Robust Filtering for Deterministic Systems with Implicit Outputs

A. Pedro Aguiar [*,1]

Institute for Systems and Robotics, Instituto Superior Técnico, Lisbon, Portugal

João P. Hespanha

Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106, USA

Abstract

This paper addresses the state estimation of a class of continuous-time affine systems with implicit outputs. We formulate the problem in the deterministic H_∞ filtering setting by computing the value of the state that minimizes the induced $\mathcal{L}_2$-gain from disturbances and noise to estimation error, while remaining compatible with the past observations. To avoid weighting the distant past as much as the present, a forgetting factor is also introduced. We show that, under appropriate observability assumptions, the optimal estimate converges globally asymptotically to the true value of the state in the absence of noise and disturbance. In the presence of noise, the estimate converges to a neighborhood of the true value of the state. We apply these results to the estimation of position and attitude of an autonomous vehicle using measurements from an inertial measurement unit (IMU) and a monocular charged-coupled-device (CCD) camera attached to the vehicle.

Keywords: H_∞ Estimation; Systems with implicit outputs; Robotics

1. Introduction

Consider a continuous-time system described by

$$\dot{x} = A(x,u) + G(u)w, \tag{1}$$

$$E(x,u,v)y = C(x,u) + v, \tag{2}$$

where $A(x,u)$ and $C(x,u)$ are affine functions in x, $x \in \mathbb{R}^n$ denotes the state of the system, $u \in \mathbb{R}^m$ its control input, $y \in \mathbb{R}^q$ its measured output, $w \in \mathbb{R}^{n_w}$ an input disturbance that cannot be measured, and $v \in \mathbb{R}^p$

* Corresponding author. Tel: +351 21 841 8056; Fax: +351 21 841 8291.
Email addresses: `pedro@isr.ist.utl.pt` (A. Pedro Aguiar), `hespanha@ece.ucsb.edu` (João P. Hespanha).
[1] Research supported in part by projects DENO / FCT-PT (PTDC/EEA-ACR/67020/2006), the FREESUBNET RTN of the CEC, the NSF Grant 0720842ARO, and the ARO grant DAAD19-03-D-0004.

measurement noise. The initial condition $x(0)$ of (1) and the signals w and v are assumed deterministic but unknown. The measured output y is only defined implicitly through (2) and the map $E(x,u,v)$ satisfies the property that for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $v \in \mathbb{R}^p$, $i \in \{1,\ldots,n\}$, $j \in \{1,\ldots,p\}$

$$\frac{\partial}{\partial x_i}E(x,u,v) = E_{x_i}(u), \quad \frac{\partial}{\partial v_j}E(x,u,v) = E_{v_j}, \tag{3}$$

where each $E_{x_i}(u)$ is a $p \times q$ matrix-valued function that may depend on u but not on x and v, and each E_{v_j} is a constant $p \times q$ matrix.

We call (1)–(2) a *state-affine system with implicit outputs*, or for short simply a *system with implicit outputs*. These type of systems are motivated by applications in dynamic vision such as the estimation of the motion of the camera from a sequence of images. In particular, we shall see in Section 4 that the system (1)–(2) arises when one needs to estimate the pose (position and attitude) of autonomous vehicles using measurements from an inertial measurement unit (IMU) and a monocular charged-coupled-device (CCD) camera attached to the vehicle. It can also be seen as a generalization of perspective systems introduced by Ghosh et al. [7]. The reader is referred to [8, 28] for several other examples of perspective systems in the context of motion and shape estimation. The system with implicitly defined outputs described in [20] and the state-affine systems with multiple perspective outputs considered in [2] are also special cases of (1)–(2).

In this paper we design a state-estimator for (1)–(2) using a deterministic H_∞ approach. Given an initial estimate and the past controls and observations collected up to time t, the optimal state estimate $\hat{x}$ at time t is defined to be the value that minimizes the induced $\mathcal{L}_2$-gain from disturbances to estimation error. To avoid weighting the distant past as much as the present, a forgetting factor λ is also introduced.

Over the last two decades the H_∞ criterion has been applied to filtering problems, cf., e.g., [3, 4, 16, 23, 25, 29]. Closely related to H_∞ filtering are the minimum-energy estimators, which were first proposed by Mortensen [22] and further refined by Hijab [10]. Game theoretical versions of these estimators were proposed by McEneaney [21] and Fleming [6]. In [2], minimum-energy estimators were derived for systems with perspective outputs and input-to-state stability like properties with respect to disturbances were presented. These results were applied to the estimation of position and orientation of a wheeled mobile robot that only uses a CCD camera mounted on-board to observe the apparent motion of stationary points.

It is worth pointing out that in general either minimum-energy or H_∞ state estimators for nonlinear systems lead to infinite dimensional observers with state evolving according to a first-order nonlinear partial differential equation (PDE) of Hamilton-Jacobi type driven by the observations. The main contribution of this paper is a closed-form solution that is filtering-like and iterative, continuously improving estimates as more measurements become available, and that is robust to noise and disturbances. More precisely, under appropriate observability assumptions, we show that the state-estimator proposed has the desirable property that the state estimate converges asymptotically to the true value of the state in the absence of noise and disturbance. In the presence of bounded noise, the estimate converges to a neighborhood of the true value of the state. We can therefore use this state-estimator to design output-feedback controllers by using the estimated state to drive state-feedback controllers.

Another contribution of the paper is the application of these results to the estimation of position and attitude of an autonomous vehicle using measurements from an IMU and a monocular CCD camera attached to the vehicle. The problem of estimating the position and orientation of a camera mounted on a rigid body from the apparent motion of point features has a long tradition in the computer vision literature (cf., e.g., [5, 14, 15, 19, 24, 27] and references therein). In [24], rigid-body pose estimation using inertial sensors and a monocular camera is considered. A locally convergent observer where the states evolve on $SO(3)$ is proposed (the rotation estimation is decoupled from the position estimation). In the area of wheeled mobile robots, Ma et al. [17] addressed the problem of tracking an arbitrarily shaped continuous ground curve by formulating it as controlling the shape of the curve in the image plane. An application for landing an unmanned air vehicle using vision in the control loop is described in [26]. In [15], the autonomous aircraft landing problem based on measurements provided by airborne vision and inertial sensors is addressed. The authors cast the problem in a linear parametrically varying framework and solve it using tools that borrows from the theory of linear matrix inequalities. These results are extended in [9] to deal with the so-called out-of-frame events.

The organization of the paper is as follows: Section 2 formulates the state estimation problem using a H_∞ deterministic approach. Section 3 presents the main results of the paper. In Section 3.1 we derive, using dynamic programming, the equations for the optimal observer. In Section 3.2 we determine under what conditions the state estimate $\hat{x}$ converges to the true state x. An application to the estimation of the position and attitude of an autonomous vehicle using measurements from an IMU and a monocular CCD camera on-board in Section 4 illustrates the results. Concluding remarks are given in Section 5.

This paper builds upon and extends previous results by the authors that were presented in [1].

2. Problem Formulation

This section formulates the state estimation problem using a H_∞ deterministic approach. Consider the system with implicit outputs (1)–(2). Our goal is *to design and analyze an observer which estimates the state vector $x(t)$ given an initial estimate $\hat{x}_0$ and the past controls and observations $\{(u(\tau), y(\tau)) : 0 \le \tau \le t\}$, and minimize the induced $\mathcal{L}_2$-gain from disturbances and noise to estimation error.* In particular, for a given gain level $\gamma > 0$, the estimate $\hat{x}$ should satisfy the following disturbance attenuation inequality

$$\int_0^t \|x(\tau) - \hat{x}(\tau)\|^2 \, d\tau \le \gamma^2 \Big((x(0) - \hat{x}_0)' P_0^{-1} (x(0) - \hat{x}_0) + \int_0^t \|w(\tau)\|^2 + \|v(\tau)\|^2 \, d\tau \Big), \quad \forall\, t, x(0), w, v$$

where $P_0^{-1} > 0$, $\hat{x}_0$ encode a-priori information about the state. We also consider the possibility of introducing an exponential forgetting factor that decreases the weight of x, w and v from a distant past. More specifically, we address the following deterministic optimization problem:

Problem 1 (H_∞ state estimation) *Given an initial estimate $\hat{x}_0$, a gain level $\gamma > 0$, an input u and a measured output y defined on an interval $[0, t)$, compute the estimate $\hat{x}(t)$ of the state at time t defined by*

$$\hat{x}(t) := \arg\min_{z \in \mathbb{R}^n} J(z, \gamma, t) \tag{4}$$

with $J(z, \gamma, t)$ given by

$$J(z,\gamma,t) := \min_{\substack{w:[0,t]\\ v:[0,t]}} \Big\{ \gamma^2 e^{-2\lambda t}(x(0)-\hat{x}_0)' P_0^{-1}(x(0)-\hat{x}_0) + \gamma^2 \int_0^t e^{-2\lambda(t-\tau)} \Big(\|w(\tau)\|^2 + \|v(\tau)\|^2 \Big) d\tau - \int_0^t e^{-2\lambda(t-\tau)} \|x(\tau)-\hat{x}(\tau)\|^2 \, d\tau : x(t)=z, \dot{x} = A(x,u)+G(u)w, E(x,u,v)y = C(x,u)x+v \Big\}, \tag{5}$$

where the minimization is taken over all signals w and v that are square integrable in $[0,t]$, $t \geq 0$, $P_0^{-1} > 0$, and $\lambda \geq 0$ denotes a forgetting factor. □

The symmetric negative of $J(z,\gamma,t)$ is the *information state* introduced in [12, 13] and can be interpret as a measure of the likelihood of state $x = z$ at time t.

3. Main results

In this section we propose an H_∞ observer that solves Problem 1 and provide conditions under which the state estimate converges to a small neighborhood of the true values.

3.1. *The observer equations*

We propose the following observer for (1)–(2):

$$\dot{P} = (\mathbf{J}_x A(u) + \lambda I)P + P(\mathbf{J}_x A(u) + \lambda I)' - \gamma^2 P\big(\Psi(u,y) - \gamma^{-2} I\big)P + \gamma^{-2} GG', \qquad P(0) = P_0 \tag{6}$$

$$\dot{\hat{x}} = A(\hat{x},u) - \gamma^2 P(\Psi(u,y)\hat{x} + \psi(u,y)), \qquad \hat{x}(0) = \hat{x}_0 \tag{7}$$

with

$$\Psi(u,y) := \big(Y_x(u,y) - \mathbf{J}_x C\big)' \big(I - Y_v(y)Y_v^{\perp}(y)\big)' \big(I - Y_v(y)Y_v^{\perp}(y)\big)\big(Y_x(u,y) - \mathbf{J}_x C\big),$$
$$\psi(u,y) := \big(Y_x(u,y) - \mathbf{J}_x C\big)' \big(I - Y_v(y)Y_v^{\perp}(y)\big)\big(I - Y_v(y)Y_v^{\perp}(y)\big)(E(0,u,0)y - C(0,u)),$$

where $Y_x(u,y) := \big[E_{x_1}(u)y | E_{x_2}(u)y | \cdots | E_{x_n}(u)y\big]$, $Y_v(y) := \big[E_{v_1}y | E_{v_2}y | \cdots | E_{v_p}y\big]$, $(\cdot)^{\perp}$ denotes the pseudo-inverse, and $\mathbf{J}_x A(u)$ the Jacobian of $A(x,u)$ with respect to x. The following result solves Problem 1.

Theorem 1 (H_∞ state estimator) *Assuming that* (6)–(7) *has a solution on* $[0,T)$, $T \in [0,\infty]$, *then the* H_∞ *state estimate defined by* (4)–(5) *can be obtained from* (6)–(7). *Furthermore, the cost function* $J(z;t)$ *defined in* (5) *is quadratic and can be written as*

$$J(z,\gamma,t) = \big(z - \hat{x}(t)\big)' Q(t) \big(z - \hat{x}(t)\big) + c(t), \tag{8}$$

where $Q(t) = P^{-1}(t)$ *and* $c(t)$ *satisfies an appropriate ODE (cf.* (13) *below).*

Proof. The function $J(z,\gamma,t)$, $z \in \mathbb{R}^n$, $t \geq 0$ defined in (5) can be viewed as a cost-to-go and computed using dynamic programming. To derive the dynamic programming operator we can consider an elementary time interval dt and write

$$
\begin{aligned}
J(z,\gamma,t) = \min_{\substack{w:[0,t]\\ v:[0,t]}} \Big\{ &\gamma^2 e^{-2\lambda dt}\big(\|w\|^2 + \|v\|^2 - \frac{1}{\gamma^2}\|x-\hat{x}\|^2\big)dt \\
&+ \gamma^2 e^{-2\lambda dt}\Big[e^{-2\lambda(t-dt)}(x(0)-\hat{x}_0)' P_0^{-1}(x(0)-\hat{x}_0) \\
&+ \int_0^{t-dt} e^{-2\lambda(t-dt-\tau)}\Big(\|w\|^2 + \|v\|^2\Big)d\tau \\
&- \frac{1}{\gamma^2}\int_0^{t-dt} e^{-2\lambda(t-dt-\tau)}\|x-\hat{x}\|^2 d\tau\Big] : \\
&x(t) = z,\; x(t-dt) = z - (A(z,u) + G(u)w)dt, \\
&\dot{x} = A(x,u) + G(u)w,\; E(x,u,v)y = C(x,u) + v\Big\}, \\
= \min_{\substack{w:[t-dt,t]\\ v:[t-dt,t]}} \Big\{ &\gamma^2 e^{-2\lambda dt}\big(\|w\|^2 + \|E_0(z,u,y) + Y_v(y)v - C(z,u)\|^2 - \frac{1}{\gamma^2}\|x-\hat{x}\|^2\big)dt \\
&+ \gamma^2 e^{-2\lambda dt}\min_{\substack{w:[0,t-dt]\\ v:[0,t-dt]}}\Big[e^{-2\lambda(t-dt)}(x(0)-\hat{x}_0)' P_0^{-1}(x(0)-\hat{x}_0) \\
&+ \int_0^{t-dt} e^{-2\lambda(t-dt-\tau)}\Big(\|w\|^2 + \|v\|^2\Big)d\tau \\
&- \frac{1}{\gamma^2}\int_0^{t-dt} e^{-2\lambda(t-dt-\tau)}\|x-\hat{x}\|^2 d\tau\Big] : \\
&x(t) = z,\; x(t-dt) = z - (A(z,u) + G(u)w)dt, \\
&\dot{x} = A(x,u) + G(u)w,\; E(x,u,v)y = C(x,u) + v\Big\},
\end{aligned}
$$

where we have used the fact that from (3), the term $E(x,u,v)y$ can be written as $E(x,u,v)y = E_0(x,u,y) + Y_v(y)v$, with $E_0(x,u,y) := Y_x(u,y)x + E(0,u,0)y$. For simplicity we also have dropped the argument t of the signals outside the integral and the argument τ inside the integral.

We can now recognize the inner minimization to be precisely $J(z-(A(z,u)+G(u)w)dt,\gamma,t-dt)$, which leads to the following equation:

$$
\begin{aligned}
J(z,\gamma,t) = \min_{\substack{w:[t-dt,t]\\ v:[t-dt,t]}} \Big\{ &\gamma^2 e^{-2\lambda dt}\Big(\|w\|^2 + \|E_0(z,u,y) + Y_v(y)v - C(z,u)\|^2 - \frac{1}{\gamma^2}\|z-\hat{x}\|^2\Big)dt \\
&+ e^{-2\lambda dt} J(z-(A(z,u)+G(u)w)dt,\gamma,t-dt)\Big\}.
\end{aligned}
$$

Subtracting $J(z,\gamma,t-dt)$ from both sides of the above equation, dividing by dt, and taking the limit as $dt \to 0$ provided that all the derivatives exist, leads to

$$
\begin{aligned}
J_t(z,\gamma,t) &= \min_{w(t),v(t)} \Big\{\gamma^2\Big(\|w\|^2 + \|E_0(z,u,y) + Y_v(y)v - C(z,u)\|^2 - \frac{1}{\gamma^2}\|z-\hat{x}\|^2\Big) \\
&\qquad - J_z(z,\gamma,t)(A(z,u)+G(u)w) - 2\lambda J(z,\gamma,t)\Big\} \\
&= \min_{w(t),v(t)} \Big\{\gamma^2\|w - \frac{1}{2\gamma^2}G'(u)J_z'(z,\gamma,t)\|^2 - \frac{1}{4\gamma^2}\|G'(u)J_z'(z,\gamma,t)\|^2 \\
&\qquad + \gamma^2\|E_0(z,u,y) + Y_v(y)v - C(z,u)\|^2 - \|z-\hat{x}\|^2 - J_z(z,\gamma,t)A(z,u) - 2\lambda J(z,\gamma,t)\Big\} \\
&= -\frac{1}{4\gamma^2}\|G'(u)J_z(z,\gamma,t)\|^2 + \gamma^2\|\big(I - Y_v(y)Y_v^{\perp}(y)\big)\big(E_0(z,u,y) - C(z,u)\big)\|^2 \\
&\quad - \|z-\hat{x}\|^2 - J_z(z,\gamma,t)A(z,u) - 2\lambda J(z,\gamma,t),
\end{aligned} \tag{9}
$$

where J_t and J_z denote the partial derivatives of J with respect to t and z, respectively. The value of $J(z,\gamma,t)$ can be determined from the linear PDE (9) with initial condition

$$J(z,\gamma,0)=(z-\hat{x}_0)'P_0^{-1}(z-\hat{x}_0), \qquad z\in\mathbb{R}^n. \tag{10}$$

It turns out that there exists a solution to (9)–(10) which is differentiable with respect to z and can be written as (8) for appropriately defined signals $\hat{x}(t)$ and $c(t)$. The signal $\hat{x}$ is then precisely the estimate for the state x of (1)–(2). Moreover, matching (10) with (8) we conclude that $P(0)=P_0$, $\hat{x}(0)=\hat{x}_0$, and $c(0)=0$. To verify that the solution to (9)–(10) can indeed be written as (8), we substitute it into (9) and obtain

$$\begin{aligned}-2(z-\hat{x})'Q\dot{\hat{x}}+(z-\hat{x})'\dot{Q}(z-\hat{x})+\dot{c}=&-\frac{1}{\gamma^2}(z-\hat{x})'QGG'Q(z-\hat{x})\\&+\gamma^2\|\big(I-Y_v(y)Y_v^{\perp}(y)\big)\big(E_0(z,u,y)-C(z,u)\big)\|^2\\&-\|z-\hat{x}\|^2-2(z-\hat{x})'QA(z,u)-2\lambda(z-\hat{x})'Q(z-\hat{x})-2\lambda c.\end{aligned}$$

Using the fact that $A(x,u)=\mathbf{J}_xA(u)x+A(0,u)$ and $E_0(x,u,y)=Y_x(u,y)x+E(0,u,0)y$, we get

$$\begin{aligned}&z'\big[\dot{Q}+\frac{1}{\gamma^2}QGG'Q+Q\mathbf{J}_xA+\mathbf{J}_xA'Q+2\lambda Q-\gamma^2\Psi(u,y)+I\big)z\\&\quad+2z'\big[-Q\dot{\hat{x}}-\dot{Q}\hat{x}-\frac{1}{\gamma^2}QGG'Q\hat{x}-\mathbf{J}_xA'Q\hat{x}+QA(0,u)-2\lambda Q\hat{x}-\gamma^2\psi(u,y)-\hat{x}\big]\\&\quad+\dot{c}+2\hat{x}'Q\dot{\hat{x}}+\hat{x}'\dot{Q}\hat{x}+\frac{1}{\gamma^2}\hat{x}'QGG'Q\hat{x}-\hat{x}'QA(0,u)+2\lambda\hat{x}'Q\hat{x}+2\lambda c\\&\quad-\gamma^2\|\big(I-Y_v(y)Y_v^{\perp}(y)\big)\big(E(0,u,0)y-C(0,u)\big)\|^2+\|\hat{x}\|^2=0.\end{aligned}$$

Since this equation must hold for all $z\in\mathbb{R}^n$ we conclude that

$$\dot{Q}+\frac{1}{\gamma^2}QGG'Q+Q\mathbf{J}_xA+\mathbf{J}_xA'Q+2\lambda Q-\gamma^2\Psi(u,y)+I=0 \tag{11}$$

$$-Q\dot{\hat{x}}-\dot{Q}\hat{x}-\frac{1}{\gamma^2}QGG'Q\hat{x}-\mathbf{J}_xA'Q\hat{x}+QA(0,u)-2\lambda Q\hat{x}-\gamma^2\psi(u,y)-\hat{x}=0 \tag{12}$$

$$\begin{aligned}&\dot{c}+2\hat{x}'Q\dot{\hat{x}}+\hat{x}'\dot{Q}\hat{x}+\frac{1}{\gamma^2}\hat{x}'QGG'Q\hat{x}-\hat{x}'QA(0,u)+2\lambda\hat{x}'Q\hat{x}+2\lambda c\\&\quad-\gamma^2\|\big(I-Y_v(y)Y_v^{\perp}(y)\big)\big(E(0,u,0)y-C(0,u)\big)\|^2+\|\hat{x}\|^2=0\end{aligned} \tag{13}$$

Substituting (11) in (12), we obtain

$$-\dot{Q}=Q(\mathbf{J}_xA+\lambda I)+(\mathbf{J}_xA+\lambda I)'Q+\gamma^{-2}QGG'Q-\gamma^2\Psi(u,y)+I \tag{14}$$

$$Q\dot{\hat{x}}=QA(\hat{x},u)-\gamma^2\Psi(u,y)\hat{x}-\gamma^2\psi(u,y) \tag{15}$$

Since the solution P to (6) is positive definite (cf. Lemma 4 in Appendix), then P^{-1} is defined on $[0,T)$. Using the fact that $\dot{P}^{-1}=-P^{-1}\dot{P}P^{-1}$, it is straightforward to show that both P^{-1} and Q satisfy (14). Since $P(0)=P_0$, by unicity of solution, $Q(t)=Q^{-1}(t)$, $\forall t\in[0,T)$. Therefore (14)–(15) and (6)–(7) are equivalent. □

To guarantee that the H_∞ state estimate has a global solution ($T=\infty$), the value of γ should be sufficiently large. In particular, a sufficient condition for this is given by the following observability condition.

Lemma 2 (Observability condition) *The H_∞ estimator (6)–(7) has a global solution and*

$$Q(t) \geq \delta I > 0, \qquad \forall t \geq 0, \tag{16}$$

for some $\delta > 0$, if there exists a sufficiently large $\gamma > 0$ such that the following condition

$$\gamma^2 W_0(t) \geq \int_0^t \Phi(\tau,t)'\Phi(\tau,t)d\tau + \delta I \quad \forall t \geq 0 \tag{17}$$

holds, where

$$W_0(t) := \int_0^t \Phi(\tau,t)'\Psi(u,y)\Phi(\tau,t)d\tau, \tag{18}$$

and $\Phi(t,\tau)$ denotes the state transition matrix of $\dot{z} = (\mathbf{J}_x A + \gamma^{-2}GG'Q + \lambda I)z$.

Proof. To prove this lemma we observe that (14) can also be written as

$$\dot{Q} = -Q(\mathbf{J}_x A + \gamma^{-2}GG'Q + \lambda I) - (\mathbf{J}_x A + \gamma^{-2}GG'Q + \lambda I)'Q + \gamma^{-2}QGG'Q + \gamma^2\Psi(u,y) - I,$$

and, therefore,

$$Q(t) = \Phi(0,t)'P_0^{-1}\Phi(0,t) + \int_0^t \Phi(\tau,t)'\big(\gamma^{-2}QGG'Q + \gamma^2\Psi(u,y) - I\big)\Phi(\tau,t)d\tau. \tag{19}$$

This can be verified by taking derivatives of the candidate expression (19) for Q. Now, since $\Phi(t,0)P_0^{-1}\Phi(t,0)' > 0$ and $\gamma^{-2}QGG'Q \geq 0$, from (19) and (17)–(18) we conclude that $Q(t) \geq \Phi(0,t)'P_0^{-1}\Phi(0,t) + \delta I$ for all $t \geq 0$. Therefore, the smallest singular value of Q remains strictly positive for every finite time t, which implies that $P(t) = Q^{-1}(t)$ remains bounded for every finite t. Global existence of solutions to (6) follows. □

3.2. *Estimator convergence*

We are now interested in determining under what conditions does the state estimate $\hat{x}$ converges to the true state x. The following result provides an input-to-state (ISS) stability condition for the estimation error.

Theorem 3 (Convergence) *Assuming that the solutions to the system with implicit outputs (1)–(2) and to state estimator (6)–(7) exist on $[0,T)$, $T \in [0,\infty]$, $Q(t) \geq \delta I$, and $\lambda > 0$, then there exist positive constants $c, \kappa, \gamma_w, \gamma_v$ such that*

$$\|\tilde{x}(t)\| \leq ce^{-\kappa t}\|\tilde{x}(0)\| + \gamma_w \sup_{\tau\in[0,t)} \|w(\tau)\| + \gamma_v \sup_{\tau\in[0,t)} \|v(\tau)\|, \quad \forall t \in [0,T) \tag{20}$$

where $\tilde{x}(t) := \hat{x}(t) - x(t)$ denotes the state estimation error.

Proof. From (1) and (7) we conclude that

$$\begin{aligned}
\dot{\tilde{x}} &= (\mathbf{J}_x A - \gamma^2 P\Psi(u,y))\tilde{x} - \gamma^2 P(\Psi(u,y)x + \psi(u,y)) - Gw \\
&= (\mathbf{J}_x A - \gamma^2 P\Psi(u,y))\tilde{x} - \gamma^2 P\big[(Y_x(u,y) - \mathbf{J}_x C)'(I - Y_v(y)Y_v^\perp(y))'(I - Y_v(y)Y_v^\perp(y))(v - Y_v(y)v) - Gw \\
&= (\mathbf{J}_x A - \gamma^2 P\Psi(u,y))\tilde{x} - \gamma^2 P\big[(Y_x(u,y) - \mathbf{J}_x C)'(I - Y_v(y)Y_v^\perp(y))'(I - Y_v(y)Y_v^\perp(y))v - Gw
\end{aligned} \tag{21}$$

Defining $V(\tilde{x}) := \tilde{x}'Q\tilde{x}$, $Q := P^{-1}$, computing its time-derivative, and using (14), we obtain

$$
\begin{aligned}
\dot{V} &= \tilde{x}'(\dot{Q} + Q\mathbf{J}_x A + \mathbf{J}_x A' Q - 2\gamma^2 \Psi(u,y))\tilde{x} - 2\tilde{x}' Q G w \\
&\quad - 2\gamma^2 (Y_x(u,y) - \mathbf{J}_x C)'(I - Y_v(y) Y_v^{\perp}(y))'(I - Y_v(y) Y_v^{\perp}(y)) v \\
&= -\tilde{x}'\Big(\frac{1}{\gamma^2} Q G G' Q + 2\lambda Q + \gamma^2 \Psi(u,y) + I\Big)\tilde{x} \\
&\quad - 2\tilde{x}' Q G w - 2\gamma^2 (Y_x(u,y) - \mathbf{J}_x C)'(I - Y_v(y) Y_v^{\perp}(y))'(I - Y_v(y) Y_v^{\perp}(y)) v
\end{aligned}
$$

By completing the squares, we further conclude that

$$
\begin{aligned}
\dot{V} &= -\tilde{x}'(2\lambda Q + I)\tilde{x} - \frac{1}{\gamma^2}\|G' Q \tilde{x} + \gamma^2 w\|^2 + \gamma^2 \|w\|^2 \\
&\quad - \gamma^2 \|(I - Y_v(y) Y_v^{\perp}(y))(Y_x(u,y) - \mathbf{J}_x C)\tilde{x} + v\|^2 + \gamma^2 \|v\|^2 \\
&\leq -(2\lambda + 1/\lambda_{max}(Q))V + \gamma^2 \|w\|^2 + \gamma^2 \|v\|^2 \\
&\leq -(2\lambda + 1/\lambda_{max}(Q))(1-\theta)V, \quad \text{for all } V \geq \frac{\gamma^2}{(2\lambda + 1/\lambda_{max}(Q))\theta}(\|w\|^2 + \|v\|^2),
\end{aligned}
$$

where $\theta \in (0,1)$. It is now straightforward to conclude that the input-to-state stability bound (20) holds with $c = \sqrt{\frac{\lambda_{max}(P_0^{-1})}{\delta}}$, $\kappa = (\lambda + \frac{1}{2\lambda_M})(1-\theta)$, $\gamma_w = \gamma_v = \frac{\gamma}{\sqrt{(2\lambda + 1/\lambda_M)\theta}}$, where $\lambda_M = \sup_{\tau \in [0,T)} \lambda_{max}(Q(\tau))$. $\square$

4. Autonomous vehicles motion estimation using CCD cameras and inertial sensors

In this section we show how one can estimate the position and attitude of an autonomous vehicle with respect to an inertial coordinate frame defined by visual landmarks using both measurements from an inertial measurement unit (IMU) and a monocular charged-coupled-device (CCD) camera mounted on-board. We do this by reducing the problem to the estimation of the state of a system with implicit outputs of the form (1)–(2).

4.1. *Kinematic equations of motion*

Let $\{I\}$ be an inertial coordinate frame and $\{B\}$ a body-fixed coordinate frame whose origin is located e.g. at the center of mass of the vehicle. The configuration of the vehicle $\left({}^I_B R,\ {}^I p_B\right)$ or for simplicity of notation (R,p), is an element of the Special Euclidean group $\mathrm{SE}(3) := \mathrm{SO}(3) \times \mathbb{R}^3$, where $R \in \mathrm{SO}(3) := \{R \in \mathbb{R}^{3\times 3} : RR' = I_3, \det(R) = +1\}$ is a rotation matrix that describes the orientation of the vehicle by mapping body coordinates into inertial coordinates, and $p \in \mathbb{R}^3$ is the position of the origin of $\{B\}$ in $\{I\}$. Denoting by $v \in \mathbb{R}^3$ and $\omega \in \mathbb{R}^3$ the linear and angular velocities of the vehicle relative to $\{I\}$ expressed in $\{B\}$, respectively, the following kinematic relations apply:

$$
\dot{p} = Rv, \tag{22}
$$

$$
\dot{R} = RS(\omega), \tag{23}
$$

where $S(\cdot)$ is a function from $\mathbb{R}^3$ to the space of skew-symmetric matrices $\mathcal{S} := \{M \in \mathbb{R}^{3\times 3} : M = -M'\}$ defined by

$$
S(x) := \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \quad \forall x := (x_1, x_2, x_3) \in \mathbb{R}^3.
$$

Let $\{V\}$ be another inertial coordinate frame, but this one defined by visual landmarks. The objective is to determine the position ${}^{V}p_B \in \mathbb{R}^3$ and orientation ${}^{V}_{B}R \in \mathrm{SO}(3)$ of the vehicle with respect to the visual coordinate system $\{V\}$. It is assumed that the position and orientation of $\{I\}$ with respect to the visual coordinate frame $\{V\}$ are unknown.

4.2. *Sensor Measurements*

We consider that the IMU provides the vehicle's linear velocity v, angular velocity ω, and pose (position and attitude) with respect to some inertial coordinate frame $\{I\}$. The measurements are denoted by

$$\begin{aligned} \zeta_1 &= v, \\ \zeta_2 &= \omega, \\ \zeta_3 &= p, \qquad (24) \\ \zeta_4 &= R, \qquad (25) \end{aligned}$$

where $\zeta_1 \in \mathbb{R}^3, \zeta_2 \in \mathbb{R}^3, \zeta_3 \in \mathbb{R}^3$, and $\zeta_4 \in \mathrm{SO}(3)$.

We also suppose that there is a camera attached to the vehicle that sees N points $q_i = (x_i, y_i, z_i)'$, $i = 1, 2, \ldots, N$ with known coordinates in the visual coordinate system $\{V\}$. Denoting by $\zeta_{i+4} \in \mathbb{R}^3$ the homogeneous image coordinates provided by the camera of the point q_i, the following relationships apply:

$$\mu_{i+4}\zeta_{i+4} = F\,{}^{C}q_i, \qquad (26)$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \zeta_{i+4} = 1, \quad \forall i \in \{1, 2, \ldots, N\} \qquad (27)$$

where ${}^{C}q_i$ is the position of q_i expressed in the camera's frame $\{C\}$, $\mu_{i+4} \in \mathbb{R}$ captures the depth of the point ${}^{C}q_i$ (which is unknown), and F is an upper triangular matrix with the camera's intrinsic parameters, of the form

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ 0 & f_{22} & f_{23} \\ 0 & 0 & 1 \end{bmatrix},$$

where each f_{ij} denotes a scalar [18, Chapter 3].

Given the measurements ζ_i, $i = 1, \ldots, N + 4$, we now proceed with the formulation of a system with implicit outputs.

4.3. *System with implicit outputs*

Let ${}^{V}q_1$ and ${}^{B}q_1$ be the coordinates of a point q_1 in the frames $\{V\}$ and $\{B\}$, respectively. Then, the following holds:

$${}^{V}q_1 = {}^{V}p_B + {}^{V}_{B}R\,{}^{B}q_1. \qquad (28)$$

From this and (22)–(23), we obtain the state equations

$${}^{B}\dot{q}_1 = {}^{V}_{B}R'\,{}^{V}\dot{q}_1 - S(\omega)\,{}^{B}q_1 - v, \qquad (29)$$

$${}^{V}_{B}\dot{R} = -S(\omega)\,{}^{V}_{B}R'. \qquad (30)$$

To obtain the output equations for the vision system, we first note that if ${}^{V}q_j$ and ${}^{B}q_j$ denote the coordinates of another point q_j in the frames $\{V\}$ and $\{B\}$, respectively, we conclude that

$$\begin{aligned}{}^{B}q_j &= {}^{V}_{B}R'\,{}^{V}q_j - {}^{V}_{B}R'\ {}^{V}p_B \\ &= {}^{V}_{B}R'\big({}^{V}q_j - {}^{V}q_1\big) + {}^{B}q_1.\end{aligned}$$

Using now (26) and the fact that ${}^{C}q_i = {}^{C}p_B + {}^{C}_{B}R\,{}^{B}q_i$, we obtain the output equations

$$\mu_{i+4}\zeta_{i+4} = F\Big({}^{C}p_B + {}^{C}_{B}R\,{}^{V}_{B}R'\big({}^{V}q_i - {}^{V}q_1\big) + {}^{C}_{B}R\,{}^{B}q_1\Big), \quad \forall i \in \{1, 2, \ldots, N\} \tag{31}$$

where $\big({}^{C}_{B}R,\ {}^{C}p_B\big) \in \mathrm{SE}(3)$ denotes the configuration of the frame $\{B\}$ with respect to the camera's frame $\{C\}$.

We will regard ζ_1 and ζ_2 as inputs to the implicit output system. The dynamics of (24)–(25) can be written as

$$\overset{\cdot}{\overbrace{{}^{V}_{B}R'\ {}^{V}p_I}} = -S(\omega)\,{}^{V}_{B}R'\ {}^{V}p_I, \tag{32}$$

$${}^{V}_{I}\dot{R}' = 0, \tag{33}$$

with the output equations

$$\zeta_4'\zeta_3 = {}^{V}_{B}R'\,{}^{V}q_1 - {}^{B}q_1 - {}^{V}_{B}R'\ {}^{V}p_I, \tag{34}$$

$$\zeta_4\,{}^{V}_{I}R' = {}^{V}_{B}R'. \tag{35}$$

Thus, our implicit output system is composed by (32)–(33), (29)–(30), (34)–(35) and (31). We now need to rewrite it in the form (1)–(2).

To proceed we use the following notation: Given an $m \times n$-matrix M, we denote by $\mathrm{stack}(M)$ the mn-vector obtained from stacking the columns of M one on top of each other, with the first column on top. Given two matrices $M_i \in \mathbb{R}^{m_i \times n_i}$, $i \in \{1, 2\}$ we denote by $M_1 \otimes M_2 \in \mathbb{R}^{m_1 m_2 \times n_1 n_2}$ the Kronecker product of M_1 by M_2. Using the fact that given three matrices A, B, X with appropriate dimensions, $\mathrm{stack}(A\,X\,B) = (B' \otimes A)\,\mathrm{stack}(X)$ [11], we can rewrite (32)–(33), (29)–(30), (31), (34)–(35) as follows:

$$\overset{\cdot}{\overbrace{{}^{V}_{B}R'\ {}^{V}p_I}} = -S(\omega)\,{}^{V}_{B}R'\ {}^{V}p_I, \tag{36}$$

$$\mathrm{stack}({}^{V}_{I}\dot{R}') = 0_{9\times 1} \tag{37}$$

$${}^{B}\dot{q}_1 = -S(\omega)\,{}^{B}q_1 - v + \big({}^{V}\dot{q}_1' \otimes I_{3\times 3}\big)\,\mathrm{stack}({}^{V}_{B}R'), \tag{38}$$

$$\mathrm{stack}({}^{V}_{B}\dot{R}') = \big(I_{3\times 3} \otimes -S(\omega)\big)\,\mathrm{stack}({}^{V}_{B}R'), \tag{39}$$

$$\zeta_4'\zeta_3 = \big({}^{V}q_1' \otimes I_{3\times 3}\big)\,\mathrm{stack}({}^{V}_{B}R') - {}^{B}q_1 - {}^{V}_{B}R'\ {}^{V}p_I, \tag{40}$$

$$\big({}^{V}_{I}R \otimes I_{3\times 3}\big)\,\mathrm{stack}(\zeta_4) = \mathrm{stack}({}^{V}_{B}R'), \tag{41}$$

$$\mu_{i+4}\zeta_{i+4} = F\,{}^{C}p_B + \big[\big({}^{V}q_i - {}^{V}q_1\big)' \otimes F\,{}^{C}_{B}R\big]\,\mathrm{stack}({}^{V}_{B}R') + F\,{}^{C}_{B}R\,{}^{B}q_1 \tag{42}$$

Thus, defining the vectors $x \in \mathbb{R}^{24}$, $y \in \mathbb{R}^{12+N}$, and $u \in \mathbb{R}^{6}$ as

$$x := \begin{bmatrix} {}^{V}_{B}R'\ {}^{V}p_I \\ \mathrm{stack}({}^{V}_{I}R') \\ {}^{B}q_1 \\ \mathrm{stack}({}^{V}_{B}R') \end{bmatrix}, \quad y := \begin{bmatrix} \zeta_4'\zeta_3 \\ \mathrm{stack}(\zeta_4) \\ \zeta_5 \\ \vdots \\ \zeta_{4+N} \end{bmatrix}, \quad u := \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix},$$

it follows that system (36)–(42) can be expressed in the form (1)–(2) with

$$A(x,u) := \begin{bmatrix} -S(\omega) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -S(\omega) & {}^{V}\dot{q}_1' \otimes I_{3\times 3} \\ 0 & 0 & 0 & I_{3\times 3} \otimes -S(\omega) \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -v \\ 0 \end{bmatrix},$$

$$C(x,u) := \begin{bmatrix} -I & 0 & -I & {}^{V}q_1' \otimes I_{3\times 3} \\ 0 & 0 & 0 & I \\ 0 & 0 & F\,{}^{C}_{B}R & 0 \\ 0 & 0 & F\,{}^{C}_{B}R & ({}^{V}q_2 - {}^{V}q_1)' \otimes F\,{}^{C}_{B}R \\ 0 & 0 & F\,{}^{C}_{B}R & ({}^{V}q_3 - {}^{V}q_1)' \otimes F\,{}^{C}_{B}R \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & F\,{}^{C}_{B}R & ({}^{V}q_{2+N} - {}^{V}q_1)' \otimes F\,{}^{C}_{B}R \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ F\,{}^{C}p_B \\ \vdots \\ F\,{}^{C}p_B \end{bmatrix}$$

To determine $E(x,u,v)$ we introduce additive noise to (26), i.e.,

$$\mu_{i+4}\zeta_{i+4} = F\,{}^{C}q_i + v_i, \quad i = 1,\ldots,N \tag{43}$$

Note that noise does not destroy the constraint (27).

From (43), (27) and (31) we conclude that

$$\mu_{i+4} = [\,0\ 0\ 1\,]\,F\left[\,{}^{C}p_B + {}^{C}_{B}R\,{}^{V}_{B}R'\left({}^{V}q_i - {}^{V}q_1\right) + {}^{C}_{B}R\,{}^{B}q_1\right] + v_i, \qquad \forall i \in \{1,2,\ldots,N\}.$$

Thus,

$$E(x,u,v) := \begin{bmatrix} I & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & {}^{V}_{I}R \otimes I_{3\times 3} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \mu_3 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \mu_4 & \cdots & 0 \\ & & & & \ddots & \\ 0 & \cdots & \cdots & \cdots & & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \mu_{2+N} \end{bmatrix}$$

and in particular

$$\begin{aligned} E_{v_1} &= \mathrm{diag}\{0_{3\times 3}, 0_{9\times 9}, 1, 0, \ldots, 0\}, \\ E_{v_j} &= \mathrm{diag}\{0_{3\times 3}, 0_{9\times 9}, 0..., 1, .., 0\}, \\ E_{v_N} &= \mathrm{diag}\{0_{3\times 3}, 0_{9\times 9}, 0, \ldots, 0, 1\}. \end{aligned}$$

We can now use the results given in the previous sections to compute $\hat{x}$. From ${}^{B}\hat{q}_1$ and ${}^{V}_{B}\hat{R}'$, the position ${}^{V}p_B$ can also be estimated using

$${}^{V}\hat{P}_B = {}^{V}q_1 - {}^{V}_{B}\hat{R}\,{}^{B}\hat{q}_1.$$

4.4. *Simulation Results*

We now illustrate the performance of the proposed estimator through computer simulation. The autonomous vehicle starts at the origin ${}^{V}p_B = 0$ with orientation ${}^{V}_{B}R = I$ and follows a circular path with a camera looking up at four non-coplanar points. The linear velocity is $v = [0.3, 0, 0]'\,m/s$ and the angular velocity is $\omega = [0, 0, 0.2]'\,rad/s$. The measurements were corrupted with additive Gaussian noise with standard deviation equal to roughly 5% of the measurements.

Fig. 1 displays the time evolution of the estimation errors. It can be seen that the estimated pose without noise converges to zero (see Fig. 1(a)) and in the presence of noise tend to a small neighborhood of the true value (see Fig. 1(b)).

5. Conclusions

We considered the problem of estimating the state of a system with implicit outputs. We designed estimators using a deterministic H_∞ approach that are globally convergent under appropriate observability assumptions and can therefore, be used to design output-feedback controllers. We apply these results to the estimation of position and attitude of an autonomous vehicle using measurements from an inertial measurement unit and a monocular charged-coupled-device camera attached to the vehicle. The estimation problem in the presence of latency and intermittency of the observations is a topic of current research. Another issue for future research is the study of the complexity and performance of the proposed estimator algorithm as the number of features N increase.

Appendix

Lemma 4 *Assuming that* (6) *has a solution on* $[0,T)$, $T \in [0,\infty]$, *then* $P(t)$ *is positive definite for all* $t \in [0,T)$.

Proof. Observe that (6) can also be written as

$$\dot{P} = \big[\mathbf{J}_x A(u) + \lambda I - \frac{\gamma^2}{2} P(\Psi(u,y) - \gamma^{-2} I)\big] P + P\big[\mathbf{J}_x A(u) + \lambda I - \frac{\gamma^2}{2} P(\Psi(u,y) - \gamma^{-2} I)\big]' + \gamma^{-2} GG',$$

and, therefore,

$$P(t) = \Phi(0,t)' P_0 \Phi(0,t) + \gamma^{-2} \int_0^t \Phi(\tau,t)' GG' \Phi(\tau,t) d\tau, \quad t \in [0,T), \tag{44}$$

where $\Phi(t,\tau)$ denotes the state transition matrix of $\dot{z} = -(\mathbf{J}_x A(u) + \lambda I - \frac{\gamma^2}{2} P(\Psi(u,y) - \gamma^{-2} I))z$. This can be verified by taking derivatives of the candidate expression (44) for P, i.e.,

$$\begin{aligned}
\dot{P} &= (\mathbf{J}_x A(u) + \lambda I - \frac{\gamma^2}{2} P(\Psi(u,y) - \gamma^{-2} I))' \Phi(0,t)' P_0 \Phi(0,t) \\
&\quad + \Phi(0,t)' P_0 \Phi(\tau,t)(\mathbf{J}_x A(u) + \lambda I - \frac{\gamma^2}{2} P(\Psi(u,y) - \gamma^{-2} I)) \\
&\quad + \gamma^{-2} GG' + \gamma^{-2}(\mathbf{J}_x A(u) + \lambda I + \frac{\gamma^2}{2} P(\Psi(u,y) - \gamma^{-2} I))' \int_0^t \Phi(\tau,t)' GG' \Phi(\tau,t) d\tau \\
&\quad + \gamma^{-2} \int_0^t \Phi(\tau,t)' GG' \Phi(\tau,t) d\tau (\mathbf{J}_x A(u) + \lambda I - \frac{\gamma^2}{2} P(\Psi(u,y) - \gamma^{-2} I)) \\
&= \big[\mathbf{J}_x A(u) + \lambda I - \frac{\gamma^2}{2} P(\Psi(u,y) - \gamma^{-2} I)\big] P + P\big[\mathbf{J}_x A(u) + \lambda I - \frac{\gamma^2}{2} P(\Psi(u,y) - \gamma^{-2} I)\big]' + \gamma^{-2} GG'.
\end{aligned}$$

Here, we used the fact that, for every fixed τ,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \Phi(\tau,t) &= -\Phi(t,\tau)^{-1} \Big(\frac{\mathrm{d}}{\mathrm{d}t} \Phi(t,\tau)\Big) \Phi(t,\tau)^{-1} \\
&= \Phi(t,\tau)^{-1} (\mathbf{J}_x A(u) + \lambda I - \frac{\gamma^2}{2} P(\Psi(u,y) - \gamma^{-2} I)) \Phi(t,\tau) \Phi(t,\tau)^{-1} \\
&= \Phi(\tau,t)(\mathbf{J}_x A(u) + \lambda I - \frac{\gamma^2}{2} P(\Psi(u,y) - \gamma^{-2} I)).
\end{aligned}$$

Now, since $\Phi(t,0) P_0 \Phi(t,0)' > 0$ and $GG' \geq 0$, from (44) we conclude that $P(t)$ remains positive definite for all $t \in [0,T)$. □

References

[1] A. P. Aguiar and J. P. Hespanha. State estimation for systems with implicit outputs for the integration of vision and inertial sensors. In *Proc. of the 44th Conf. on Decision and Contr.*, Dec. 2005.

[2] A. P. Aguiar and J. P. Hespanha. Minimum-energy state estimation for systems with perspective outputs. *IEEE Trans. on Automat. Contr.*, 51(2):226–241, Feb. 2006.

[3] T. Başar and P. Bernhard. *H^∞ Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach.* Birkhäuser, Boston, MA, 2$^{\text{nd}}$ edition, 1995.

[4] R. K. Boel, M. R. James, and I. R. Petersen. Robustness and risk-sensitive filtering. *IEEE Trans. on Automat. Contr.*, 47(3):451–461, Mar. 2002.

[5] A. Chiuso, P. Favaro, H. Jin, and S. Soatto. Structure from motion causally integrated over time. *IEEE Trans. on Pattern Anal. Mach. Intell.*, 24(4):523–535, Apr. 2002.

[6] W. H. Fleming and W. M. McEneaney. A max-plus-based algorithm for a Hamilton–Jacobi–Bellman equation of nonlinear filtering. *SIAM J. Contr. Optimization*, 38(3):683–710, 2000.

[7] B. K. Ghosh, M. Jankovic, and Y. T. Wu. Perspective problems in system theory and its application in machine vision. *J. Math. Syst. Estimation Contr.*, 4(1):3–38, 1994.

[8] B. K. Ghosh and E. P. Loucks. A perspective theory for motion and shape estimation in machine vision. *SIAM J. Contr. Optimization*, 33(5):1530–1559, 1995.

[9] J. Hespanha, O. Yakimenko, I. Kaminer, and A. Pascoal. Linear parametrically varying systems with brief instabilities: an application to integrated vision/IMU navigation. In *Proc. of the 40th Conf. on Decision and Contr.*, Dec. 2001.

[10] O. J. Hijab. *Minimum Energy Estimation.* PhD thesis, University of California, Berkeley, 1980.

[11] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press., Cambridge, 1994.

[12] M. James, J. Baras, and R. Elliott. Output feedback risk-sensitive control and differential games for continuous-time nonlinear systems. In *Proc. of the 32nd Conf. on Decision and Contr.*, pages 3357–3360, San Antonio, TX, USA, 1993.

[13] M. James, J. Baras, and R. Elliott. Risk-sensitive control and dynamic games for partially observed discrete-time nonlinear systems. *IEEE Trans. on Automat. Contr.*, 39(4):780792, 1994.

[14] M. Jankovic and Ghosh. Visually guided ranging from observations of points, lines and curves via an identifier based nonlinear observer. *Syst. & Contr. Lett.*, 25:63–73, 1995.

[15] I. Kaminer, A. M. Pascoal, W. Kang, and O. Yakimenko. Integrated vision/inertial navigation systems design using nonlinear filtering. *IEEE Trans. Aerospace and Electronic Syst.*, 37(1):158–172, Jan. 2001.

[16] A. J. Krener. Necessary and sufficient conditions for nonlinear worst case (H-infinity) control and estimation. *J. Math. Syst. Estimation Contr.*, (7):81106, 1997.

[17] Y. Ma, J. Kosecka, and S. Sastry. Vision guided navigation for a nonholonomic mobile robot. *IEEE Trans. Robot. Automat.*, 15(3):521–536, 1999.

[18] Y. Ma, J. Kosecka, S. Soatto, and S. Sastry. *An Invitation to 3–D Vision.* Springer Verlag, 2003.

[19] L. Matthies, T. Kanade, and R. Szeliski. Kalman filter-based algorithms for estimating depth from image sequences. *Int. J. of Comput. Vision*, 3:209–236, 1989.

[20] A. Matveev, X. Hu, R. Frezza, and H. Rehbinder. Observers for systems with implicit output. *IEEE Trans. on Automat. Contr.*, 45(1):168–173, Jan. 2000.

[21] W. M. McEneaney. Robust/H-infinity filtering for nonlinear systems. *Syst. & Contr. Lett.*, (33):315–325, 1998.

[22] R. E. Mortensen. Maximum likelihood recursive nonlinear filtering. *J. Opt. Theory and Applications*, 2:386–394, 1968.

[23] K. M. Nagpal and P. P. Khargonekar. Filtering and smoothing in an H_∞ setting. *IEEE Trans. on Automat. Contr.*, 36(2):152–166, Feb. 1991.

[24] H. Rehbinder and B. K. Ghosh. Pose estimation using line-based dynamic vision and inertial sensors. *IEEE Trans. on Automat. Contr.*, 48(2):186–199, Feb. 2003.

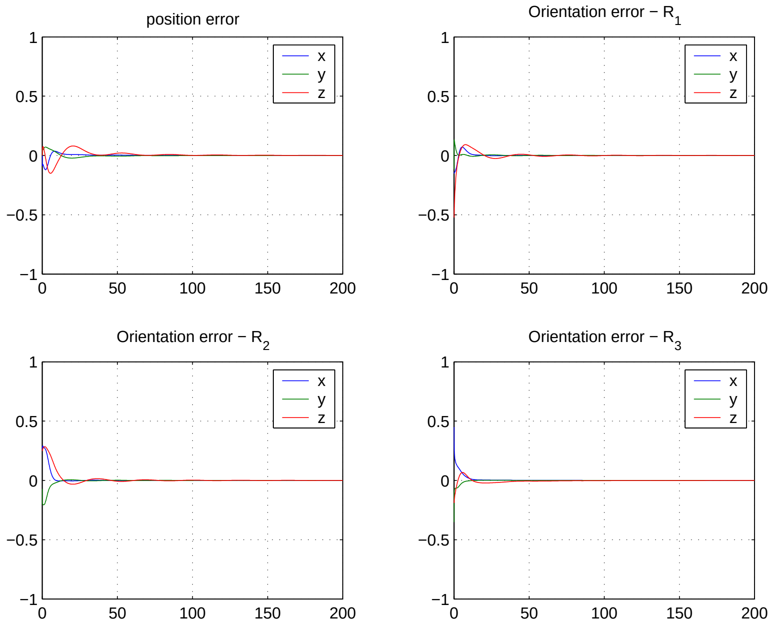
[25] A. H. Sayed. A framework for state-space estimation with uncertain models. *IEEE Trans. on Automat. Contr.*, 46(7):9981013, July 2001.

[26] O. Shakernia, Y. Ma, T. Koo, and S. Sastry. Landing an unmanned air vehicle: vision based motion estimation and nonlinear control. *Asian J. Contr.*, 1(3):128–145, 1999.

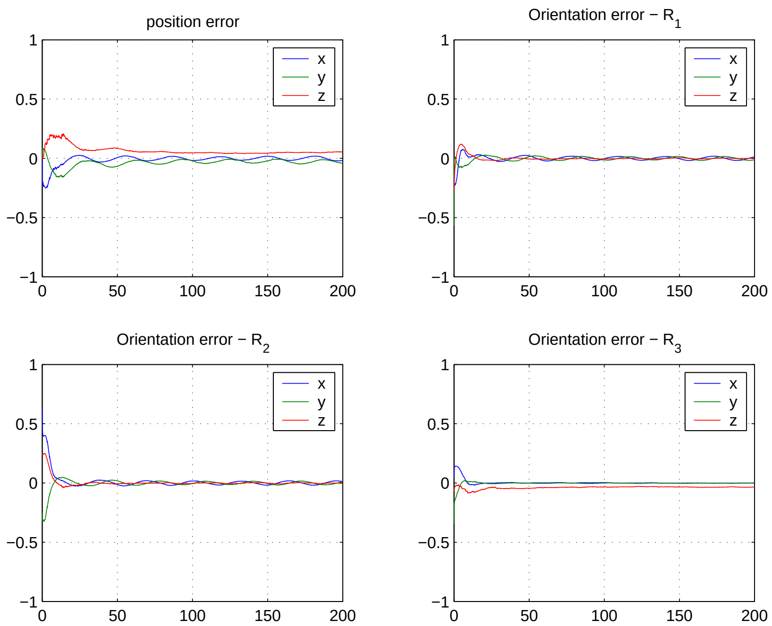
[27] S. Soatto, R. Frezza, and P. Perona. Motion estimation via dynamic vision. *IEEE Trans. on Automat. Contr.*, 41(3):393–413, Mar. 1996.

[28] S. Takahashi and B. K. Ghosh. Motion and shape parameters identification with vision and range. In *Proc. of the 2001 Amer. Contr. Conf.*, volume 6, pages 4626–4631, June 2001.

[29] L. Xie, Y. C. Soh, and C. E. de Souza. Robust Kalman filtering for uncertain discrete-time systems. *IEEE Trans. on Automat. Contr.*, 29(6):1310–1314, 1994.



(a) Without noise and disturbances



(b) With noise and disturbances

Fig. 1. Time evolution of the estimation errors in position and orientation. The orientation errors labeled R_1, R_2, and R_3 correspond to the estimation errors for the first, second, and third columns of ${}^{V}_{B}R$, respectively.