Minimum Robust Sensor Placement for Large Scale Linear Time-Invariant Systems: A Structured Systems Approach ⋆

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Abstract: This paper addresses the problem of robust sensor placement for large scale linear time-invariant systems. Two different concepts of robustness are analyzed: 1) the robustness with respect to one sensor failure, and 2) the robustness with respect to one link failure. We show that both aforementioned problems can be posed as a set cover problem. Due economic constraints we may be interested in considering the minimum robust sensor placement, a much harder problem, which is partially addressed in this paper. Additionally, we provide the relation between robust sensor placement with respect to one link failure and the notion of a spanning tree. Finally, some illustrative examples are presented.

Keywords: Structure systems, Dynamic observability, Structural observability, Sensor failure, Link failure

1. INTRODUCTION

Complex systems infrastructures are commonly required to be reliable, they must be designed to be redundant and/or robust with respect to adverse scenarios. Among the possible failures that affect a dynamic infrastructure we have the 1) malfunction/loss of sensors and 2) link failures. For example, consider the electrical power grid where 1) the phasor measure units (PMUs) or other kind of sensors can become unreliable or stop forwarding the data to a central entity, and 2) transmission lines can suddenly fail. These scenarios have been previously addressed to ensure static or topological observability of the system, see for instance Nuqui and Phadke (2005) and Li et al. (2012). Both, static and topological observability consider steady state system models, which is a reasonable assumption if the system variables present slow dynamics. However, the integration of renewable energy sources in the electrical power grid of the future, such as wind farms, represent a shift in the paradigm where steady state models no longer can be considered (Pulgar-Painemal (2009)). Wind power presents a fast, intermittent and uncertainty variation, which contributes with a relative fast dynamics when compared with the old fashion electrical power grids. Hence, an higher degree of automation is required, where state estimation is only possible by considering *dynamic observability*. Dynamic observability is the standard observability definition of dynamic systems in modern control theory (Kalman (1960)), as well as the necessary requirement to infer the whole state of a dynamical system from the output measurements.

In this paper, we address the problem of ensuring dynamic observability under the adverse scenarios 1)-2). In fact, we take into consideration the fact that for an actual system, it is often very hard to obtain the real-time system parameters precisely, except the zero parameters that denote the absence of connection between components of the system, see Liu et al. (2011). Because most of the parameters of the system are unknown or considered with uncertainty, we consider the notion of *structural observability* [1] (see Dion et al. (2002)), where only the structure of zero/non-zeros of the system plant is considered. This notion has been previously explored in several contexts, for instance, Boukhobza and Hamelin (2011); Boukhobza et al. (2007); Pasqualetti et al. (2011), just to name a few. Boukhobza et al. (2007) extends the work of Lin (1974) to linear systems where unknown input is considered, and provides graphical conditions that ensure structural observability. Some years later, Boukhobza and Hamelin (2011) extended the results to the case of dynamical systems in descriptor form, with unknown input. Later, the relation between structural observability and observability in dynamical systems in descriptor form was address by Pasqualetti et al. (2011), and applied to the analysis of vulnerabilities of the electrical power systems. Pequito et al. (2013a) addressed the problem of obtaining a minimal subset of state variables

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[1] A pair (A, C) is said to be structurally observable if there exists a pair (A', C') with the same structure as (A, C), i.e., same locations of zeroes and non-zeroes, such that (A', C') is observable. By density arguments, it may be shown that if a pair (A, C) is structurally observable, then almost all (with respect to the Lebesgue measure) pairs with the same structure as (A, C) are observable. In essence, structural observability is a property of the structure of the pair (A, C) and not the specific numerical values.

that need to be measured to obtain structural observability, and in Pequito et al. (2013b) a full description of the minimum configurations of state variables that ensure structural observability by assigning to each variable an output is presented. In the present paper we extend the results in Pequito et al. (2013b) to the case where a robust subset of variables has to be measured to obtain structural observability.

The main contributions of this paper are twofold: firstly, we show that the structural observability with respect to one sensor node failure and one link failure can be reformulated as set cover problems, which are efficiently solved. Secondly, we explore the minimum robust sensor placement with respect to one link failure and its connection with the tree structure.

This paper is organized as follows: Section 2 presents preliminary results from structurally observability theory and minimum sensor placement for linear time-invariant systems. Section 3 formulates mathematically the robust sensor placement problem. Section 4, presents the main results, which consist in efficient procedures to identify subset of variables that need to be measure to either ensure structural observability with respect to one sensor failure or one link failure. In section 5, some examples are presented in synthetic model(s) and a small power grid model to illustrate the concept and methodology. Conclusions and further research directions are presented in Section 6.

2. PRELIMINARIES AND TERMINOLOGY

In this section we recall some classical concepts in structural systems, introduced in Lin (1974).

Let

$$\dot{x} = Ax, \qquad y = Cx, \qquad x(0) = x_0 \tag{1}$$

represent the large scale LTI dynamical system of interest, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$ denotes the state variable and measured output, respectively. The structural pattern of the system in (1) is given by the binary matrices $\bar{A}$ and $\bar{C}$, where an entry in $\bar{A}$ (or $\bar{C}$) is one if and only if the corresponding entry in A (or C) is non-zero.

Given a dynamical system (1), an efficient approach to the analysis of its structural properties is to associate it with a digraph (i.e., a directed graph) $\mathcal{D} = (V, E)$, in which V denotes a set of *vertices* and E represents a set of *edges*, such that, an edge (v_j, v_i) is directed from vertex v_j to vertex v_i. Denote by $\mathcal{X} = \{x_1, \cdots, x_n\}$ and $\mathcal{Y} = \{y_1, \cdots, y_p\}$ the set of state vertices and output vertices, respectively. Denote by $\mathcal{E}_{\mathcal{X},\mathcal{X}} = \{(x_i, x_j) : [\bar{A}]_{ji} \neq 0\}$ and $\mathcal{E}_{\mathcal{X},\mathcal{Y}} = \{(x_i, y_j) : [\bar{C}]_{ji} \neq 0\}$, to define $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ and $\mathcal{D}(\bar{A}, \bar{C}) = (\mathcal{X} \cup \mathcal{Y}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{Y},\mathcal{X}})$. In addition, we will require the following graph theoretic notions: Given a graph $\mathcal{D} = (V, E)$, $\mathcal{D}_S = (V_S, E_S)$ is a *subgraph* of $\mathcal{D}$ if $V_S \subset V$ and $E_S \subset E$. A digraph $\mathcal{D}$ is said to be strongly connected if there exists a directed path between any pair of vertices, see Able et al. (2001). A *strongly connected component* (SCC) is a maximal subgraph $\mathcal{D}_S = (\mathcal{V}_S, \mathcal{E}_S)$ of $\mathcal{D}$ such that for every $v, w \in \mathcal{V}_S$ there exists a path from v to w and from w to v. In addition, we denote by $|V|$ the number of elements in the set V.

An undirected graph $\mathcal{U} = (V, E_U)$ can be represented as a digraph $\mathcal{D}_U(\bar{M}) = (V, E_{D_U})$ associated with a symmetric matrix $\bar{M}$. In other words, for some $v, w \in V$ we have that an undirected edge $(v, w) \in E_U$ if and only if the directed edges $(v, w), (w, v) \in E_{D_U}$. Therefore, we can say that an undirected graph U spans a directed graph $\mathcal{D}$, if the digraph associated with the undirected graph $\mathcal{U}$, i.e., $\mathcal{D}_U$ spans the digraph $\mathcal{D}$. A particular instance of an undirected graph is *a tree*, denoted by $\mathcal{T} = (V_T, E_T)$ with $|V_T|$ vertices, $|E_T| = |V_T| - 1$ undirected edges and such that every vertex $v \in V_T$ belongs to an edge in E_T. If there exists a tree $\mathcal{T}$ that spans a digraph $\mathcal{D}$ we say that $\mathcal{T}$ is a spanning tree. In this paper we assume that the digraph have no self-loop, which implies that the undirected graph is loopless. Finally, given an undirected graph $\mathcal{U} = (V, E_U)$ we introduce the notion of a *degree* of a node $v \in V$, denoted by $deg(v) = |\{(v, w) \in E_U : w \in V\}|$. In addition, we say that $v \in V$ is *a leaf* (of an undirected graph) if and only if $deg(v) = 1$.

For any two vertex sets $S_1, S_2 \subset V$, we define the *bipartite graph* $\mathcal{B}(S_1, S_2, E_{S_1,S_2})$ associated with $D = (V, E)$, to be a directed graph (bipartite), whose vertex set is given by $S_1 \cup S_2$ and the edge set E_{S_1,S_2} by $E_{S_1,S_2} = \{(s_1, s_2) \in E \ : \ s_1 \in S_1, s_2 \in S_2 \ \}$.

Given $\mathcal{B}(S_1, S_2, E_{S_1,S_2})$, a matching M corresponds to a subset of edges in E_{S_1,S_2} that do not share vertices, i.e., given edges $e = (s_1, s_2)$ and $e' = (s_1', s_2')$ with $s_1, s_1' \in S_1$ and $s_2, s_2' \in S_2$, $e, e' \in M$ only if $s_1 \neq s_1'$ and $s_2 \neq s_2'$. A maximum matching M^* may then be defined as a matching M that has the largest number of edges among all possible matchings. The maximum matching problem may be solved efficiently in $\mathcal{O}(\sqrt{|S_1 \cup S_2|}|E_{S_1,S_2}|)$. Vertices in S_1 and S_2 are matched vertices if they belong to an edge in the maximum matching M^*, otherwise, we designate the vertices as *unmatched vertices*. If there are no unmatched vertices, we say that we have a *perfect match*. A maximum matching M^* may not be unique.

The term *left-unmatched vertices* (w.r.t. $\mathcal{B}(S_1, S_2, E_{S_1,S_2})$) will refer to only those vertices in S_1 that do not belong to a matched edge in M^*.

Our results in Section 4.1 and Section 4.2 consist in constructing and solving the *set cover problem* (Feige, 1998), that may be described as follows: given a finite collection of k sets $\mathcal{W}_i$ with $i \in I = \{1, \cdots, k\}$, and a set $\mathcal{Z}$ find a minimum number of sub-collections $\{\mathcal{W}_i\}_{i \in \mathcal{J}}$, with $\mathcal{J} \subset \mathcal{I}$, of subsets $\mathcal{W}_i$ that cover $\mathcal{Z}$, i.e.,

$$\mathcal{Z} \subset \bigcup_{j \in \mathcal{J}} \mathcal{W}_j. \tag{2}$$

such that $\mathcal{J} \subset \mathcal{I}$.

2.1 Previous results on structural system design

Recall that a *dedicated output* is an output (sensor) that measures the state of a single state variable. For a system consisting of a single SCC, consider the notions on dedicated output configurations (measurement configurations that consist of collections of dedicated outputs). These can be obtained as follows.

Definition 1. ((Pequito et al., 2013b)). A *feasible dedicated output configuration* $\mathcal{S}_y \subset \mathcal{X}$ is a collection of state variables such that assigning dedicated outputs (sensors) to the variables in $\mathcal{S}_y$ ensures structural observability. In other words, denoting by $\bar{C}$ the structural output matrix corresponding to the assignment of dedicated outputs to the state variables in $\mathcal{S}_y$, the configuration $\mathcal{S}_y$ is said to be a feasible dedicated output configuration if the pair $(\bar{A}, \bar{C})$ is structurally observable. □

A feasible dedicated output configuration with the minimal number of state variables is said to be a *minimal feasible dedicated output configuration.*

Theorem 2. (Pequito et al. (2013b)). (*Minimal Number and Placement of Dedicated Outputs*) Let the system digraph $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ be an SCC and $\mathcal{B} \equiv \mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ its bipartite representation. Let $S_y \subset \mathcal{X}$, then the following statements are equivalent:

(1) The set $\mathcal{S}_y$ is a feasible dedicated output configuration
(2) There exists a subset $\mathcal{U}_L \subset \mathcal{S}_y$ corresponding to the set of left-unmatched vertices of some maximum matching of $\mathcal{B}$, and a subset $\mathcal{A}_y \subset \mathcal{S}_y$ comprising one state variable from $\mathcal{D}(\bar{A})$.

In particular, if $\mathcal{U}_L \neq \emptyset$, then $S_y = \mathcal{U}_L$ corresponds to a minimal feasible dedicated output configuration else $\mathcal{S}_u = \{x\}$ for some $x \in \mathcal{X}$ is a minimal feasible dedicated output configuration. □

Note that minimal feasible dedicated output configurations are not unique due to the non-uniqueness of maximum matchings (and hence left-unmatched vertices).

3. PROBLEM FORMULATION

Let

$$\dot{x} = Ax,$$

be a given system plant, where $x \in \mathbb{R}^n$ denotes the state, and let $\bar{A}$ denote the structural pattern (i.e., location of zero-nonzero entries) of A. In this paper, we are interested in two robust output placement problems, namely P1 and P2, corresponding to preserving structural observability in the face of sensor failures and link failures respectively. Formally, these problems may be described as follows:

P1 Robustness with respect to sensor failure:

Design a (canonical) $p \times n$ output matrix C, with p being some positive integer (preferably $p \ll n$), such that $(\bar{A}, \bar{C})$ is structurally observable and the pair $(\bar{A}, \bar{C}_{-i})$ is structurally observable for each $i \in [1, \cdots, N]$.

In the above, $\bar{C}_{-i} = [\bar{c}_1^T \ \cdots \bar{c}_{i-1}^T \ \bar{c}_{i+1}^T \ \cdots \ \bar{c}_p^T \ ]^T$ corresponds to the structure of the reduced $(p-1) \times n$ output matrix resulting from the removal of the row $\bar{c}_i$, i.e., loss of sensor (output) i.

P2 Robustness with respect to link failure:

Design a (canonical) $p \times n$ output matrix C, with p being some positive integer (preferably $p \ll n$), such that $(\bar{A}, \bar{C})$ is structurally observable and the pair $(\bar{A}_{i' \sim j'}, C)$ is structurally observable for each possible (link) pair (i', j').

In the above,

$$\bar{A}_{i' \sim j'} = \begin{cases} \bar{A}_{ij} \ \text{if} \bar{A}_{ij} \neq 0 \text{ and } (i,j) \neq (i',j'), (j',i') \\ 0 \quad \text{otherwise,} \end{cases}$$

corresponds to the structure resulting from the loss of the phsyical coupling (link) (i', j').

The following mild assumptions on the system structure will be imposed:

A1 The matrix $\bar{A}$ is symmetric, i.e., the system digraph $\mathcal{D}(\bar{A})$ is in fact an undirected graph.
A2 The digraph $\mathcal{D}(\bar{A})$ is strongly connected.

Note that, in most large scale systems of interest the coupling between states (agents) is bi-directional, justifying **A1**. Also, note that under **A1**, $\mathcal{D}(\bar{A})$ is an undirected graph, and hence **A2**, i.e., strong connectedness, reduces to connectivity of $\mathcal{D}(\bar{A})$. Finally, note that as long as **A1** holds, **A2** may be relaxed. Indeed, by relaxing **A2** we may get multiple disconnected components (with no coupling) and our design methodology may be applied independently to each of these components. However, we still enforce **A2** as it simplifies the presentation.

4. MAIN RESULTS

In this section we present the main results of the paper addressing the solutions of **P1** and **P2**. Specifically, in Section 4.1 we elaborate on the solution to **P1**. The solution to **P2** is presented in Section 4.2-4.3, where we present sufficient design criteria for general systems in Section 4.2 and investigate in detail more specific structures in Section 4.3.

4.1 Robustness with respect to sensor failure

We note upfront that under Assumptions **A1**-**A2** the problem **P1** has a trivial (but interesting nonetheless) solution if the bipartite graph $\mathcal{B} \equiv \mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ associated with $\bar{A}$ has a perfect match. In fact, in this case, by Theorem 2 it follows that robust structural observability (in the sense of **P1**) may be achieved by placing dedicated outputs at any two state variables; moreover, such placements of two dedicated outputs are also minimal for robust structural observability with respect to a single sensor (dedicated output) failure.

Hence, in the following, we focus on the (non-trivial) scenario in which the system bipartite graph $\mathcal{B}$ does not have a perfect match.

In this section we show that the solution to **P1** may be cast as a set cover problem.

Now, consider a minimal feasible dedicated output configuration $\mathcal{S} = \{v_1, \cdots, v_p\}$ and note that (since, by assumption, $\mathcal{B}$ has no perfect match), by Theorem 2, $\mathcal{S}$ corresponds to the set of left-unmatched vertices of a maximum matching of $\mathcal{B}$. In addition, consider the following sets:

$$\Omega_{\mathcal{S}}^i = \{x \in \mathcal{X} \backslash \{v_i\} : \mathcal{S}_{v_i}(x) \equiv \mathcal{S} \backslash \{v_i\} \cup \{x\} \text{ is a set of left-unmatched vertices w.r.t. max. matching of } \mathcal{B}\} \tag{3}$$

where $i \in \mathcal{I} = \{1, \cdots, |\mathcal{S}|\}$.

Note that, by Theorem 2, we can construct new minimal feasible dedicated output configurations using (3) - in other words, $\mathcal{S}_{v_i}(x)$ is a minimal feasible dedicated output configuration for any $i \in \mathcal{I}$ and $x \in \Omega_{\mathcal{S}}^i$. Now, consider the following scenario: suppose there exists $x^* \in \mathcal{X}$ such that $x^* \in \Omega_{\mathcal{S}}^i$ for all $i \in \mathcal{I}$, then $\mathcal{S}^* = \mathcal{S} \cup \{x^*\}$ is an *s-robust feasible dedicated output configuration*, i.e., a feasible dedicated output configuration robust w.r.t. sensor failures. In other words, $\mathcal{S}^* \backslash \{x\}$ is a feasible dedicated output configuration for all $x \in \mathcal{S}^*$. Moreover, in such a scenario, $\mathcal{S}^*$ is also a *minimal* s-robust feasible dedicated output configuration since any s-robust feasible dedicated output configuration should have at least $|\mathcal{S}|+1$ dedicated outputs.

In general, let $\mathcal{X} = \{x_1, \cdots, x_n\}$ and denote by

$$\mathcal{V}_j = \{i \in \mathcal{I} : x_j \in \Omega_{\mathcal{S}}^i\}, \quad j = 1, \cdots, n \tag{4}$$

the indices of the sets $\Omega_{\mathcal{S}}^{\cdot}$'s to which the state variable x_j belongs to. We then have the following result.

Theorem 3. Let $\mathcal{S}$ be a minimal feasible dedicated output configuration and consider the sets (3)-(4). If there exists $\mathcal{J} \subset \{1, \cdots, n\}$ such that

$$\mathcal{I} \subset \bigcup_{j\in\mathcal{J}} \mathcal{V}_j,$$

i.e., the family $\{\mathcal{V}_j\}_{j\in\mathcal{J}}$ covers $\mathcal{I}$, then

$$\mathcal{S}^* = \bigcup_{j\in\mathcal{J}} \{x_j\} \ \cup \ \mathcal{S}$$

is an s-robust feasible dedicated output configuration. □

Proof. It suffices to verify that $\mathcal{S}^* - \{x\}$ is a feasible dedicated output configuration for all $x \in \mathcal{S}^*$. To this end, we consider two cases, as to whether $x \in \mathcal{S}^*\backslash\mathcal{S}$ or $x \in \mathcal{S}$:

- Let $x \in \mathcal{S}^*\backslash\mathcal{S}$. Then, clearly $\mathcal{S} \subset \mathcal{S}^* - \{x\}$ and $\mathcal{S}$; since, $\mathcal{S}$ is a feasible dedicated output configuration, we conclude that $\mathcal{S}^* - \{x\}$ is also a feasible dedicated output configuration.
- Let $x \in \mathcal{S}$ and let $\mathcal{S}$ be given by $\mathcal{S} = \{v_1, v_2, \cdots, v_{|\mathcal{I}|}\}$. Suppose that $x \equiv v_i$, then, note that, by the hypothesis that $\bigcup_{j\in\mathcal{J}} \mathcal{V}_j$ is a set cover for $\mathcal{I}$, there exists a state variable $y \in \Omega^i_{\mathcal{S}} \cap \bigcup_{j\in\mathcal{J}}\{x_j\}$. Clearly, $\mathcal{S}\backslash\{x\} \cup \{y\} \subset \mathcal{S}^* - \{x\}$ and, hence, $\mathcal{S}^* - \{x\}$ is a feasible dedicated output configuration since $\mathcal{S}\backslash\{x\} \cup \{y\}$ is a feasible dedicated output configuration (by definition of $\Omega^i_{\mathcal{S}}$). ■

We now show that an s-robust feasible dedicated output configuration, i.e., a solution to **P1**, may be determined efficiently.

Theorem 4. (Complexity). An s-robust feasible dedicated output configuration may be computed using a polynomial complexity (in the number of state variables) algorithm.□

Proof. From Pequito et al. (2013b), we have that a minimal feasible dedicated output configuration $\mathcal{S}$ can be efficiently determined using a polynomial complexity algorithm (see Theorem 5 in Pequito et al. (2013b)), as well as $\Omega^j_{\mathcal{S}}$ for $j = 1, \cdots, |\mathcal{S}|$. Remark that $\mathcal{V}_i$ can be efficiently implemented in at most $\mathcal{O}(|\mathcal{X}|^2)$ since it consists in verifying if each of the $|\mathcal{X}|$ state variables belongs to at most $|\mathcal{X}|$ sets $\Omega^j_{\mathcal{S}}$'s. Finally, remark that finding a set cover may easily be implemented by a polynomial complexity algorithm (Able et al., 2001). ■

Remark that in practice we may want to measure the smallest number of state variables, which implies that we want the minimum set cover with the smallest number of state variables. Nevertheless, this problem is known to be NP-hard (Cormen et al., 2001), and even a simpler formulation as the minimum set cover is NP-complete. There exist approximation algorithms that achieve (with theoretical guarantees) bounded errors, see for instance Feige (1998) and Levin (2006). Although our solution does not guarantee a minimal s-robust dedicated output configuration, illustrative examples provided in Section 5 show that our set cover based design approach leads to s-robust dedicated output configurations with number of outputs much smaller than the total number of states variables.

4.2 Robustness w.r.t. link failure: set cover approach

This is the first of two sections where we explore the robustness with respect to link failure. Let a link between x_i and x_j be denoted by $\mathcal{L}_{i,j}$, given by the subset of edges $\mathcal{L}_{i,j} = \{(x_i, x_j), (x_j, x_i)\}$. Provided the original system digraph $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$, once a link fails, a new system system $\bar{A}_{i\sim j}$ is obtained. The digraph representation $\mathcal{D}(\bar{A}_{i\sim j}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}} - \mathcal{L}_{i,j})$ and the associated bipartite graph given by $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}} - \mathcal{L}_{i,j})$. Note that, although $\mathcal{D}(\bar{A})$ is strongly connected by **A1**, the digraph $\mathcal{D}(\bar{A}_{i\sim j})$ may lead to a digraph comprising two SCCs (connected components in this case as all structures are undirected), which results from the fact that $\mathcal{L}_{i,j}$ is the only link connecting the two different SCCs.

Now, starting with a minimal feasible dedicated output configuration $\mathcal{S}$, a link failure may or may not preserve structural observability. Therefore, let us introduce the notion of *sensitive link* w.r.t. $\mathcal{S}$: given a minimal feasible dedicated output configuration $\mathcal{S}$ for $\bar{A}$, $\mathcal{L}_{i,j}$ is a sensitive link w.r.t. $\mathcal{S}$ if and only if $\mathcal{S}$ is not a minimal feasible dedicated output configuration for $\bar{A}_{i\sim j}$. In other words, $\mathcal{L}_{i,j}$ is not a sensitive link if $\mathcal{S}$ continues to be a minimal feasible dedicated output configuration for $\bar{A}_{i\sim j}$. In order to determine if a link $\mathcal{L}_{i,j}$ is sensitive or not w.r.t. $\mathcal{S}$, we consider the bipartite graphs $\mathcal{B}_{\mathcal{S}} \equiv \mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}} - \{(x, .) : x \in \mathcal{S}\})$ and $\mathcal{B}^{ij}_{\mathcal{S}} \equiv \mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}} - \{(x, .) : x \in \mathcal{S}\} - \mathcal{L}_{ij})$, where the removal of all the edges starting in a specific vertex, enforces that same vertex to be a left-unmatched vertex in all possible maximum matchings w.r.t. the corresponding bipartite graph. Hence, we should state conditions about these maximum matchings by, first, recovering Theorem 2, and second, considering the SCCs that compose the system $\bar{A}_{i\sim j}$ as stated in the next result.

Lemma 1. A link $\mathcal{L}_{i,j}$ is not a sensitive link w.r.t. a minimal feasible dedicated output configuration $\mathcal{S}$ if and only if there exists a common maximum matching $M^{*\prime}$ of $\mathcal{B}_{\mathcal{S}}$ and $\mathcal{B}^{ij}_{\mathcal{S}}$ such that each SCC of $\mathcal{D}(\bar{A}_{i\sim j})$ has at least one left-unmatched vertex from $M^{*\prime}$. □

Proof. [⇐] If there exists a common maximum matching $M^{*\prime}$ of $\mathcal{B}_{\mathcal{S}}$ and $\mathcal{B}^{ij}_{\mathcal{S}}$ such that each SCC of $\mathcal{D}(\bar{A}_{i\sim j})$ has at least one left-unmatched vertex from $M^{*\prime}$ then we have two cases: 1) there is only one SCC and the set left-unmatched vertices can be considered as the minimal feasible output configuration $\mathcal{S}$ as consequence of Theorem 2; and 2) there exist two SCCs, then because these are disjoint we can consider $M^{*\prime} = M^{*\prime}_1 \cup M^{*\prime}_2$, where $M^{*\prime}_i$ (for $i = 1, 2$) corresponds to the edges belonging to the ith SCC. Therefore, we can associate to each maximum matching $M^{*\prime}_i$ a set of left-unmatched vertices $\mathcal{U}_i$ (non-empty since there exists at least one left-unmatched vertex in each SCC), which implies that $\mathcal{S}_i = \mathcal{U}_i$ is a possible minimal feasible output configuration (by Theorem 2). Hence, $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ is a minimal feasible output configuration and by definition of sensitive link, follows that $\mathcal{L}_{i,j}$ is not a sensitive link.

[⇒] To verify that this implication holds, we just need to verify that the contra positive holds. Which follows by reasoning as in the reverse implication, and by arguing that Theorem 2 does not hold. ■

The above characterization provides us with an intuitive procedure to obtain an *l-robust feasible dedicated output configuration*, i.e., a feasible dedicated output configuration that is robust w.r.t. the failure of a single (but arbitrary) link, starting with a minimal feasible dedicated output configuration.

Similarly to the procedure in Section 4.1, given a minimal feasible dedicated output configuration $\mathcal{S}$, we will reduce the problem of constructing an l-robust feasible dedicated output configuration (not necessarily minimal) to a set cover problem. For each of the (possibly two) SCCs in

$\mathcal{D}(\bar{A}_{i\sim j})$ we verify if $\mathcal{S}$ is a feasible dedicated output configuration. In fact, due to a sensitive link failure (as consequence of Lemma 1) we have two cases:

(i) A state vertex that is a left-unmatched vertex with respect to all possible maximum matchings (but not belonging to $\mathcal{S}$) in the SCC has to be considered to place a dedicated output to make the system structurally observable, see the example in Fig. 1-b);

(ii) An SCC has no left-unmatched vertex belonging to $\mathcal{S}$ and w.r.t. any possible maximum matching. Hence, any state variables in the SCC may be considered to place a dedicated output to make the system structurally observable (as consequence of Theorem 2), see the example in Fig. 1-c).

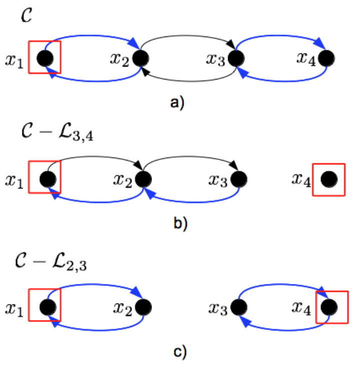

Fig. 1. This figure depicts a chain $\mathcal{C}$ where the edges blue represent the edges belonging to some maximum matching of the digraph represented. The red squares represents the state variables belonging a possible feasible output configuration. In a) we show that $\mathcal{S} = \{x_1\}$ is a minimal feasible output configuration; b) under the failure $\mathcal{L}_{3,4}$, the original chain originates two SCC, where x_4 is a stand alone vertex which is required to belong to the l-robust feasible output configuration; finally, c) depicts the scenario where a failure $\mathcal{L}_{2,3}$ occurs and the SCC comprising $\{x_3, x_4\}$ requires one of its variables to belong to a l-robust feasible output configuration.

Now, consider the collection of all possible SCCs satisfying conditions (i)-(ii), which we refer from now as *non structurally observable SCCs*, that may be originated due to a linked failure. Let γ be the number of such SCCs and denote these SCCs by $\{\Delta_i\}_{i\in\mathcal{I}}$ with $\mathcal{I} = \{1, \cdots, \gamma\}$. Also, recall $\mathcal{X} = \{x_1, \cdots, x_n\}$ and define the subsets

$$\mathcal{V}_j = \{i \in \mathcal{I} : x_j \in \Delta_i\}, \quad j = 1, \cdots, n. \tag{5}$$

We have the following result.

Theorem 5. Let $\mathcal{S}$ be a minimal feasible dedicated output configuration, $\{\Delta_i\}_{i\in\mathcal{I}}$ with $\mathcal{I} = \{1, \cdots, \gamma\}$ be the collection of all possible non-structurally observable SCCs and consider the sets $\{\mathcal{V}_j\}_{j=1}^n$ defined in (5). If there exists $\mathcal{J} \subset \{1, \cdots, n\}$ such that

$$\mathcal{I} \subset \bigcup_{j\in\mathcal{J}} \mathcal{V}_j$$

i.e., the family $\{\mathcal{V}_j\}_{j\in\mathcal{J}}$ covers $\mathcal{I}$, then

$$\mathcal{S}^* = \bigcup_{j\in\mathcal{J}} \{x_j\} \cup \mathcal{S}$$

is an l-robust feasible dedicated output configuration. □

Proof. Since $\mathcal{S}^* \subset \mathcal{S}$, it suffices to show that for each sensitive link $\mathcal{L}_{ij}$ (w.r.t. $\mathcal{S}$) the set $\mathcal{S}^*$ is a feasible dedicated output configuration for $\bar{A}_{i\sim j}$, which follows by construction. To this end consider a sensitive link $\mathcal{L}_{ij}$ and note that, by the classification in (i)-(ii), the failure of $\mathcal{L}_{ij}$ leads to at most two SCCs (say, indexed by γ_1, γ_2) that require additional state variables to be measured, i.e., to which we need to assign additional dedicated outputs to ensure structural observability. From the set cover we know that there exist sets $\mathcal{V}_p$ with $p \in \mathcal{P} \subset \mathcal{J}$ that cover the SCCs γ_1, γ_2; therefore we only need to assign additional dedicated outputs to state variables x_p with $p \in \mathcal{P}$. Hence, $\bigcup_{p\in\mathcal{P}}\{x_p\} \cup \mathcal{S}$ is a feasible dedicated output configuration. The same reasoning applies for any arbitrary link sensitive failure, hence the result follows. ■

The discussion in Section 4.1 about the complexity easily extends to the case of the l-robust feasible output dedicated configuration. Similar to Section 4.1, we still have that a minimal feasible dedicated output configuration $\mathcal{S}$ can be efficiently determined using a polynomial complexity algorithm. Also, determining the set of sensitive links, which is equivalent to determining the structural observability of at most $|\mathcal{E}_{\mathcal{X},\mathcal{X}}|$ systems, can be done in polynomial time, since verifying if a system is structural observable can be done in polynomial time Murota (1987). Remark that $\mathcal{V}_i$ can be efficiently implemented in at most $\mathcal{O}(|\mathcal{X}|^3)$ since it consists in verifying if each of the $|\mathcal{X}|$ state variables belongs to at most $2|\mathcal{E}_{\mathcal{X},\mathcal{X}}|$ sets Δ_i's. Finally, remark that finding a set cover may easily be implemented by a polynomial complexity algorithm Able et al. (2001).

4.3 Robustness w.r.t. link failure: spanning tree approach

In this section we provide upper and lower bounds on the size of l-robust feasible dedicated output configurations. We first provide results for specific system structures (trees) and then extend the bounds to general structures. Specifically, upper and lower bounds on the size of an l-robust feasible dedicated output configuration for tree structured system are presented in Theorem 7, whereas, bounds for general system structures are obtained in Theorem 8 which shows that the number of state variables in an l-robust feasible dedicated output configuration is lower and upper bounded by the number of leaves of the system graph and those of a tree that spans it, respectively. Since a spanning tree may be determined with polynomial implementation complexity, the bounds obtained in Theorem 8 are efficiently computable. Moreover, the proofs are constructive, in that, while providing a spanning tree based upper bound, we explicitly construct an l-robust feasible dedicated output configuration (with polynomial complexity). Some of the results in this section are inspired by techniques developed in Deri et al. (2013), although in a different context.

In a sense, based on the spanning tree arguments described above, the best upper bound may be obtained by considering the spanning tree with the minimum number of leaves; however, we also illustrate that such an upper bound (obtained by minimizing over all possible spanning trees) is not necessarily tight in general. To provide better intuition into our construction, we first consider the simplest tree structured system: $\mathcal{D}(\bar{A})$ corresponds to a *chain*, i.e., an elementary undirected path (without vertex repetitions) that covers all the vertices. We first show that by assigning dedicated outputs to the end vertices of the chain, we obtain an l-robust feasible dedicated output configuration that is also minimal, since no other l-robust feasible dedicated output configuration can have less than two monitored state variables.

In this section we present the following results: first 1) we show that the minimum number of state variables in a l-robust feasible dedicated output configuration must be comprised between the number of leaves of the graph and those of a tree that spans it. Second, although the intuition would say that we need to find the spanning tree with the minimum number of leaves, we show that it only provides an upper-bound.

Lemma 2. Let the system graph $\mathcal{C} \equiv \mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ corresponds to a chain with $|\mathcal{X}| = n$, where $\mathcal{X} = \{x_1, \cdots, x_n\}$ and $E = \{(x_i, x_{i+1}) : i = 1, \cdots, n-1\} \cup \{(x_{i+1}, x_i) : i = 1, \cdots, n-1\}$. Then

(i) $\mathcal{S}_y^1 = \{x_1\}$ and $\mathcal{S}_y^2 = \{x_n\}$ are minimal feasible dedicated output configurations;
(ii) $S_y^r = \{x_1, x_n\}$ is a minimal l-robust feasible dedicated output configuration. □

Proof. Let $\mathcal{B} \equiv \mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ denote the bipartite representation of $\mathcal{C}$. We have two cases:

• If $\mathcal{C}$ has even length, then $n = 2k+1$ for some $k \in \mathbb{N}$ and we can consider as a maximum matching the set $M = \{(x_{2i-1}, x_{2i}) : i = 1, \cdots, \lfloor \frac{n}{2} \rfloor\} \cup \{(x_{2i}, x_{2i-1}) : i = 1, \cdots, \lfloor \frac{n}{2} \rfloor\}$, which has size n. Hence, all vertices in $\mathcal{X}$ belong to a matched edge and the set of left-unmatched vertices is empty. Therefore, from Theorem 2, it follows that $\mathcal{S}_y = \{x\}$ for any $x \in \mathcal{X}$ is a minimal feasible dedicated output configuration. In particular, $\mathcal{S}_y^1 = \{x_1\}$ and $\mathcal{S}_y^2 = \{x_n\}$ are feasible dedicated output configurations.

• If $\mathcal{C}$ has odd length, then $n = 2k$ for some $k \in \mathbb{N}$ and we can consider as a maximum matching the sets: 1) $M^1 = \{(x_{i+1}, x_i) : i = 1, \cdots, n-1\}$, which has size $N-1$ and the set of left-unmatched vertices consists of $=\{x_1\}$. Hence, from Theorem 2 we have that $\mathcal{S}_y^1 = \{x_1\}$ is a feasible dedicated output configuration; and 2) $M^2 = \{(x_i, x_{i+1}) : i = 1, \cdots, n-1\}$, which has size $N-1$,which has size $n-1$ and the set of left-unmatched vertices consists of $\mathcal{V} = \{x_n\}$. Hence, from Theorem 2 we have that $\mathcal{S}_y^2 = \{x_n\}$ is a feasible dedicated output configuration.

Remark that both $\mathcal{S}_y^1$ and $\mathcal{S}_y^2$ are also minimal by Theorem 2. Up to this point, we have shown that in a chain, by considering its ending vertices, we can obtain a feasible dedicated output configuration. Therefore, by considering $S_y^r = \{x_1, x_n\}$ we have an l-robust feasible dedicated output configuration, since the elimination of any link $\mathcal{L}_{i,i+1} = \{(x_i, x_{i+1}), (x_{i+1}, x_i)\}$ originates two chains, where each has one of its ending vertices given by $\mathcal{S}_y^1$ and $\mathcal{S}_y^2$, i.e., feasible dedicated output configuration. Finally, remark that there cannot exist a smaller l-robust feasible dedicated output configuration. Suppose on the contrary it is possible to have a smaller l-robust feasible dedicated output configuration $\tilde{\mathcal{S}}_y^r$, then we have only one state variable in $\tilde{\mathcal{S}}_y^r = \{x\}$ for some $x \in x \in \mathcal{X}$. By considering a link failure $\mathcal{L}_{i,i+1}$ we have two chains: one has x belonging to it and the other does not have any of its state variables belonging to $\mathcal{S}_y^l$, hence leading to a contradiction by Theorem 2. ■

We now extend the previous result to a general tree as follows.

Theorem 6. Let the system graph $\mathcal{T} \equiv \mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ correspond to a tree and let $L_{\mathcal{T}} \subset \mathcal{X}$ be its set of leaves. Then,

(i) For each $x \in L_{\mathcal{T}}$, $\mathcal{S}_y = L_{\mathcal{T}} - \{x\}$ is a feasible dedicated output configuration;
(ii) $\mathcal{S}_y^r = L_{\mathcal{T}}$ is an l-robust feasible dedicated output configuration. □

Proof. Note that in a tree, given any two leaves, there exists a chain with those leaves as the end vertices, and any tree can be decomposed into a union of disjoint chains where each chain has at least one vertex in $L_{\mathcal{T}}$. To prove (i), fix $x \in L_{\mathcal{T}}$, and consider a disjoint spanning decomposition of $\mathcal{T}$ given by $\mathcal{T} = \mathcal{C} \cup \bigcup_{k \in \mathcal{K}} \mathcal{C}_k$ with $\mathcal{K} \subset \mathbb{N}$, such that $\mathcal{C}$ is a chain whose end vertices are in $L_{\mathcal{T}}$ with one of the end vertices being x, and $\{\mathcal{C}_k\}$ is a disjoint family of chains (also disjoint from $\mathcal{C}$) each of which has one end vertex in $L_{\mathcal{T}}$. By Lemma 1 and Theorem 2, we just need to assign one of the end points in each chain with a dedicated output to obtain a feasible dedicated output configuration. Note that, by construction (the decomposition of $\mathcal{T}$), the configuration $\mathcal{S}_y = L_{\mathcal{T}} - \{x\}$ achieves this, and hence $\mathcal{S}_y$ is a feasible dedicated output configuration.

Now, to prove (ii), let $\mathcal{L}_{ij} \subset E$ be an arbitrary link failure. Therefore, consider the following disjoint spanning decomposition of $\mathcal{T}$: $\mathcal{T}_{ij} = \mathcal{C}_{ij} \cup \bigcup_{k \in \mathcal{K}} \mathcal{C}_k$ where $\mathcal{C}_{ij}$ is a chain that contains $\mathcal{L}_{ij}$ with vertices in the leaves of $\mathcal{T}$ (i.e., $L_{\mathcal{T}}$), and $\{\mathcal{C}_k\}$ is a disjoint family of chains (also disjoint from $\mathcal{C}$) each of which has one end vertex in $L_{\mathcal{T}}$. Therefore, if link failure $\mathcal{L}_{ij}$ occurs, we have that $\mathcal{C}_{ij}$ originates two disjoint chains $\mathcal{C}_{ij}^1, \mathcal{C}_{ij}^2$ each of which one of its end vertices in $L_{\mathcal{T}}$. Hence, each of the chains $\mathcal{C}_{ij}^1, \mathcal{C}_{ij}^2, \mathcal{C}_k$ for $k \in \mathcal{K}$ is structurally observable by invoking Lemma 1 (if dedicated outputs are assigned to all the leaves) and hence by Theorem 1 the configuration $\mathcal{S}_u^r = L_{\mathcal{T}}$ preserves structural observability when link $\mathcal{L}_{ij}$ fails. Noting that the same reasoning applies to any link failure, we conclude that $\mathcal{S}_y^l$ is an l-robust feasible dedicated output configuration. ■

Following Theorem 6, we can now characterize minimal l-robust feasible output configurations for general system structures in terms of spanning trees of the associated system graph.

Theorem 7. Let the system digraph $\mathcal{G} \equiv \mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ satisfy assumptions **A1-A2** and let $\mathcal{T}$ be a spanning tree of $\mathcal{D}(\bar{A})$. Then, a minimal l-robust feasible dedicated output configuration $\mathcal{S}_y^*$ satisfies the following inequalities

$$|L_{\mathcal{G}}| \leq |\mathcal{S}_y^*| \leq |L_{\mathcal{T}}|$$

where $L_{\mathcal{G}}, L_{\mathcal{T}} \subset V$ represent the set of leaves of $\mathcal{G}$ and $\mathcal{T}$, respectively. □

Proof. Note that $|\mathcal{S}_y^*| \leq |L_{\mathcal{T}}|$ follows directly from Theorem 6. By definition of leaf we have that $deg(x_i) = 1$ for $x_i \in L_{\mathcal{G}}$, which implies that $\mathcal{L}_{ij} \in \mathcal{E}_{\mathcal{X},\mathcal{X}}$ for some $x_j \in \mathcal{X}$ and hence, x_i becomes isolated if the link $\mathcal{L}_{ij}$ fails. Therefore x_i must belong to $\mathcal{S}_y^*$ for $\mathcal{S}_y^*$ to be l-robust feasible. Hence $|L_{\mathcal{G}}| \leq |\mathcal{S}_y^*|$. ■

In particular, we have the following result

Corollary 8. Let the digraph $\mathcal{G} \equiv \mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ satisfy assumptions **A1-A2**, $\mathcal{T}$ be a spanning tree of $\mathcal{D}(\bar{A})$ and denote by L_G and $L_\mathcal{T}$ the corresponding sets of leaves. Then if $|L_\mathcal{G}| = |L_\mathcal{T}|$ then $\mathcal{S}_y^* = L_\mathcal{G}$ is a minimal l-robust feasible dedicated output configuration. □

Intuitively, to obtain a minimal l-robust feasible dedicated output configuration we need to obtain a spanning tree with the minimum number of leaves. However, this intuition is wrong, since it only provides an upper bound (not necessarily tight) to the minimal l-robust feasible dedicated output configuration. This fact is illustrated by the example depicted in Fig. 2, where any possible spanning tree achieves the minimum number of leaves (in a total of three leaves) is depicted by the blue edges, but it turns out that by assigning dedicated outputs only to the leaves of the original graph (two in number) we obtain a minimal l-robust feasible dedicated output configuration. Finally, remark that we can find spanning trees efficiently (i.e., using algorithm with polynomial complexity in the number of state variables in the original graph), but to find a spanning tree with the minimum number of leaves is an NP-complete problem (Salamon and Wiener, 2008). Nevertheless, as we illustrate in Section 5.3, we can find spanning trees with a number of leaves much smaller than the total number of state variables.

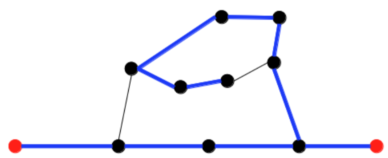

Fig. 2. This example depicts a system with a minimal l-robust feasible dedicated output configuration comprising the state variables in red, whereas the spanning tree (with edges represented in blue) with the minimum number of leaves, has three leaves which implies that a l-robust feasible dedicated output configuration that comprises three state variables (its leaves) is not minimal.

5. ILLUSTRATIVE EXAMPLE

In this section we provide some illustrative examples, where the main results are used. As in Section 4, we first explore the set cover problems to achieve a s-robust and l-robust dedicated output configuration in Section 5.1 and 5.2, respectively. Followed by Section 5.3 where we use the spanning tree construction to obtain a l-robust dedicated output configuration.

5.1 Robustness with respect to sensor failure

We start by providing a s-robust feasible dedicated output configuration to the case of a five-vertex star network, depicted in Fig. 3-a). From Theorem 2, since a star network is an SCC, we first need to compute a maximum matching, for instance $M = \{(x_1, x_3), (x_3, x_2)\}$ (depicted in Fig. 3-b)), with set of left-unmatched vertices given by $\mathcal{U}_L = \{x_2, x_4, x_5\}$. Hence, $\mathcal{S}_y = \{x_2, x_4, x_5\}$ is a minimal feasible dedicated output configuration.

We now construct the sets $\Omega_\mathcal{S} = \{\Omega^1_\mathcal{S}, \Omega^2_\mathcal{S}, \Omega^3_\mathcal{S}\}$ as we defined in (3) and given by: $\Omega^1_\mathcal{S} = \{x_1\}$, $\Omega^2_\mathcal{S} = \{x_1\}$ and $\Omega^3_\mathcal{S} = \{x_1\}$.

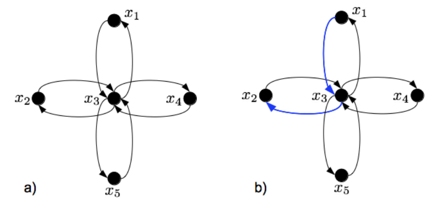

Fig. 3. In a) we depict the five-vertex star network and in b) the blue edges consist of a possible maximum matching associated with the five-vertex star network bipartite graph.

Invoking Theorem 3, observe that exist three left-unmatched vertices, hence $\mathcal{I} = \{1, 2, 3\}$ and the covering sets are given as in (4) by $\mathcal{V}_1 = \{1, 2, 3\}$ and $\mathcal{V}_3 = \{\}$.

As a result, $\mathcal{V}_1$ is a solution to the set cover problem in Theorem 3, which implies that $\mathcal{S}^* = \{x_1, x_2, x_4, x_5\}$ is a s-robust feasible dedicated output configuration.

5.2 Robustness with respect to link failure

Consider a IEEE standard 14-bus power system depicted in Fig. 4-a). From Theorem 2, we first need to compute a maximum matching. One of the possible maximum matching with respect to the associated bipartite graph given by the blue edges is depicted in Fig. 4-b), with set of left-unmatched vertices given by $\mathcal{U}_L = \{x_8\}$. Hence, $\mathcal{S}_y = \{x_8\}$ is a minimal feasible dedicated output configuration.

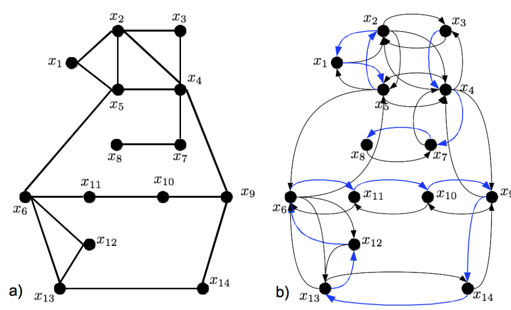

Fig. 4. In a) we depict the 14-bus system and in b) we depict in blue the edges belonging to a possible maximum matching associated with the bipartite graph representation of the 14-bus system.

The set of sensitive links corresponding to $\mathcal{S}_y$, as we defined in Section 4.2, is:

$$L = \{L_{7,8}, L_{4,7}, L_{9,10}, L_{10,11}, L_{13,12}, L_{14,13}, L_{9,14}\}$$

Invoking Theorem 5, we have that the number of SCCs originated due to a linked failure is 9, hence $\mathcal{I} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (only the failure of $L_{7,8}$ and $L_{4,7}$ creates two SCCs, and only one SCC is created resulting from the failure of other sensitive links). Denote the two SCCs created by the failure of $L_{7,8}$ as Δ_1 (the smaller one) and Δ_2. Denote the two SCCs created by the failure of $L_{4,7}$ as Δ_3 (the smaller one) and Δ_4. And denote the SCCs created by the failure of $L_{9,10}, L_{10,11}, L_{13,12}, L_{14,13}, L_{9,14}$ as Δ_5, Δ_6, Δ_7, Δ_8, Δ_9, respectively.

And we can construct the covering sets $\{\mathcal{V}_j\}$, $j \in \{1, 2, ..., 14\}$, as given in 5:

$\mathcal{V}_1 = \{5,9\},\ \mathcal{V}_2 = \{4,5,9\},\ \mathcal{V}_3 = \{2,5,9\},\ \mathcal{V}_4 = \{4\},$
$\mathcal{V}_5 = \{5,9\},\ \mathcal{V}_6 = \emptyset,\ \mathcal{V}_7 = \{2,3,5\},\ \mathcal{V}_8 = \{1,3,5\},$
$\mathcal{V}_9 = \{2,4,5,7,8,9\},\ \mathcal{V}_{10} = \{2,4,5,6,8\},$
$\mathcal{V}_{11} = \{2,4,5,6,7,8,9\},\ \mathcal{V}_{12} = \{2,4,5,6,7,8,9\},$
$\mathcal{V}_{13} = \{7,8\},\ \mathcal{V}_{14} = \{7,8,9\}.$

The solution for this set cover problem as given in Theorem 5 is $\{\mathcal{V}_8,\mathcal{V}_{12}\}$ or $\{\mathcal{V}_8,\mathcal{V}_{11}\}$. Hence we will have $\mathcal{J} = \{8,12\}$ (or $\{8,11\}$) such that the family $\{\mathcal{V}_j\}_{j\in\mathcal{J}}$ covers $\mathcal{I}$.

According to Theorem 5, the l-robust sensor placement configuration is $\bigcup_{j\in\mathcal{J}}\{x_j\} \ \cup\ \mathcal{S}$. Two possible l-robust feasible dedicated output configuration are: $\mathcal{S}_1^r = \{x_8, x_{12}\}$ and $\mathcal{S}_2^r = \{x_8, x_{11}\}$.

5.3 Robustness with respect to link failure: spanning tree structure

Let's try to solve the problem of Section 5.2 in a different way, illuminated by the property of spanning tree introduced in Section 4.3.

According to Theorem 7, any minimal l-robust feasible dedicated output configurations of a system $\mathcal{S}_y^*$ satisfies: $|L_G| \leq |\mathcal{S}_y^*| \leq |L_\mathcal{T}|$. For the IEEE standard 14-Bus power system shown in Fig. 4, $L_G = \{x_8\}$, $|L_G| = 1$, hence $|\mathcal{S}_y^*| \geq 1$. For this system, we can easily find that $|\mathcal{S}_y^*| > 1$, i.e. $|\mathcal{S}_y^*| \geq 2$, by simply checking the situation of the failure of $L_{7,8}$ in Fig. 4-a).

Since $|S_y^*| \leq |L_\mathcal{T}|$, let's consider the lower bound of $|L_\mathcal{T}|$, namely, consider the spanning trees that have minimum number of leaves. One structure of the spanning trees with minimum number of leaves is shown in Fig. 5.

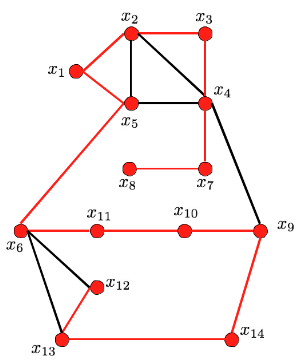

Fig. 5. Depicts the 14-bus system and the spanning tree with the minimum number of leaves composed by the red edges.

In this spanning tree of the original system, $L_\mathcal{T} = \{x_8, x_{12}\}$. Hence $|L_\mathcal{T}| = 2$, which is equal to the lower bound of $|\mathcal{S}_y^*|$, as we found in our previous discussion. According to Theorem 7, $|L_\mathcal{T}|$ is also the upper bound of $\mathcal{S}_y^*$. As a result, we can come to the conclusion that any minimal l-robust feasible dedicated output configurations of the IEEE standard 14-Bus power system $\mathcal{S}_y^*$ satisfies $|\mathcal{S}_y^*| = 2$, and one minimal l-robust feasible dedicated output configurations is $\{x_8, x_{12}\}$.

Although the result is not general, we can tell from this example that the property of spanning tree can help us design the minimal l-robust feasible dedicated output configurations, or at least l-robust feasible dedicated output configurations, in some cases.

6. CONCLUSIONS AND FURTHER RESEARCH

This paper provided systematic methods to reformulate the robust observability problem w.r.t. one sensor failure and one link failure as a set cover problem. Additionally, we showed how to find l-robust feasible dedicated output configurations by determined the spanning tree of the system graph. Both reductions allow us to use efficient methods to find a solution to the proposed problems. In addition, we have shown that such reduction that finding the minimal robust configuration to either sensor/link failure implies to solve NP-hard problems. A natural extension consists in the combination of two kinds of failure (i.e., considering one sensor node failure and one link failure at the same time) and to find the solution to a more general class of graphs, i.e., where assumptions **A1-A2** are waived.

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