, σ_{j−1} + 1, ..., σ_j
           [Σ, x̄] = ekfUpdateTime(Σ, x̄, Σ_ω^l);
       end for
       compute x̂_k using formula (6)
               with Π^{−1}_{T−N|T−N−1} = Σ^{−1};
   end for
```

where ekfUpdateMeasurement($\Sigma, \bar{x}, y_{j-1}, R_{j-1}$) implements the measurement update EKF formulas.

5. HIV MODEL

The proposed algorith is illustrated in the estimation of the concentration of HIV-1 virus, infected T-CD4+ cells and healthy cells. Our model is the following (Perelson and Nelson (1999)):

$$\begin{cases} \dfrac{dT}{dt} = s - dT - e^{-u_1}\beta T\nu \\ \dfrac{dT^*}{dt} = e^{-u_1}\beta T\nu - \mu_2 T^* \\ \dfrac{d\nu}{dt} = e^{-u_2}kT^* - \mu_1\nu \end{cases} \tag{7}$$

where T is the concentration of healthy T-CD4+ cells, T^* is the concentration of infected cells and ν is the concentration of free virus particles, all in units per $[mm^3]$. The quantities u_1 and u_2 are the manipulated variables related to the quantities of drugs administered. The other symbols, which are described in Table 1, are assumed to be constant parameters that are related to each individual, see references in Barão and Lemos (2007).

Par.	Description	U. Value
d	Mortality rate for healthy cells	0.02
k	Production rate of virus by infected cells	100
s	Production rate of healthy cells	10
β	Infection rate coefficient	24×10^{-5}
μ_1	Elimination rate for the virus	2.4
μ_2	Elimination rate for infected cells	0.24

Table 1. HIV model parameters description and used values.

To apply the MHE method described before we will use the discretization with the following update formula given by Euler's method:

$$\phi_k(T,T^*,\nu,u_1,u_2) = \begin{bmatrix} T + t_s(s - dT - e^{-u_1}\beta T\nu) \\ T^* + t_s(e^{-u_1}\beta T\nu - \mu_2 T^*) \\ \nu + t_s(e^{-u_2}kT^* - \mu_1\nu) \end{bmatrix} \tag{8}$$

where t_s is the time interval between two consecutive points in the discretization.

We consider that we have discrete measurements given by

$$y_{\sigma_k} = h_k(T,T^*,\nu) = \nu$$

Figure 1 shows the time evolution of the state (T,T^*,ν) and the estimated state using the Extended Kalman Filter (EKF) and the MH estimator. In the simulation, the unit of time is day, but the sampling time is $t_s = 0.1$. The measurement data $y_{\sigma_k} = \nu$ is only provided at times (in days) 0, 1, 3, 5, 7, 10, 15, 30, and 50. The values of the

Parameter	Value
Σ_{ω}^{i}	$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}^2$
R_{σ_k}	200^2
x_0	$\begin{bmatrix} 1000 \\ 0 \\ 5 \end{bmatrix}$
$\bar{x}_0$	$\begin{bmatrix} 500 \\ 0 \\ 0 \end{bmatrix}$
$\Sigma_{\bar{x}_0}$	$\begin{bmatrix} 1000 & 0 & 0 \\ 0 & 1000 & 0 \\ 0 & 0 & 1000 \end{bmatrix}^2$
α	1

Table 2. Filter and system parameters.

parameters described in Section 4 are described in Table 2. From the figure it can be seen that the MH algorithm performs slightly better than the EKF (in particular at the transient phase). This fact is more relevant when the measurements are less frequent. See Fig. 2 that display the curve of the mean estimation error as a function of the size of the interval of time between measurements. The EKF performance degrades faster when the interval between measurements increases.

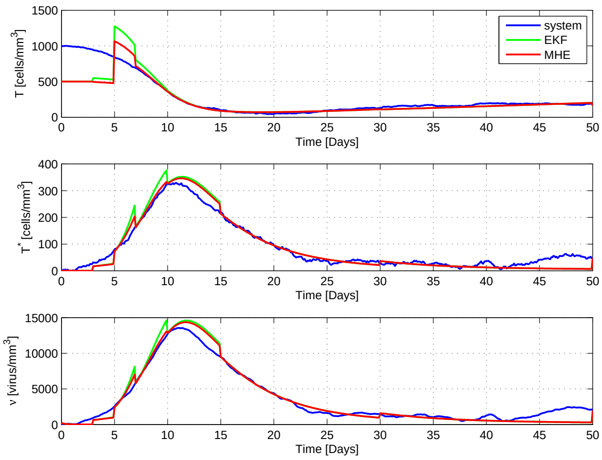

Fig. 1. Evolution of the state of the system and the estimates given by the EKF and the MHE, with $N = 3$ and measurements at times: 0, 1, 3, 5, 7, 10, 15, 30, and 50

6. CONCLUSIONS

We considered the problem of moving horizon state estimation of discrete lumped nonlinear systems subject to constraints and whose measurements of the observed variables are not available at every sampling instant (decimated observations). We proposed an estimation algorithm together with results on its convergence. It was shown that, under suitable assumptions, the estimation error is bounded in the presence of bounded disturbances and noise. In particular, if these ones vanish, the error convergences to zero.

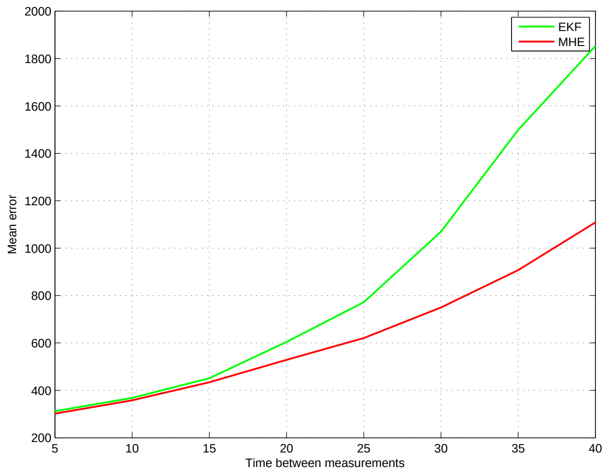

Fig. 2. Performance comparison of the MHE versus EKF for different interval of measurements. The measurements are provided periodically with a period that ranges .5, 1, 1.5, ..., 4. The simulations end at time 15.

The algorithm features were illustrated by simulations concerning the application to state estimation in a model of the HIV-1 infection. From the simulations we could concluded that the MH estimator exhibits superior performance over the extended Kalman filter. This difference of performance increases with the growth of the time interval between consecutive measurements.

REFERENCES

A. Alessandri, M. Baglieto and G. Battistelli (2008). Moving-horizon state estimation for nonlinear discrete-time systems: New stability results and approximation schemes. *Automatica*, 44:1753-1765.

M. Barão and J. M. Lemos (2007). Nonlinear control of HIV-1 infection with a singular perturbation model. *Biomedical Signal Processing and Control*, 2:248-257.

Rui F. Barreiro, A. Pedro Aguiar, and João M. Lemos (2010). Moving Horizon Estimation with Decimated Observations. *Institute for Systems and Robotics, Lisbon, Portugal, Tech. Rep.*, Sept. 2010.

Graham C. Goodwin, José A. De Doná and María M. Seron (2005). Constrained Control and Estimation. *Springer*.

N. Haverbeke, M. Diehl and B. De Moor (2009). A structure exploiting interior-point method for moving-horizon estimation. *Proc. Joint 48th IEEE CDC and 28th CCC*, Shangai, P. R. China, December 16-18, 2009, 1273-1278.

A. Perelson and P. Nelson (1999). Mathematical analysis of HIV-1 dynamics *in vivo*. *SIAM Rev.*, 41(1):3-44.

C. V. Rao, Rawlings, J. B. Rawlings and D. Mayne (2003). Constrained state estimation for nonlinear discrete-time systems: Stability and moving horizon approximations. *IEEE Trans. Autom. Control*, 44(4):714-728.

D. Robertson, J. Lee and J. Rawlings (1996). A moving horizon-based approach for least-squares estimation. *AIChE J.*, 42:2209-2224.

Y.A. Thomas (1975). Linear quadratic optimal estimation and control with receding horizon. *Electron. Lett.*, 11:19-21.