An Input-to-State-Stability approach to Economic Optimization in Model Predictive Control

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***Abstract*—This paper presents a Model Predictive Control (MPC) scheme where a combination of a stabilizing stage cost and an economic stage cost is employed to allow the minimization of an economic performance index while still guaranteeing convergence toward a desired steady-state. Input-to-state-stability (ISS) with respect to the economic stage cost is provided. More precisely, for the case of an economic stage cost converging to zero, the economic optimization only affects the transient behavior of the closed-loop trajectories preserving the convergence to the desired steady-state. Alternatively, if the economic stage cost is merely bounded, or convergent to a bound, the closed-loop state trajectory is ultimately bounded around the desired steady-state with the size of the bound being monotonically increasing with the magnitude of the economic stage cost. The loosening of the closed-loop guarantees, i.e., moving from convergence to ultimate boundedness, gives space to the increase of economic performance. Numerical results illustrate the effectiveness of the proposed method on an energy efficient trajectory-tracking control problem of a marine robotic vehicle navigating in the presence of water currents.**

***Index Terms*—Economic model predictive control; input-to-state-stability; nonlinear predictive control; constrained control; nonlinear systems; maritime control**

I. Introduction

The two key features that determined the success of MPC strategies in many control applications are the ability to satisfy input and state constraints and to explicitly optimize a given performance index. Depending on the meaning of the performance index, we identify two classes of MPC schemes: tracking MPC and economic MPC.

In a tracking MPC scheme, the performance index is chosen to penalize the distance from the current state to a desired one, therefore driving the state vector to a desired steady-state or state trajectory. For an overview of the methods that utilize the so-called terminal set and terminal cost, we refer the reader to, e.g., [1], [2], [3], [4], for the discrete-time case, and to, e.g., [5], [6], [7], for the continuous-time counterpart. In contrast, the work [8] avoids the use of these terminal elements and focusses on the computation of a sufficiently long horizon to guarantee closed-loop stability.

In an economic MPC scheme, the term "economic" is used to highlight that the performance index is not designed as a distance from a desired state, but it rather represents an index of interest to be minimized, e.g., an economic performance index. Here, the main challenge stems from the fact that the standard value function, used in tracking MPC, is generally not decreasing as the state approaches the economically optimal steady-state or state trajectory. A solution to this problem is proposed in [9], and later generalized in [10], [11], [12], where a rotated value function, which satisfies the desired decrease, is designed using a dissipativity property of the system. Whereas the above-mentioned methods employ a terminal set and a terminal cost, in [13] the author avoids these elements and provides conditions on horizon length and prediction for closed-loop state convergence to an arbitrarily small neighborhood of the optimal steady-state. In [14] the authors extend their tracking MPC scheme [15], [16], designed to address the stability and recursive feasibility for the case of jumps of the desired set point, to the case of jumps on the economic stage costs, and therefore, on the associated economically optimal steady-state. The work [12] considers generic time-varying economic stage costs and convergence to possibly time-varying economically optimal state trajectories in the continuous-time framework.

Other approaches to economic MPC developed without the use of dissipativity properties of the system has been reported in the literature. In [17], [18] a generalized terminal set, consisting of all the feasible steady-states, and a constraint on the decrease of the terminal cost from one solution of the MPC optimization problem to the other, are employed to ensure closed-loop state convergence to a steady-state of the system. In [19] the proposed controller constantly optimizes the economic stage cost and, at an arbitrarily given time, the convergence to a desired steady-state is enforced by constraining the state within shrinking level sets of a given control Lyapunov function defined over the whole region of interest. In [20], [21] a combination of a stabilizing and an economic stage cost is adopted while still guaranteeing stability and convergence, respectively, to a desired steady-state.

Despite of the active research in the field, most of the contributions focus on the convergence to a steady-state, which is either the economically optimal one [9], [10], [11], [14], [13], a generic one [17], [18], or one that is given [19], [20], [21]. Some exceptions are the works [22], [11], [23] that focus on the convergence to an optimal periodic state trajectory and [12] that addresses convergence to a pre-computed economically optimal time-dependent state trajectories. This general trend rules out a set of interesting behaviors, e.g., bounded and non-periodic trajectories, that arise in many practical applications where the compromise between economic performance and tracking performance is the desired behavior. Further, often, the economically optimal behavior cannot be calculated a priori and the use of other specific assumptions (like the dissipativity assumption of the dissipativity-based Economic MPC, e.g., [9], [10], [11], [12]) can be challenging to verify, but one would still require to have some guarantees on the closed-loop behavior of the system that is driven to minimize an economic objective.

Combining an economic and a stabilizing performance index is a direct and appealing technique used in many practical control problems. The idea of adding a "stabilizing" term to an economic stage cost function to obtain closed-loop guarantees is not new in the technical literature. Such term is intended as a regularization term in, e.g., in [22] to prove convergence to periodic orbits, or in [9] to satisfy a dissipativity property and therefore certify convergence to an economically optimal steady-state. In [24] a similar term is added to increase the closed-loop performance in the transient phase.

In this work, we propose a different approach by proving that the system in closed-loop with an MPC controller is ISS with respect to the economic stage cost. Using this property, it is shown that for the case where of economic stage cost is uniformly bounded over time, or converges to a bound, the closed-loop state trajectory of the system converges to a bounded set around the desired steady-state, with the size of the bound being monotonically increasing with the magnitude of the economic stage cost. We build on our previous works [25], [26], where the result in [26] is first extended to the case of time-varying dynamical models, constraints sets, and stage costs, and then merged with [25] in a unified framework. The theory is further extended from convergent-economic-cost-convergent-state to ISS with respect to the economic cost.

The remainder of this paper is organized as follows: the problem definition is given in Section II, followed by the main result in Section III, and in Section IV with a discussion on possible design methods. Section V presents an application of the proposed approach to energy efficient control of a marine vehicle navigating through water currents. Section VI closes the paper with conclusions.

Notation. For a generic continuous-time trajectory $\boldsymbol{x}$, the term $\boldsymbol{x}([t_1,t_2])$ denotes the trajectory considered in the time interval $[t_1,t_2]$ and $x(t)$ the trajectory evaluated at a specific time t. The notation $x(\tau;t,z)$ is used whenever we want to make explicit the dependence of $x(\tau)$ on the optimization problem parameters t and z, and we adopt the bar notation $\bar{\boldsymbol{x}}$ to refer to a predicted trajectory, i.e., a feasible future trajectory of the MPC optimization problem. For a generic function $g:\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}$, with time and state vectors as parameters, and scalar $\gamma\geq 0$, the term $\mathcal{L}(t;g,\gamma)$ denotes the time-varying γ-sublevel set of $g(\cdot)$, i.e., $\mathcal{L}(t;g,\gamma) := \{\, x : g(t,x)\leq\gamma \,\}$. If the function $g(\cdot)$ takes only the state as parameter, then $\mathcal{L}(g,\gamma) := \{\, x : g(x)\leq\gamma \,\}$ denotes the associated time invariant γ-sublevel set. The term $\mathcal{B}(r)$ denotes the closed ball set of radius r defined as $\mathcal{B}(r) := \{x : \|x\|\leq r\}$. A function continuous $\alpha:[0,a)\to\mathbb{R}_{\geq 0}$, for some $a>0$, is said to belong to class $\mathcal{K}$ if it is zero at zero and strictly increasing. Moreover, $\alpha(\cdot)$ is said to belong to class $\mathcal{K}_\infty$ if it is a class-$\mathcal{K}$ function with $a=\infty$, and it is radially unbounded, i.e., $\alpha(x)\to\infty$ as $x\to\infty$. A function $\beta:\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{KL}$ if for each fixed s the mapping $\beta(r,s)$ belongs to class-$\mathcal{K}$ with respect to r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s)\to 0$ as $s\to\infty$. For a given $n\in\mathbb{N}$, $SE(n)$ denotes the Cartesian product of $\mathbb{R}^n$ with the group $SO(n)$ of $n\times n$ rotation matrices and $se(n)$ denotes the Cartesian product of $\mathbb{R}^n$ with the space $so(n)$ of $n\times n$ skew-symmetric matrices. The term $\mathcal{C}(a,b)$ and $\mathcal{PC}(a,b)$ denotes the space of continuous and piecewise continuous trajectories, respectively, defined over $[a,b]$ or $[a,+\infty)$ for the case where $b=+\infty$. For sake of simplicity, the dependence on time and parameters is dropped whenever it is clear from the context.

II. Problem definition

Consider the continuous-time dynamical system

$$\dot{x}(t) = f(t,x(t),u(t)), \qquad x(t_0)=x_0, \qquad t\geq t_0 \tag{1}$$

and let the state and input vectors $x(t)\in\mathbb{R}^n$ and $u(t)\in\mathbb{R}^m$ be constrained as

$$(x(t),u(t))\in\mathcal{X}(t)\times\mathcal{U}(t), \qquad t\geq t_0, \tag{2}$$

where the set-valued maps $\mathcal{X}:\mathbb{R}\rightrightarrows\mathbb{R}^n$ and $\mathcal{U}:\mathbb{R}\rightrightarrows\mathbb{R}^m$ denote the time-varying state and input constraint sets, and t_0 and $x_0=x(t_0)$ are the initial-time and initial-state vectors, respectively.

Definition 1 (Open-loop MPC problem)**.** *Given a pair $(t,z)\in\mathbb{R}_{\geq t_0}\times\mathbb{R}^n$ and an horizon length $T\in\mathbb{R}_{>0}$, the open-loop MPC optimization problem $\mathcal{P}(t,z)$ consists in finding the optimal control signal $\bar{\boldsymbol{u}}^*\in\mathcal{PC}(t,t+T)$ that solves*

$$\begin{aligned} J_T^*(t,z) = &\min_{\bar{\boldsymbol{u}}\in\mathcal{PC}(t,t+T)} J_T(t,z,\bar{\boldsymbol{u}}) \\ \textit{s.t.}\quad & \dot{\bar{x}}(\tau) = f(\tau,\bar{x}(\tau),\bar{u}(\tau)), \qquad \forall\tau\in[t,t+T] \\ & (\bar{x}(\tau),\bar{u}(\tau))\in\mathcal{X}(\tau)\times\mathcal{U}(\tau), \quad \forall\tau\in[t,t+T] \\ & \bar{x}(t)=z, \\ & \bar{x}(t+T)\in\mathcal{X}_{aux}(t+T) \end{aligned}$$

with

$$J_T(t,z,\bar{\boldsymbol{u}}) := \int_t^{t+T} l(\tau,\bar{x}(\tau),\bar{u}(\tau))d\tau + m(t+T,\bar{x}(t+T))$$

where the finite horizon cost $J_T(\cdot)$*, which corresponds to the* performance index *of the MPC controller, is composed of the* stage cost $l:\mathbb{R}_{\geq t_0}\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}$ *and the* terminal cost $m:\mathbb{R}_{\geq t_0}\times\mathbb{R}^n\to\mathbb{R}$*, which is defined over the time-varying* terminal set $\mathcal{X}_{aux}:\mathbb{R}_{\geq t_0}\rightrightarrows\mathbb{R}^n$*. In the sequel, we denote by $k_{aux}:\mathbb{R}_{\geq t_0}\times\mathbb{R}^n\to\mathbb{R}^m$ an* auxiliary control law *defined over the terminal set, such that $k_{aux}(t,x)\in\mathcal{U}(t)$ for all $t\geq t_0+T$ and $x(t)\in\mathcal{X}_{aux}(t)$.* □

For a given pair $(\hat{t},\hat{x})\in\mathbb{R}_{\geq t_0}\times\mathbb{R}^n$ we say that the open-loop MPC problem $\mathcal{P}(\hat{t},\hat{x})$ is feasible if it admits a feasible solution.

In a *sampled-data MPC* approach, the control input is computed at the discrete-time samples

$$\mathcal{T} := \{t_0,t_1,\dots\}, \qquad \delta_k := t_{k+1}-t_k\in[\underline{\delta},\bar{\delta}] \tag{4}$$

for two positive scalars $\underline{\delta}$ and $\bar{\delta}$ with $\underline{\delta}<\bar{\delta}$. Concatenating the solution of the MPC optimization problem with the auxiliary control law results in the following notion of extended state and input trajectories.

Definition 2 (Extended trajectories)**.** *The* extended input trajectory $u_{e_i}(t)$ *for* $t \in [t_i, \infty)$ *at* (t_i, x_i)*, with* $x_i := x(t_i)$*, is defined to be the concatenation of the open-loop MPC solution* $\bar{u}^*(t; t_i, x_i)$ *with the auxiliary control law* $k_{aux}(\cdot)$ *as follows:* $u_{e_i}(t) = k_{ext}(t, x; t_i, x_i)$ *with*

$$k_{ext}(t, x; t_i, x_i) \ := \ \begin{cases} \bar{u}^*(t; t_i, x_i), & t \in [t_i, t_i + T] \\ k_{aux}(t, x), & t > t_i + T \end{cases}.$$

The extended state trajectory $x_{e_i}(t)$ *is the related closed-loop state trajectory starting at* $(t_i, x_i) \in \mathbb{R}_{\geq t_0} \times \mathcal{X}(t_i)$*.* □

The following mild assumptions on the dynamical model are required to guarantee, together with the boundedness of the state vector, the existence and uniqueness of the solution of the system in closed-loop with the MPC controller.

Assumption 1. *The function* $f(\cdot)$*, introduced in* (1)*, is locally Lipschitz continuous in* x*, piecewise continuous in* u *and* t *in the region of interest, and without loss of generality,* $f(t, 0, 0) = 0$*, for all* $t \geq t_0$*. Moreover,* $f(\cdot)$ *is bounded for bounded states, i.e., the set*

$$\{\|f(t, x, u)\| : t \geq t_0, x \in \hat{\mathcal{X}}, u \in \mathcal{U}(t)\} \tag{5}$$

is bounded for any bounded set $\hat{\mathcal{X}} \subset \mathcal{R}^n$*.* □

The condition on the boundedness of $f(\cdot)$ for bounded x, also adopted in [7], is always satisfied in the case of time-invariant systems with input constraints set $\mathcal{U}(\cdot)$ being uniformly bounded over time.

The sampled-data MPC control law is defined as

$$u(t) = k_{MPC}(t, x) := k_{ext}(t, x; \lfloor t \rfloor, x(\lfloor t \rfloor)), \tag{6}$$

where $\lfloor t \rfloor$ is the maximum sampling instant $t_i \in \mathcal{T}$ smaller than or equal to t, i.e., $\lfloor t \rfloor = \max_{i \in \mathbb{N}_{\geq 0}} \{t_i \in \mathcal{T} : t_i \leq t\}$. Note that the control input (6) is well defined even when the distance between two consecutive sampling instants of time is greater than the horizon length, i.e., $t_{i+1} - t_t > T$. In this case, the closed-loop input trajectory results in a combination of optimal trajectories and auxiliary control inputs.

The following technical assumption is required to guarantee the existence of the solution associated with the auxiliary control law.

Assumption 2. *The closed-loop systems* (1) *with* $u(t) = k_{aux}(t, x)$*, from Definition 1, admits a unique solution defined for all* $t \geq t_0 + T$*.* □

The goal of this paper is to analyze the properties of the closed-loop system (1)-(6) for the case that the stage cost is chosen to be a combination of a standard stabilizing stage cost, from tracking MPC, and an economic stage cost. Toward this goal, the stage cost is decomposed as

$$l(t, x, u) = l_s(t, x, u) + l_e(t, x, u) \tag{7}$$

where the functions $l_s : \mathbb{R}_{\geq t_0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $l_e : \mathbb{R}_{\geq t_0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ denote the *stabilizing stage cost* and the *economic stage cost*, respectively.

III. MAIN RESULT

Before stating the main result, we introduce the following assumptions that are common in the tracking MPC literature.

Assumption 3 (Initial feasibility)**.** *The optimization problem* $\mathcal{P}(t_0, x_0)$ *admits a feasible solution.* □

Assumption 4 (Stabilizing stage cost)**.**

(i) *The state constraint set* $\mathcal{X}(t)$ *and the terminal set* $0 \in \mathcal{X}_{aux}(t) \subseteq \mathcal{X}(t)$ *are closed, connected, and contain the origin for all* $t \geq t_0$*. The input constraint set* $\mathcal{U}(t)$ *is such that* $0 \in \mathcal{U}(t)$ *for all* $t \geq t_0$*.*

(ii) *The stabilizing stage cost satisfies* $l_s(t, 0, 0) = 0$ *and there is a class-*$\mathcal{K}_\infty$ *function* $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *such that* $l_s(t, x, u) \geq \alpha(\|x\|)$ *for all* $(t, x, u) \in \mathbb{R}_{\geq t_0} \times \mathbb{R}^n \times \mathbb{R}^m$*.*

(iii) *The function* $m(\cdot)$ *is positive semi-definite.*

(iv) *For any given pair* $(\hat{x}, \hat{u}) \in \mathbb{R}^n \times \mathbb{R}^m$ *the functions* $l(t, \hat{x}, \hat{u})$ *and* $m(t, \hat{x})$ *are uniformly bounded over time.*

(v) *There exists a feasible auxiliary control law* $k_{aux} : \mathbb{R}_{\geq t_0} \times \mathbb{R}^n \to \mathbb{R}^m$*, defined over the terminal set* $\mathcal{X}_{aux}(\cdot)$*, such that, for the closed-loop system* (1) *with* $u(t) = k_{aux}(t, x)$*, with initial time and state pair* $(\hat{t}, \hat{x}) \in \mathbb{R}_{\geq t_0 + T} \times \mathcal{X}_{aux}(\hat{t})$*, the state and input vectors satisfy* $x(t) \in \mathcal{X}_{aux}(t)$ *and* $u(t) \in \mathcal{U}(t)$*, respectively, and the condition*

$$\begin{aligned} &m(\hat{t} + \delta, x(\hat{t} + \delta)) - m(\hat{t}, \hat{x}) \\ &\qquad \leq - \int_{\hat{t}}^{\hat{t}+\delta} l_s(t, x, k_{aux}(t, x)) dt \end{aligned} \tag{8}$$

holds for any $\delta > 0$*.*

Assumption 5. *Consider the constrained system* (1)-(2) *and the open-loop MPC problem from Definition 1. For all* $(\hat{t}, \hat{x}) \in \mathbb{R}_{\geq t_0} \times \mathbb{R}^n$ *with* $\mathcal{P}(\hat{t}, \hat{x})$ *feasible there exists a control law* $\mathbf{u}_f \in \mathcal{PC}(\hat{t}, \hat{t} + T)$ *such that, the closed-loop system* (1) *with* $u(t) = u_f(t)$*,* $t_0 = \hat{t}$*, and* $x_0 = \hat{x}$*, has feasible state and input trajectories, i.e., satisfying* (2)*, and the inequality*

$$\int_{\hat{t}}^{\hat{t}+T} l_s(\tau, x, u_f) d\tau + m(\hat{t} + T, x(\hat{t} + T)) \leq \alpha_c(\|x(\hat{t})\|)$$

holds for a class-$\mathcal{K}_\infty$ *function* $\alpha_c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$*.* □

This assumption was employed in [3] (Assumption SA4) for the discrete-time case, for a specific $\boldsymbol{u}_f$ to guarantee closed-loop stability of the origin. For a suitable choice of stage and terminal cost, this assumption coincides with the existence of a control input that renders the origin of the closed-loop system an asymptotically stable equilibrium point. We refer the reader to Section III of [3] for an interesting discussion on the topic.

For the sake of generality, Assumptions 1-5 and the Definition 1 of the open-loop MPC problem consider piecewise continuous input signals. Although, we highlight that the result of this paper still holds if restricting the input to be piecewise constant, as long as all the assumptions are satisfied within this framework, i.e., including the piecewise constant auxiliary control law. This can be the case in many practical applications where continuous-time dynamical systems are controlled by digital MPC controllers.

The main theorem of the paper is the following:

Theorem 1 (ISS-based Economic MPC)**.** *Let Assumptions 1-5 hold. Then, the system* (1) *in closed-loop with* (6)*, where $l(\cdot)$ is decomposed as in* (7)*, is ISS with respect to the economic stage cost, i.e., there exists a class-$\mathcal{KL}$ function $\beta(\cdot)$ and a class-$\mathcal{K}$ function $\gamma(\cdot)$ such that for any initial state $x(t_0) \in \mathcal{X}(t_0)$ and any bounded B defined as*

$$B := \sup_{\tau \geq t_0} (\|l_e(\tau, x(\tau), u(\tau))\|) \tag{9}$$

the solution $x(t)$ exists and satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma(B). \tag{10}$$

for all $t \geq t_0$. □

Remark 1 (Transient economic optimization)**.** *It is worth noting that Theorem 1 can be applied to cases where the economic optimization is only performed during a transient phase, while still guaranteeing convergence to the desired steady-state. This is due to the fact that, in general, the ISS property of a system implies the convergent-input-convergent-state property. Therefore, if the economic stage cost converges to zero, then also the state will converge to the desired steady-state. For further discussions on transient optimization in a similar framework, we refer the reader to [24]. Depending on the intensity of the economic stage cost, the latter work shows how it is possible to preserve closed-loop stability of the set-point, as proposed in [20], or allow to renounce to stability for convergence in order to leave more space to perform economic optimization, as proposed in [25], [24] and allowed by the framework presented in this paper.* □

Remark 2 (Uniform and asymptotic boundedness)**.** *The use of the terminal set in Assumption 4 is enough to guarantee the recursive feasibility of the open-loop MPC optimization problem. Therefore, uniform boundedness of the state trajectory could be enforced using state constraints for any arbitrary performance index. Although, as the main drawback, the region of attraction of the controller would clearly be limited within such constraints. This is in contrast with the approach proposed in this paper, where the bound is not imposed by state constraints, i.e., uniformly for all $t \geq t_0$, but derives from the ISS property of Theorem 1, and is reached asymptotically as $t \to \infty$, from all initial states of the region of attraction. Since no further state constraints are required, the proposed MPC controller can guarantee convergence to an ultimate bound and still potentially have a global region of attraction. This is the case of the simulation results of Section V, where a vehicle is driven to an ultimate bound around the desired trajectory starting from any initial condition.* □

Remark 3 (Quality of the bound B)**.** *As well known in the MPC literature, adding a term that is not a function of the optimization variables $\bar{\boldsymbol{x}}$ and $\bar{\mathbf{u}}$ to the performance index does not influence the optimizer of the open-loop MPC optimization problem. Therefore, it is important to remove such terms before computing the value of B to avoid a conservative estimate of the ultimate bound.* □

A. Proof of Theorem 1

The proof is structured as follows: we start by introducing a *shifted value function*. In contrast to the standard value function from tracking MPC, an offset term is introduced to capture the effect of the economic stage cost. Upper and lower bounds of such function are derived in Lemmas 1-3. Lemma 4 provides a bound on the decrease of the shifted value function evaluated along the closed-loop state trajectory. Using this latter result, Lemma 5 shows that the sampling of the closed-loop value function evaluated at the time instants $\mathcal{T}$ is ISS with respect to B, where we say that a sampling is ISS when the discrete-time system associated with the evolution of such samples is ISS. Using Lemma 4 and Lemma 5, Lemma 6 demonstrates that the same ISS property applies to the whole continuous time evolution of the value function. Combining this last result with Lemma 3 concludes the proof of Theorem 1. For the sake of clarity, the proofs of the lemmas are presented in Appendix.

Definition 3 (Shifted value function)**.** *Consider the open-loop MPC problem from Definition 1 and the bound B from* (9)*. Moreover, consider a pair $(\hat{t}, \hat{x}) \in \mathbb{R}_{\geq t_0} \times \mathcal{X}(\hat{t})$ with $\mathcal{P}(\hat{t}, \hat{x})$ feasible. Using the minimizer $\bar{\mathbf{u}}^* \in \mathcal{PC}(\hat{t}, \hat{t} + T)$, and the associated state trajectory $\bar{\boldsymbol{x}}^* \in \mathcal{C}(\hat{t}, \hat{t} + T)$, the shifted value function is defined as*

$$\begin{aligned} V(\hat{t}, \hat{x}) := \int_{\hat{t}}^{\hat{t}+T} & l(\tau, \bar{x}^*, \bar{u}^*) d\tau \\ & + m(\hat{t} + T, \bar{x}^*(\hat{t} + T)) - TB. \end{aligned} \tag{11}$$

□

Lemma 1 (Upper bounds on shifted value function)**.** *Consider the open-loop MPC problem from Definition 1, the bound B from* (9)*, and let Assumptions 1-5 hold. Then, for all $(\hat{t}, \hat{x}) \in \mathbb{R}_{\geq t_0} \times \mathcal{X}(\hat{t})$ with $\mathcal{P}(\hat{t}, \hat{x})$ feasible, the shifted value function $V(\cdot)$ from Definition 3 satisfies*

$$V(\hat{t}, \hat{x}) \leq \alpha_c(\|\hat{x}\|). \tag{12}$$

□

Before computing a lower bound for the value function, we first present the following lemma.

Lemma 2. *Consider the system* (1) *and let Assumption 1 hold. Then, for any class-$\mathcal{K}_\infty$ function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and constant $\Delta > 0$ there exists a class-$\mathcal{K}_\infty$ function $\alpha_\Delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that*

$$\int_t^{t+\delta} \alpha(\|x(\tau)\|) d\tau \geq \delta \alpha_\Delta(\|x(t)\|) \tag{13}$$

holds for all $\delta \in [0, \Delta]$, $x(t) \in \mathbb{R}^n$, and $t \geq t_0$. □

Lemma 3 (Lower bounds on shifted value function)**.** *Consider the open-loop MPC problem from Definition 1, the bound B from* (9)*, and let Assumptions 1-5 hold. Then, for all $(\hat{t}, \hat{x}) \in \mathbb{R}_{\geq t_0} \times \mathcal{X}(\hat{t})$ with $\mathcal{P}(\hat{t}, \hat{x})$ feasible and for any $\Delta \leq T$, the shifted value function $V(\cdot)$ from Definition 3 satisfies*

$$V(\hat{t}, \hat{x}) \geq \delta_v \alpha_\Delta(\|\hat{x}\|) - 2TB \tag{14}$$

for any scalar $\delta_v \in (0, \Delta]$, *and for a class-*$\mathcal{K}_\infty$ *function* $\alpha_\Delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. □

The previous lemmas describe the properties of the shifted value function at a generic pair $(\hat{t}, \hat{x}) \in \mathbb{R}_{\geq t_0} \times \mathbb{R}^n$. Next, we focus on the evolution of the latter evaluated along the state trajectory of the closed-loop (1) with (6).

Lemma 4. *Consider the shifted value function* $V(\cdot)$ *from Definition 3, the bound* B *from* (9)*, and let Assumptions 1-5 hold. Then, along the closed-loop* (1) *with* (6)*, the inequality*

$$V(t_i+\delta, x(t_i+\delta)) \leq V(t_i, x(t_i)) - \int_{t_i}^{t_i+\delta} \alpha(\|x(\tau)\|)d\tau + 2\delta B \quad (15)$$

holds for any $t_i \in \mathcal{T}$ *and* $\delta > 0$. □

Consider the sampling of the closed-loop state trajectory

$$\mathcal{S} := \{s_i = x(t_i) : t_i \in \mathcal{T}\} \quad (16)$$

at the time instants $t_i \in \mathcal{T}$ with $\mathcal{T}$ defined in (4). Then, by Lemma 2 and Lemma 4 we have

$$\begin{aligned} V(t_{i+1}, s_{i+1}) &\leq V(t_i, s_i) - \int_{t_i}^{t_{i+1}} \alpha(\|x(\tau)\|)d\tau + 2(t_{i+1}-t_i)B \\ &\leq V(t_i, s_i) - \underline{\delta}\alpha_{\bar{\delta}}(\|s_i\|) + 2\bar{\delta}B. \end{aligned} \quad (17)$$

Using standard arguments from ISS analysis of discrete-time systems [27], adapted to the shifted value function, equation (17) is used to show that the shifted value function evaluated at the sampling (16) satisfy an ISS bound with respect to B, as made explicit in the following lemma.

Lemma 5 (ISS bound on sampled value function)**.** *Consider the shifted value function* $V(\cdot)$ *from Definition 3, the bound* B *from* (9)*, and let Assumptions 1-5 hold. Then, for the sampling* $\mathcal{S}$ *of the closed-loop* (1) *with* (6)*, defined in* (16)*, there exists a class -* $\mathcal{KL}$ *functions* $\hat{\beta}_\delta(\cdot)$*, dependent on* $\underline{\delta}$ *and* $\bar{\delta}$ *and with* $\hat{\beta}_\delta(r, 0) = r$*, such that*

$$V(t_i, s_i) \leq \max\left\{\hat{\beta}_\delta(V(t_0, s_0), i), \hat{\gamma}(B)\right\} \quad (18)$$

holds with

$$\hat{\gamma}(r) := \hat{\alpha}^{-1}\left(\rho^{-1}(2\bar{\delta}r)\right), \quad (19)$$

for any integer $i \in \mathbb{N}_{\geq 0}$*, class-*$\mathcal{K}_\infty$ *function* $\rho(\cdot)$ *such that* $(Id-\rho)(\cdot)$ *belongs to class-*$\mathcal{K}_\infty$*, and class-*$\mathcal{K}_\infty$ *functions* $\hat{\alpha}(\cdot)$*, which always exists, such that* $\hat{\alpha}(r) \leq \underline{\delta}\alpha_{\bar{\delta}}(\alpha_c^{-1}(r))$*, for all* $r \geq 0$*, and* $(Id - \hat{\alpha})(\cdot)$ *belongs to class* $\mathcal{K}$. □

Since from sample to sample the value function satisfies (18) and among samples, from (15), the value function does not increase more than $2\bar{\delta}B$, the following lemma shows that an ISS bound with respect to B also applies to $V(\cdot)$ evaluated along the whole continuous-time trajectory.

Lemma 6 (ISS bound on value function)**.** *Consider the shifted value function* $V(\cdot)$ *from Definition 3, the bound* B *from* (9)*, and let Assumptions 1-5 hold. Then, there exists a class-*$\mathcal{KL}$ *functions* $\hat{\beta}(\cdot)$ *such that*

$$V(t, x(t)) \leq \max\left\{\hat{\beta}(V(t_0, x_0), t-t_0), \hat{\gamma}(B) + 2\bar{\delta}B\right\} \quad (20)$$

holds with $\hat{\gamma}(\cdot)$ *in* (19) *for all* $t \geq t_0$. □

At this point, consider Lemma 3 with $\delta_v = \Delta$. Combining the lower bound (14) with (20) and the fact that $\max\{a, b\} \leq a + b \leq \max\{2a, 2b\}$ for any $a, b \in \mathbb{R}_{\geq 0}$, leads to

$$\begin{aligned} &\Delta\alpha_\Delta(\|x(t)\|) \\ &\quad \leq \max\{\hat{\beta}(V(t_0, x_0), t-t_0), \hat{\gamma}(B) + 2\bar{\delta}B\} + 2TB \\ &\quad \leq \max\left\{2\hat{\beta}(V(t_0, x_0), t-t_0),\ 2\hat{\gamma}(B) + 4\bar{\delta}B,\ 4TB\right\} \end{aligned} \quad (21)$$

and therefore, by (12) and the monotonicity of $\alpha_\Delta^{-1}(\cdot)$, to the desired bound

$$\begin{aligned} \|x(t)\| &\leq \max\left\{\alpha_\Delta^{-1}\left(\frac{2}{\Delta}\hat{\beta}(\alpha_c(\|x(t_0)\|), t-t_0)\right)\right. \\ &\qquad \left., \ \alpha_\Delta^{-1}\left(\frac{2\hat{\gamma}(B) + 4\bar{\delta}B}{\Delta}\right),\ \alpha_\Delta^{-1}\left(\frac{4BT}{\Delta}\right)\right\} \\ &\leq \beta(\|x(t)\|, t-t_0) + \gamma(B) \end{aligned} \quad (22)$$

where $\beta(\cdot)$ and $\gamma(\cdot)$ are the class-$\mathcal{KL}$ and the class-$\mathcal{K}_\infty$ functions defined as

$$\begin{aligned} \beta(r, s) &:= \alpha_\Delta^{-1}\left(\frac{2}{\Delta}\hat{\beta}(\alpha_c(r), s)\right) \\ \gamma(r) &:= \alpha_\Delta^{-1}\left(\frac{2\hat{\gamma}(r) + 4\bar{\delta}r}{\Delta}\right) + \alpha_\Delta^{-1}\left(\frac{4rT}{\Delta}\right). \end{aligned}$$

This concludes the proof. ■

IV. Design Methods

This section addresses the design of the MPC proposed controller. Specifically, we provide guidelines for the design of the open-loop MPC optimization problem, from Definition 1, and for the estimation of the size of the asymptotic bound of the state vector.

A. Design of the open-loop MPC optimization problem

The design of the MPC controller requires the selection a suitable terminal set, terminal cost, and stage cost that satisfy the assumptions of Theorem 1.

1) Terminal set, terminal cost, and stabilizing stage cost: It is worth to notice that, by setting $l_e(\cdot) = 0$, Assumption 4 coincides with the well-known tracking MPC sufficient conditions for convergence of the closed-loop state trajectory to the origin, see, e.g., [5], [7]. Consequently, the design of the stage cost $l_s(\cdot)$ and the computation of the auxiliary elements $m(\cdot)$ and $\mathcal{X}_{aux}(\cdot)$ can be performed using any of the design techniques presented in the tracking MPC literature.

2) Economic stage cost: In many practical applications, the bound B in (9) can be obtained by physical considerations on the system under analysis. Although, for the general case where such assumption can be difficult to be a priori verified, a simple approach consists in enforcing the desired bound using a smooth saturation-like function. In particular, consider the function $\hat{l}_e : \mathbb{R}_{\geq t_0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ that we wish to minimize and a smooth saturation-like function $\mathrm{sat} : \mathbb{R} \to \mathcal{E}$, with bounded $\mathcal{E}$, e.g., the function

$$\mathrm{sat}(x) = k_1 \ \mathrm{atan}\left(\frac{1}{k_2}x\right),$$

where the constants $k_1 > 0$ and $k_2 > 0$ determine the maximum value and the smoothness of the function, respectively. Then, the saturated economic cost

$$l_e(t, x, u) = \mathrm{sat}(\hat{l}(t, x, u))$$

is trivially bounded by, e.g., $B = k_1\pi/2$.

On the same line, and recalling Remark 1, if we wish to affect only the transient behaviour of the closed-loop trajectory, the bounded economic cost can be multiplied by an integrable function $q : \mathbb{R} \to \mathbb{R}$ as follows

$$l_e(t, x, u) = q(t)\,\mathrm{sat}(\hat{l}(t, x, u))$$

with

$$\int_{t_0}^{+\infty} |q(\tau)| d\tau < +\infty.$$

This last method was employed in [25], with $q(t) = c\ e^{-\lambda(t-t_0)}$, for two positive constants $c > 0$ and $\lambda > 0$, to design highly observable closed-loop trajectories converging to the origin.

B. Estimation of the ultimate bound

Theorem 1 shows that the state vector x converges asymptotically to a set with size monotonically increasing with the value of B. Such bound can be obtained by considering (20) and taking the time t to infinity, which leads to the level set

$$\mathcal{L}(t; V, \hat{\gamma}(B) + 2\bar{\delta}B) \tag{23}$$

with $\hat{\gamma}(\cdot)$ in (19) and where, for the sake of simplicity, in this section we consider $\delta = \bar{\delta} = \underline{\delta}$. Although the set (23) is in general difficult to compute, this section shows that, under certain assumptions, it can be outer bounded by the set

$$\left\{x : \|x\| \leq \alpha^{-1}\left(\frac{\hat{\alpha}^{-1}(2\delta B) + 2\delta B + 2TB}{\delta}\right)\right\} \tag{24}$$

where $\hat{\alpha}(\cdot)$ is any class-$\mathcal{K}$ function, which always exists, such that $\hat{\alpha}(r) \leq \delta\alpha(m^{-1}(r))$, for all $r \geq 0$, and $(Id - \hat{\alpha})(\cdot)$ belongs to class $\mathcal{K}$. The function $m(\cdot)$ and $\alpha(\cdot)$ are from Assumption 4, where for the sake of simplicity, $m(\cdot)$ is assumed to be time-invariant. If this is not the case, it is always possible, by Assumption 5, to consider a time-invariant class-$\mathcal{K}_\infty$ upper bound of $m(\cdot)$.

The approximation (24) is obtained combining (23) with the functions $\alpha_{\bar{\delta}}(\cdot)$ and $\alpha_c(\cdot)$, used in (19), and the lower bound of the shifted value function $V(\cdot)$ computed next.

Computation of $\alpha_{\bar{\delta}}(\cdot)$. Notice that the function $\alpha_{\bar{\delta}}(\cdot)$ is a class-$\mathcal{K}_\infty$ function adopted in Lemma 4 to lower bound the integral of the stage cost as

$$\int_{t_i}^{t_{i+1}} l_s(\tau, x, u) d\tau \geq \int_{t_i}^{t_{i+1}} \alpha(\|x(\tau)\|) d\tau \geq \delta\alpha_{\bar{\delta}}(\|x(t)\|) \tag{25}$$

for $\boldsymbol{u} \in \mathcal{PC}(t_i, t_{i+1})$. In order to estimate the function $\alpha_{\bar{\delta}}(\cdot)$, consider the approximated performance index

$$J_T(t, z, \boldsymbol{u}) \sim \sum_{i=0}^{N} \delta l(t + \delta i, x(t + \delta i), u(t + \delta i)) + m(t + T, x(t + T)) \tag{26}$$

for $\bar{\delta} = \underline{\delta} = \delta = T/N$ with $N \in \mathbb{Z}_{\geq 1}$, or equivalently, consider the "sample-and-hold" stage cost defines as $l(\lfloor\tau\rfloor, x(\lfloor\tau\rfloor), u(\lfloor\tau\rfloor))$ with $\lfloor\tau\rfloor = \max_{i\in\mathbb{N}_{0:N}}\{\ t + \delta i\ :\ t + \delta\ i\ \leq\ \tau\}$. Note that, (26) holds with equality in many numerical implementations of MPC controllers obtained by discretizing the continuous-time performance index with a discretization step of δ. Then, the last inequality of (25) holds with equality by choosing $\alpha_{\bar{\delta}}(r) = \alpha(r)$, for all $r > 0$.

Computation of $\alpha_{\mathbf{c}}(\cdot)$. Within the terminal region, whenever present, or equivalently restricting $\mathcal{X}(t)$ to be $\mathcal{X}_{aux}(t)$, it is possible to choose $\alpha_c(r) = m(r)$, with $r \geq 0$. This is a direct consequence of the property (8) of the function $m(\cdot)$, which highlights that the terminal cost upper bounds the infinite horizon performance index (cost-to-go) over the terminal set.

Computation of lower bound on $\mathbf{V}(\cdot)$. The lower bound on the value function can be obtained by combining Lemma 3 with the choice $\alpha_\Delta(r) = \alpha(r)$, with $r \geq 0$, and $\Delta = \delta$, discussed above.

At this point, combining (23) with (19), where $\rho(r) = \theta r$ with $\theta \in (0, 1)$, approximating taking θ to 1, and selecting $\alpha_{\bar{\delta}}(r) = \alpha(r)$ and $\alpha_c(r) = m(r)$, for all $r \geq 0$, leads to (24).

Although this section provides a closed-form estimate of the asymptotic bound, it is worth noting that, in general for $B \neq 0$, the bound (24) can be conservative. This is a consequence of the fact that tight lower and upper bounds of the value function used in this section are generally difficult to compute. Moreover, the estimate of such ultimate bound is associated to the worth case scenario where the economic stage cost is always at its maximum (bounded) value, which is generally not the case. This is illustrated in the following section via numerical simulations.

V. Simulation results

The proposed MPC scheme finds applications in scenarios where one desires to i) minimize an economic objective that is conflicting with the main tracking objective and, ii) there is margin to reduce tracking accuracy to perform economic optimization. An example is the problem of energy-efficient trajectory tracking of a marine robotic vehicle in the presence of water currents. Here, similarly to [28], the tracking objective is to drive the vehicle to a pre-defined trajectory and the economic objective is to reduce the energy of the control input signal. This section analyzes the two cases of persistent penalization of the energy of the input signal, leading to an asymptotic tracking error, and its transient penalization, leading to convergence to the desired trajectory.

A. Model description

Let I be an inertial coordinate frame and B be a body coordinate frame attached to the vehicle. The pair $(p(t), R(t)) \in SE(2)$ denote the configuration of the vehicle, position and orientation, where $R(t)$ is the rotation matrix from body to inertial coordinates. Now, let $(v(t), \Omega(\omega(t))) \in se(2)$ be the twist that defines the velocity of the vehicle, linear and angular, where the matrix $\Omega(\omega(t))$ is the skew-symmetric matrix associated to the angular velocity $\omega(t)$, defined as

$$\Omega(\omega(t)) := \begin{pmatrix} 0 & -\omega(t) \\ \omega(t) & 0 \end{pmatrix}.$$

The marine vehicle is modelled as

$$\dot{p}(t) = R(t)\begin{pmatrix} v(t) \\ 0 \end{pmatrix} + c(p(t)), \quad \dot{R}(t) = R(t)\Omega(\omega(t)) \quad (27a)$$

where the vector field $c : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the contribution of the water currents to the velocity of the marine vehicle. The control input $u(t) = \begin{pmatrix} v(t) & \omega(t) \end{pmatrix}'$, consists of only the forward and the angular velocity. The vector field is randomly generated with $\|c(p)\| \le 10$ km/h for all $p \in \mathbb{R}^2$. A uniform time sampling $\mathcal{T} = \{0.04i, \ i \in \mathbb{N}_{\ge 0}\}$ is considered and the open-loop MPC optimization problems are solved using ACADO Toolbox [29] with an horizon length of 1.8 hours.

B. Computation of stabilizing stage cost, terminal set, and terminal cost

In the following examples, for any desired position trajectory $p_d : \mathbb{R}_{\ge t_0} \to \mathbb{R}^2$, parameterized over time, we use the design procedure in Appendix B, with $K_e = I_{2\times 2}$ and $\epsilon = (-0.2, 0)'$, to design the stage cost and the terminal cost of an MPC controller that steers the vehicle toward the desired position trajectory.

C. Example 1: ultimate bounded behaviour

Consider a desired circular trajectory

$$p_d(t) = 4\left(\sin\left(\frac{2\pi t}{T_{end}}\right) \quad -\cos\left(\frac{2\pi t}{T_{end}}\right)\right)' \quad (28)$$

with $T_{end} = 10$. The proposed strategy is used to optimize the energy consumption of the vehicle, while still guaranteeing boundedness of the closed-loop position around the desired trajectory. Toward this goal, following the guidelines of Section IV-A, the economic cost is designed as

$$l_e(t, x, u) = k_c \,\text{atan}\left(\frac{1.5}{k_c}\|u(t)\|^2\right),$$

and the simulation is executed four times with $k_c \in \{0, 10, 20, 30\}$, to show the effect of the economic cost on the closed-loop system.

It is worth noting that, in contrast to the standard case, where the objective is to stabilize the system around the

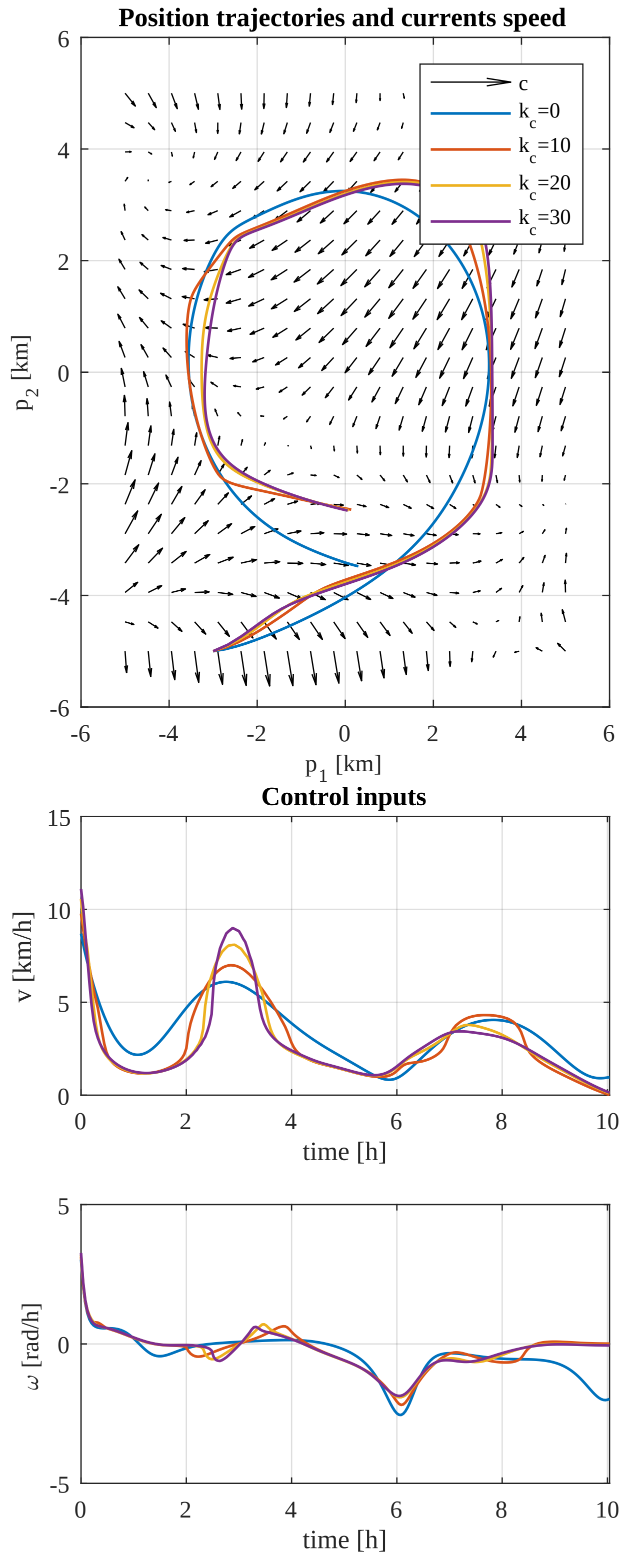

Fig. 1. Closed-loop position (top) and input (bottom) trajectories for the different values of $k_c \in \{0, 10, 20, 30\}$ associated with the Example 1.

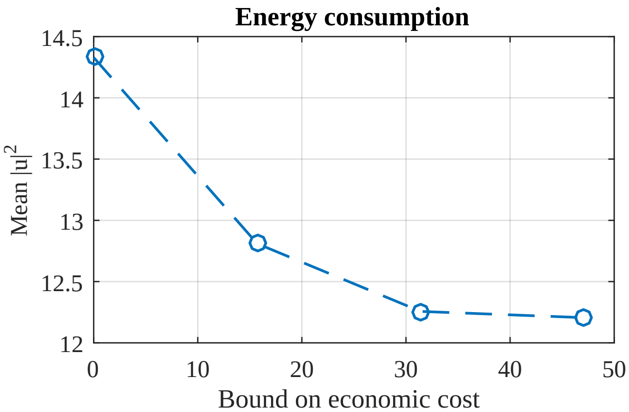

Fig. 2. Average energy consumption of different values of $k_c \in \{0, 10, 20, 30\}$ associated with the Example 1.

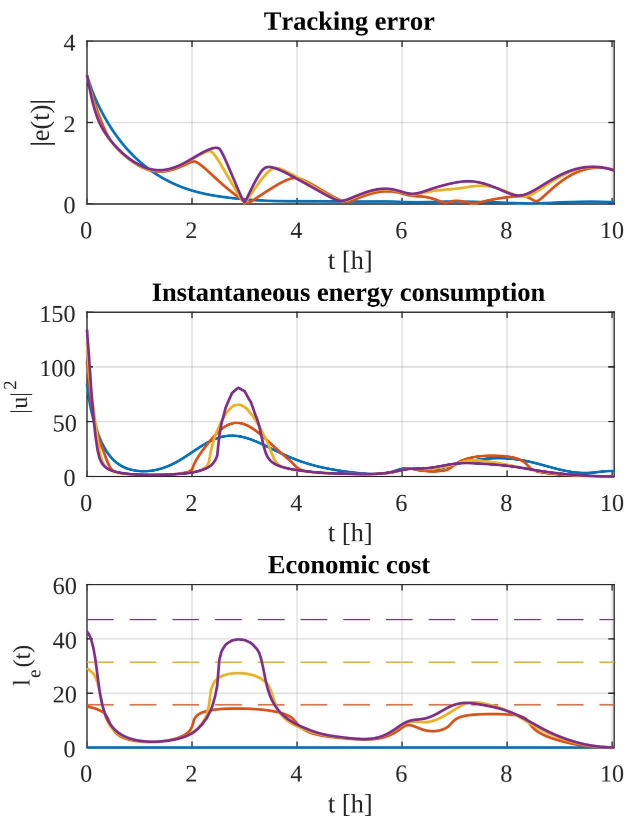

Fig. 3. Closed-loop trajectories of the tracking error (top), instantaneous energy consumption (middle), and economic stage cost (bottom) for different values of $k_c \in \{0, 10, 20, 30\}$ associated with the Example 1. The dashed-lines in the bottom figure show the maximum value of the economic stage cost allowed by the saturation function.

equilibrium $(x, u) = (0, 0)$, here the chosen economic stage cost is conflicting with the stabilizing stage cost (45). This is true because in general, at the desired behavior (tracking error equal to zero), the system still needs a non-zero input signal to keep the vehicle on the desired trajectory.

Fig. 1 shows the closed-loop state and input trajectories. The case $k_c = 0$ coincides with the nominal controller without economic optimization. We observe that by increasing the value of k_c, the closed-loop position trajectory, following the water currents, deviates from the predefined one reducing the strength of the control inputs.

Using the guidelines of Section IV-B, where Δ in (26) is chosen to be equal to $T/10$, it is possible to compute an a-priori estimate of the bound on the size of the asymptotic error. This leads to the values $\{0,\ 29.6588,\ 41.9439,\ 51.3706\}$ associated to $k_c \in \{0, 10, 20, 30\}$, respectively. As expected from the discussion in Section IV-B, Fig. 3 (top) shows the conservativeness of such bounds in the proposed example. This is also consequence of the fact that the economic stage cost, shown in Fig. 3 (bottom), is generally far below its maximum value considered in the computation.

The decrease of the instantaneous energy consumption and the associated bounded increase of tracking error is displayed in Fig. 2. The effectiveness of the proposed strategy is shown in Fig. 3, which explicits the effect of the parameter k_c on the average energy consumption described by the mean of $\|u(t)\|$ over the duration of the simulation.

D. Example 2: transient economic optimization

In this example, for the same desired trajectory of the previous case, the proposed controller is employed to reduce the consumption in a transient phase while still guaranteeing asymptotic convergence to the desired position.

Toward this goal, following the considerations in Section IV, the economic cost is designed as

$$l_e(t, x, u) \ = k_c \,\text{atan}\left(\frac{1.5}{k_c}\|u(t)\|^2\right) e^{-\frac{1}{2}t}.$$

Similarly to the previous example, the effect of the economic cost on the closed-loop system is shown executing the simulation four times with $k_c \in \{0, 10, 20, 30\}$. As expected, Fig. 4 shows that increasing the value of k_c, the closed-loop position trajectory, following the water currents, deviates from the predefined one reducing the strength of the control inputs.

Fig. 5, which explicits the effect of the parameter k_c on the average energy consumption described by the mean of $\|u(t)\|$ over the duration of the simulation.

In contrast to the case considered in Example 1, here we have an increase of the overall energy of the input signals, although due to the transient effect of the economic stage cost, convergence to the desired trajectory is guaranteed.

VI. Conclusions

This paper presents an MPC scheme that combines a stabilizing stage cost with an economic stage cost. As the main result, we prove that the state trajectory of the system in

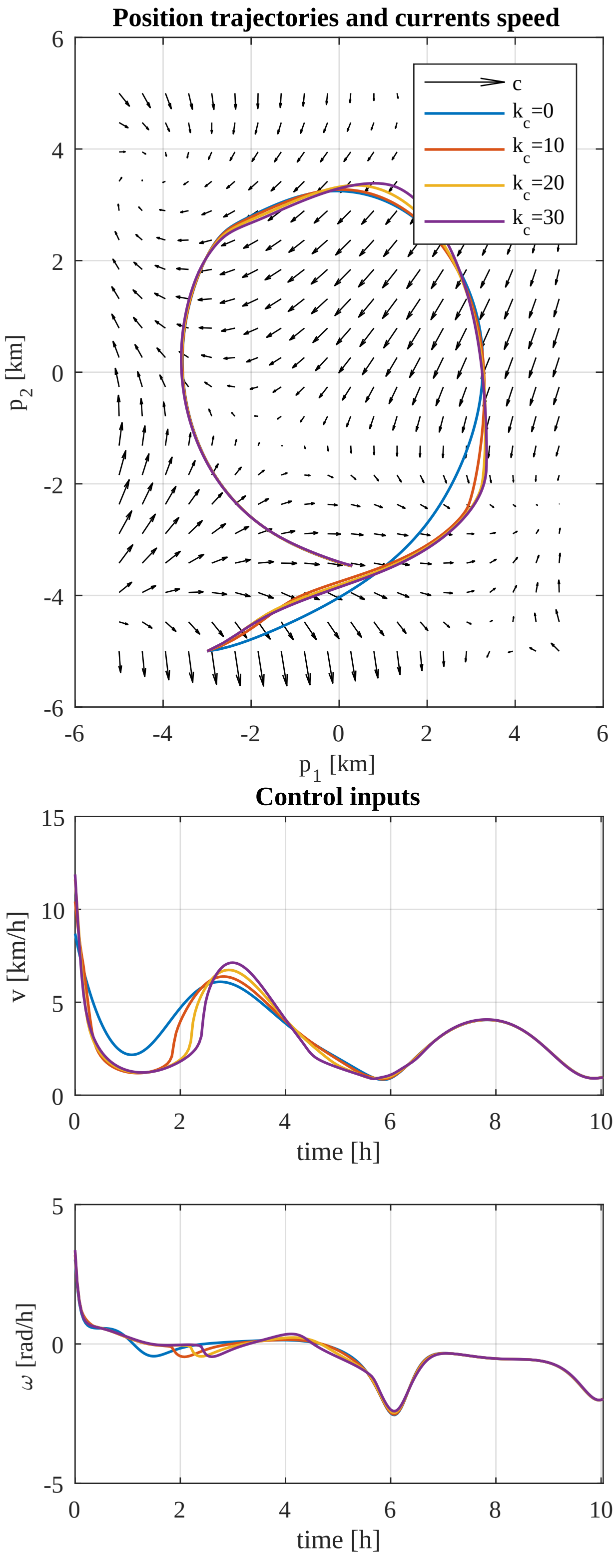

Fig. 4. Closed-loop position (top) and input (bottom) trajectories for the different values of $k_c \in \{0, 10, 20, 30\}$ associated with the Example 2.

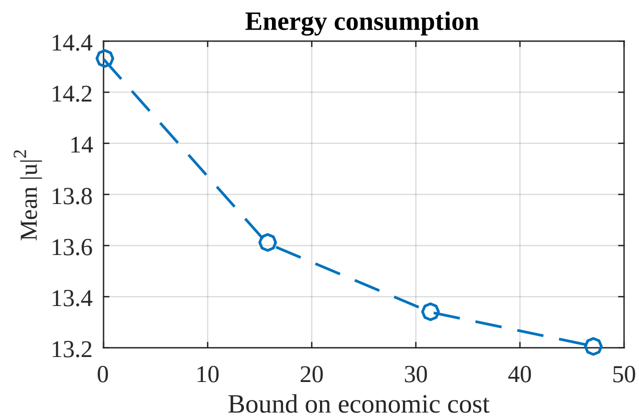

Fig. 5. Average energy consumption of different values of $k_c \in \{0, 10, 20, 30\}$ associated with the Example 2.

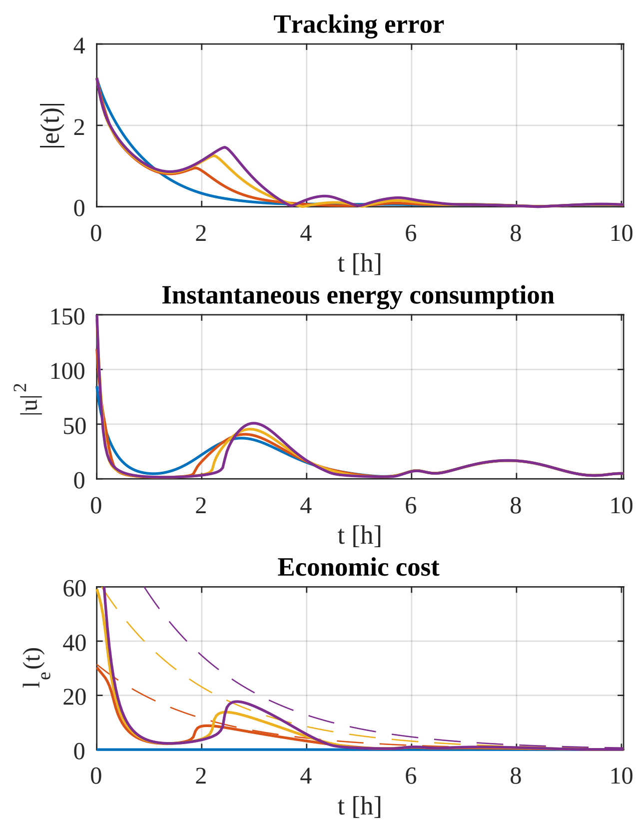

Fig. 6. Closed-loop trajectories of the tracking error (top), instantaneous energy consumption (middle), and economic stage cost (bottom) for different values of $k_c \in \{0, 10, 20, 30\}$ associated with the Example 2. The dashed-lines in the bottom figure show the maximum value of the economic stage cost allowed by the saturation function.

closed-loop with the MPC controller is ISS with respect to the economic stage cost. This guarantee can be exploited in many practical applications where the compromise between convergence to a trajectory and economic optimization represents the desired behavior. The effectiveness of the proposed scheme is demonstrated via a numerical example, where an energy efficient trajectory-tracking control problem of a marine vessel model moving through water currents is considered.

Acknowledgment

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Appendix A Proofs

A. Proof of Lemma 1

The upper bound follows immediately by plugging the feasible law $\mathbf{u}_f \in \mathcal{PC}(\hat{t}, \hat{t}+T)$ from Assumption 5 in the shifted value function from Definition 3 and using the bound on the economic cost (9). In fact, by the suboptimality of the feasible law, we have

$$\begin{aligned} V(\hat{t},\hat{x}) \leq & \int_{\hat{t}}^{\hat{t}+T} l_s(\tau,x,u_f)d\tau + m(\hat{t}+T, x(\hat{t}+T)) \\ & + \int_{\hat{t}}^{\hat{t}+T} l_e(\tau,x,u_f)d\tau - TB \\ \leq & \alpha_c(\|\hat{x}\|), \end{aligned}$$

which concludes the proof. ∎

B. Proof of Lemma 2

This proof is structured as follows: first, for a solution $\boldsymbol{x}$ of (1)-(2) starting at time $t \geq t_0$ from $x(t) \in \mathcal{X}(t)$, Assumption 1 is used to obtain a time-invariant piecewise continuous lower bound on the evolution of $\|x(\tau)\|$, for all $\tau \geq t \geq t_0$. Then, combining such bound with $\alpha(\cdot)$ and integrating from t to $t+\delta$ leads to the desired result.

We start by noticing that Assumption 1 can be used to derive the following lower bound

$$\|x(\tau)\| \geq \max(0, \|x(t)\| - (t-\tau)b(\hat{r})), \qquad \forall \tau \geq t \geq t_0 \tag{29}$$

for all $\|x(t)\| \leq \hat{r}$ and for an increasing function $b : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ positive away from the origin. The inequality (29) can be explicitly proved combining Lemma 7 with the fact that $b(\cdot)$ is an increasing function and that $\|x(t)\|$ is not negative.

At this point, consider the set of constant samples $r_i := \delta_r i$ with $i \in \mathbb{N}_{\geq 0}$ and $\delta_r > 0$. Then, evaluating (29) with $\hat{r} = r_{i+1}$ in the intervals $[r_i, r_{i+1}]$, results in a piecewise affine lower-bound $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\|x(\tau)\| \geq \beta(\|x(t)\|, \tau - t)$$

where

$$\begin{aligned} \beta(r,s) &:= \alpha_s(\max(0, \alpha_s^{-1}(r) - s)) \\ \alpha_s(s) &:= r_i + (s - s_i)b(r_{i+1}), \quad \forall s \in [s_i, s_{i+1}] \end{aligned}$$

with $s_{i+1} = s_i + \delta_r / b(r_i)$ and $s_0 = 0$. Fig. 7 provides an illustration of this bound. It is important to notice that:

1) Since $b(\cdot)$ is positive away from zero and increasing, the function $\alpha_s(\cdot)$ belongs to class-$\mathcal{K}_\infty$.
2) For all $r \in \mathbb{R}_{\geq 0}$ and $s \in \mathbb{R}_{\geq 0}$ we have $\beta(r, 0) = r$ and $\beta(r, s) > 0$ for $r \neq 0$.
3) For any two constants $\hat{r}_1$ and $\hat{r}_2$ with $\hat{r}_1 > \hat{r}_2$ we have $\beta(\hat{r}_1, s+\hat{\delta}) = \beta(\hat{r}_2, s)$ for a positive $\hat{\delta} > 0$, i.e., $\beta(\hat{r}_2, \cdot)$ is a copy of $\beta(\hat{r}_1, \cdot)$ shifted forward by $\hat{\delta}$. Specifically, this is true with $\hat{\delta} := \alpha_s^{-1}(\hat{r}_1) - \alpha_s^{-1}(\hat{r}_2)$.
4) For a given $\hat{s}$, the function $\beta(r, \hat{s})$ is non decreasing in $r \in \mathbb{R}_{\geq 0}$.
5) For a given $\hat{r}$ the function $\beta(\hat{r}, s)$ is non increasing in $s \in \mathbb{R}$.

Since $\alpha(\cdot)$ belongs to class-$\mathcal{K}_\infty$, the same conditions apply to the function $\hat{\beta}(r,s) := \alpha(\beta(r,s))$. At this point, for any $t \geq t_0$ and $\delta_v > 0$ we can write

$$\begin{aligned} \frac{\int_t^{t+\delta_v} \alpha(\|x(\tau)\|)d\tau}{\delta_v} &\geq \frac{\int_t^{t+\delta_v} \hat{\beta}(\|x(t)\|, \tau - t)d\tau}{\delta_v} \\ &= \frac{\int_0^{\delta_v} \hat{\beta}(\|x(t)\|, \tau)d\tau}{\delta_v} =: \alpha_\delta(\|x(t)\|). \end{aligned} \tag{30}$$

Note that:

- Properties 2 and 5 implies $\alpha_\delta(0) = 0$ and $\alpha_\delta(r) > 0$ for all $r \neq 0$.
- Properties 3 and 5 implies that $\alpha_\delta(\cdot)$ is non-decreasing. In fact, for any $\hat{r}_1$ and $\hat{r}_2$ with $\hat{r}_1 > \hat{r}_2$ the following holds

$$\begin{aligned} \alpha_\delta(\hat{r}_2) &= \frac{\int_0^{\delta_v} \hat{\beta}(\hat{r}_2, \tau)d\tau}{\delta_v} \\ &= \frac{\int_0^{\delta_v} \hat{\beta}(\hat{r}_1, \tau + \hat{\delta})d\tau}{\delta_v} \leq \frac{\int_0^{\delta_v} \hat{\beta}(\hat{r}_1, \tau)d\tau}{\delta_v} \\ &= \alpha_\delta(\hat{r}_1). \end{aligned}$$

- $\alpha_\delta(\cdot)$ is radially unbounded from $\alpha_s(\cdot)$ being radially unbounded.
- $\alpha_\delta(r)$ is continuous in both r, from $\hat{\beta}(\cdot)$ being continuous, and δ_v.

Now consider

$$\hat{\alpha}_v(r) := \min_{\delta_v \in [0,\Delta]} \alpha_\delta(r) \tag{31}$$

where the minimum exists from $\alpha_\delta(r)$ being continuous on δ_v with δ_v in a compact $[0, \Delta]$. Note that

- $\hat{\alpha}_v(\cdot)$ is zero at zero, positive anywhere-else, and radially unbounded since $\alpha_\delta(\cdot)$ is zero at zero, positive anywhere-else, and radially unbounded for all $\delta_v \in [0, \Delta]$.

- $\hat{\alpha}_v(\cdot)$ is non-decreasing since

$$\hat{\alpha}_v(\hat{r}_1) = \min_{\delta_v \in [0,\Delta]} \frac{\int_0^{\delta_v} \hat{\beta}(\hat{r}_1,\tau)d\tau}{\delta_v} \leq \min_{\delta_v \in [0,\Delta]} \frac{\int_0^{\delta_v} \hat{\beta}(\hat{r}_2,\tau)d\tau}{\delta_v} = \alpha_\Delta(\hat{r}_2) \quad (32)$$

  for all $\hat{r}_1 < \hat{r}_2$.
- $\hat{\alpha}_v(\cdot)$ is continuous from $\hat{\beta}(\cdot)$ being continuous.

Therefore, there always exists a function $\alpha_\Delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belonging to class $\mathcal{K}_\infty$ such that $\alpha_\Delta(r) \leq \hat{\alpha}_v(r)$ for all $r \geq 0$. Combining $\alpha_\Delta(\cdot)$ with (30) and (31) concludes the proof. ■

Lemma 7. *Consider the system* (1)-(2) *and let Assumption 1 hold. Then, the bound*

$$\|x(t+\delta)\| \geq \|x(t)\| - \delta b(\|x(t)\|)$$

holds for all $t \geq t_0$*,* $\delta > 0$*, and for an increasing function* $b : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *positive away from the origin.* □

Proof. By Assumption 1 we can write

$$\|f(t,x,u)\| \leq b(r), \qquad \forall t \geq t_0,\ x \in \mathcal{B}(r),\ u \in \mathcal{U}(t) \quad (33)$$

where, without loss of generality, the function $b : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is considered to be monotonically increasing and positive away from the origin. In fact, even if $b(\cdot)$ is not monotonically increasing (but only non-decreasing from the inclusion $\mathcal{B}(\hat{r}_1) \subset \mathcal{B}(\hat{r}_2)$ for all $0 \leq \hat{r}_1 < \hat{r}_2$) one could choose $\hat{b}(r) := b(r) + \epsilon r$, with $\epsilon > 0$ is monotonically increasing, greater than 0 away from the origin, and satisfies (33) with $b(r) = \hat{b}(r)$.

At this point, consider a solution $\boldsymbol{x}$ starting at time $t \geq t_0$ from the states $x(t)$. We identify two cases: either the solution $\boldsymbol{x}$ evolves outside $\mathcal{B}(\|x(t)\|)$, i.e., $\|x(\tau)\| > \|x(t)\|$ for all $\tau \in (t, t+\delta]$, or it spends some compact and connected interval of times I_i, with $i \in \mathbb{N}$, inside $\mathcal{B}(\|x(t)\|)$. Let the generic ith interval be defined as $I_i := [t_i, t_i+\delta_i]$ for a $t_i \geq t_0$ and a $\delta_i > 0$ where, by continuity of the solution $\boldsymbol{x}$, $\|x(t_i)\| = \|x(t)\|$. Then, for all $\tau \in I_i$ the solution satisfies

$$\begin{aligned} x(\tau) &= x(t_i) + \int_{t_i}^{\tau} f(s,x,u)ds, \\ &\Longrightarrow \\ \|x(\tau)\| &\geq \|x(t)\| - \int_{t_i}^{\tau} \|f(s,x,u)\|ds \\ &\geq \|x(t)\| - (\tau - t_i)b(\|x(t)\|) \\ &\geq \|x(t)\| - \delta_i b(\|x(t)\|) \\ &\geq \|x(t)\| - \delta b(\|x(t)\|), \end{aligned} \quad (34)$$

where we used (33) and the facts that $\|x(\tau)\| \leq \|x(t)\|$ and that, for any two vectors $v_1, v_2 \in \mathbb{R}^n$, the inequity $\|v_1 + v_2\| \geq \|v_1\| - \|v_2\|$ holds. Therefore, for all the time when $x(\tau) \in \mathcal{B}(\|x(t)\|)$ the bound (34) holds. This concludes the proof. ■

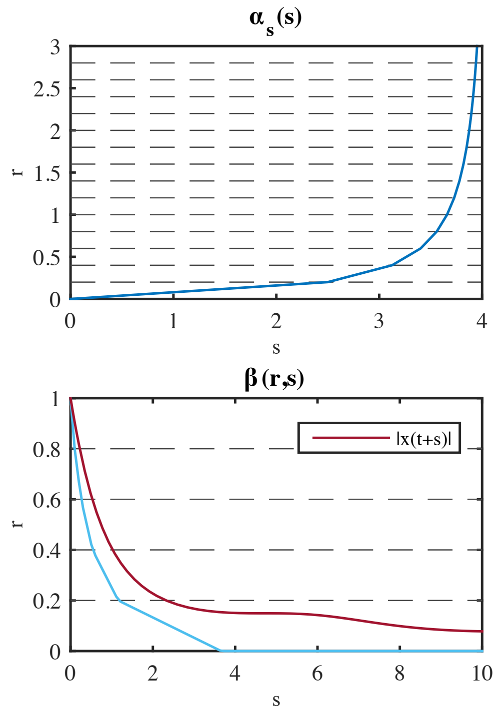

Fig. 7. Lower bound on $\|x(\tau)\|$ for the system $f(t,x,u) = (1+\sin(t))x^2u$ with $u = -\operatorname{sign}(x) \in [-1,1]$, $\alpha(r) = r$, $b(r) = 2r^2$, $\delta_r = 0.1$ and $x(t) = 0.25$.

C. Proof of Lemma 3

The desired lower bound follows combining the Definition 3 of the shifted value function, the lower bound on the stabilizing stage cost from Assumption 4-(ii), the positive semi-definiteness of $m(\cdot)$ from Assumption 4-(iii), the upper bound on the economic stage cost B from (9), and Lemma 2:

$$\begin{aligned} V(t,\hat{x}) &= \int_t^{t+T} l(\tau,\bar{x}^*,\bar{u}^*)d\tau \\ &\quad + m(t+T,\bar{x}^*(t+T)) - TB \\ &\geq \int_t^{t+T} l_s(\tau,\bar{x}^*,\bar{u}^*)d\tau + \int_t^{t+T} l_e(\tau,\bar{x}^*,\bar{u}^*)d\tau - TB \\ &\geq \int_t^{t+\delta} \alpha(\|\bar{x}^*\|)d\tau - \int_t^{t+T} \|l_e(\tau,\bar{x}^*,\bar{u}^*)\|d\tau - TB \\ &\geq \int_t^{t+\delta} \alpha(\|\bar{x}^*\|)d\tau - 2TB \geq \delta\alpha_\Delta(\|x(t)\|) - 2TB \end{aligned}$$

for all δ with $0 \leq \delta \leq \Delta \leq T$. This concludes the proof. ■

D. Proof of Lemma 4

The proof is structured as follows: first, we show that the (15) applies to any extended trajectory, and then also to any of the closed-loop trajectories, being a concatenation of extended trajectories from Definition 2.

Consider the extended state and input trajectories $\boldsymbol{x}_{e_i}$ and $\boldsymbol{u}_{e_i}$, respectively, and let at first $\delta \leq T$, then

$$\begin{aligned} V(t_i+\delta, x_{e_i}(t_i+\delta)) \leq \int_{t_i+\delta}^{t_i+T+\delta} l(\tau, x_{e_i}, u_{e_i}) d\tau - TB \\ + m(t_i+T+\delta, x_{e_i}) \\ = V(t_i, x_{e_i}(t_i)) \\ - \int_{t_i}^{t_i+\delta} l_e(\tau, x_{e_i}, u_{e_i}) d\tau + \int_{t_i+T}^{t_i+T+\delta} l_e(\tau, x_{e_i}, u_{e_i}) d\tau \\ - \int_{t_i}^{t_i+\delta} l_s(\tau, x_{e_i}, u_{e_i}) d\tau + \int_{t_i+T}^{t_i+T+\delta} l_s(\tau, x_{e_i}, u_{e_i}) d\tau \\ - m(t_i+T, x_{e_i}) + m(t_i+T+\delta, x_{e_i}) \end{aligned} \tag{35}$$

where the first inequality arises from the fact that the extended trajectory is not optimal. Combining (35) with (8) and the bound on $l_e(\cdot)$, results in

$$\begin{aligned} V(t_i+\delta, x_{e_i}(t_i+\delta)) - V(t_i, x_{e_i}(t_i)) \\ \leq - \int_{t_i}^{t_i+\delta} l_s(\tau, x_{e_i}, u_{e_i}) d\tau + 2\delta B \\ \leq - \int_{t_i}^{t_i+\delta} \alpha(\|x_{e_i}\|) d\tau + 2\delta B \end{aligned} \tag{36}$$

where the last inequality follows from Assumption 4-(ii). Using similar computations, it is easy to show that (36) also applies to the case where $\delta > T$.

To see that (36) still holds for the closed-loop (1) with (6), it is enough to notice that the associated state trajectory $\boldsymbol{x}$ is a concatenation of pieces of extended trajectories computed at the different time instants $t_i \in \mathcal{T}$, and therefore, considering $t_j = \lfloor t_i + \delta \rfloor$, results in

$$\begin{aligned} V(t_i+\delta, x(t_i+\delta)) - V(t_i, x(t_i)) \\ \leq - \sum_{k=i}^{j-1} \int_{t_k}^{t_{k+1}} \alpha(\|x_{e_k}(\tau)\|) d\tau \\ - \int_{t_j}^{t_i+\delta} \alpha(\|x_{e_j}(\tau)\|) d\tau + 2\delta B \\ = - \int_{t_i}^{t_i+\delta} \alpha(\|x(\tau)\|) d\tau + 2\delta B \end{aligned}$$

which concludes the proof. ■

E. Proof of Lemma 5

Consider the sampling $\mathcal{S}$ from (16). Combining (12) with (17) results in

$$\begin{aligned} V(t_{i+1}, s_{i+1}) \leq V(t_i, s_i) \\ - \underline{\delta} \alpha_{\bar{\delta}}(\alpha_c^{-1}(V(t_i, s_i))) + 2\bar{\delta} B. \end{aligned} \tag{37}$$

Let $\hat{\alpha}(\cdot)$ be a class-$\mathcal{K}_\infty$ function such that $\hat{\alpha}(r) \leq \underline{\delta} \alpha_{\bar{\delta}}(\alpha_c^{-1}(r))$, for all $r \geq 0$, and $(Id - \hat{\alpha})(\cdot)$ belongs to class $\mathcal{K}$, which always exists by Lemma B.1 of [27]. Combining the latter bound with (37) results in

$$V(t_{i+1}, s_{i+1}) - V(t_i, s_i) \leq -\hat{\alpha}(V(t_i, s_i)) + 2\bar{\delta} B. \tag{38}$$

At this point, following the same steps in [27], specifically form (13) to (17) of the latter work, there exists a class-$\mathcal{KL}$ functions $\hat{\beta}_\delta(\cdot)$, possibly dependent on $\bar{\delta}$ and $\underline{\delta}$, such that

$$V(t_i, s_i) \leq \max\{\hat{\beta}_\delta(V(t_0, s_0), i), \hat{\gamma}(B)\}$$

with $\hat{\gamma}(\cdot)$ defined in (19) for any class-$\mathcal{K}_\infty$ function $\rho(\cdot)$ such that $(Id - \rho)(\cdot)$ belongs to class-$\mathcal{K}_\infty$. ■

F. Proof of Lemma 6

Consider a sampling $\mathcal{S}$ from (16), then:

- From sample to sample, by Lemma 5, the bound
$$V(t_i, s_i) \leq \max\{\hat{\beta}_\delta(V(t_0, x_0), i), \hat{\gamma}(B)\} \tag{39}$$
holds for all $i \in \mathbb{N}_{\geq 0}$;
- Within the generic time interval $[t_i, t_{i+1})$, from (15) the value function never increases more than $2\bar{\delta}B$.

As a result, equation (20) can be satisfied by a continuous upper bound $\hat{\beta}(\cdot)$ of the discontinuous "sample-and-hold" function $\hat{\beta}_1(\cdot)$ defined by $\hat{\beta}_1(r,s) = \hat{\beta}_\delta(r, \lfloor s \rfloor_{\mathcal{T}}) + 2\bar{\delta}B$ with $\lfloor s \rfloor_{\mathcal{T}} := \max\{i : t_i \leq s, t_i \in \mathcal{T}\}$, e. g., $\hat{\beta}(r,s) := \hat{\beta}_1(r,0) + 2\bar{\delta}B$ for all $(r,s) \in \mathbb{R}_{\geq 0} \times [t_0, t_1]$ and $\hat{\beta}(r,s) := (1-\theta(s))\hat{\beta}_1(r, i-1) + \theta(s)\hat{\beta}_1(r,i)$ with $\theta(s) := (s - t_i)/(t_{i+1} - t_i)$ for all $(r,s) \in \mathbb{R}_{\geq 0} \times [t_i, t_{i+1}]$ with $i \geq 1$. This concludes the proof. ■

Appendix B
Design of a trajectory-tracking MPC controller

Consider the vehicle model (27) and the tracking error defined as

$$e(t) := R'(t)(p(t) - p_d(t)) - \epsilon, \tag{40}$$

where $\epsilon := (\epsilon_1, \epsilon_2)' \in \mathbb{R}^2$ is a given constant vector, arbitrary small in norm, with $\epsilon_1 \neq 0$. Computing the first time derivative of e results in the following error dynamical model

$$\begin{aligned} \dot{e} &= -SR'(p - p_d) + R'\left(R\begin{pmatrix} v \\ 0 \end{pmatrix} + c - \dot{p}_d\right) \\ &= -S(R'(p-p_d) - \epsilon) - S\epsilon + \begin{pmatrix} v \\ 0 \end{pmatrix} + R'(c - \dot{p}_d) \\ &= -Se + \Delta u + R'(c - \dot{p}_d), \ \Delta := \begin{pmatrix} 1 & \epsilon_2 \\ 0 & -\epsilon_1 \end{pmatrix}. \end{aligned} \tag{41}$$

This section, as in [21] but for the model with water currents, addresses the design of an MPC control law that drives the vector $e(t)$ to the origin as $t \to +\infty$, and thus drives $\|p - p_d\| \to \|\epsilon\|$. Specifically, we design an auxiliary control law $k_{aux}(\cdot)$, a stabilizing stage cost, a terminal set, and a terminal cost that satisfy Assumption 4, where the vector x is replaces with e and $(e,u) = (0, k_{aux}(t,0))$ is considered to be the desired steady-state.

Considering the error dynamical model (41) and noting that, from ϵ_1 being different from zero, the matrix Δ is full rank, the control input

$$u(t) = k_{aux}(t,e) = \Delta^{-1}(-K_e e - R'(c(t) - \dot{p}_d(t))) \tag{42}$$

makes the origin $e = 0$ an exponentially stable equilibrium point. This is certified by the Lyapunov function

$$V_{aux} = 0.5e'e \tag{43}$$

with the associated closed-loop decrease

$$\begin{aligned}\dot{V}_{aux} &= e'(Se + \Delta u + R'(c(t) - \dot{p}_d(t))) = e'(S - K_e)e \\ &= -e'K_e e \le -\|e\|^2\lambda_{min}(K_e) = -2\lambda_{min}(K_e)V_{aux},\end{aligned} \tag{44}$$

where we used the fact that S is a skew-symmetric matrix and therefore $v'Sv = 0$ for any $v \in \mathbb{R}^2$. Using the proposed auxiliary control law, Assumption 4-(ii) is trivially satisfied by the stage cost

$$l_s(t, x, u) = \|e\|_Q^2 + \|u - k_{aux}(t, 0)\|_O^2 \ge \|e\|_Q^2 \tag{45}$$

for any positive define matrix $Q \succ 0$ and positive semi definite matrix $O \succeq 0$. Moreover, since the proposed auxiliary control law globally exponentially stabilizes the origin of the error space, the terminal set can be omitted and the terminal cost is computed using Proposition 27 of [12].

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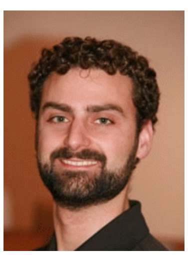

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