Performance Limitations in Reference-Tracking and Path-Following for Nonlinear Systems ⋆

A. Pedro Aguiar [a], João P. Hespanha [b], Petar V. Kokotović [b]

[a] *ISR/IST - Institute for Systems and Robotics, Instituto Superior Técnico Torre Norte 8, Av. Rovisco Pais, 1049-001 Lisbon, Portugal*

[b] *Center for Control Engineering and Computation University of California, Santa Barbara, CA 93106-9560*

Abstract

We investigate limits of performance in reference-tracking and path-following and highlight an essential difference between them. For a class of nonlinear systems, we show that in reference-tracking, the smallest achievable $\mathcal{L}_2$ norm of the tracking error is equal to the least amount of control energy needed to stabilize the zero-dynamics of the error system. We then show that this fundamental performance limitation does not exist when the control objective is to force the output to follow a geometric *path* without a timing law assigned to it. This is true even when an additional desired speed assignment is required to be satisfied asymptotically or in finite time.

Key words: Limits of performance, non-minimum phase nonlinear systems, path-following, reference-tracking, cheap-control.

1 Introduction

Fundamental performance limitations in *reference-tracking* for linear feedback systems have been quantified with classical Bode integrals and with cheap optimal control (Kwakernaak and Sivan, 1972; Middleton, 1991; Qiu and Davison, 1993; Seron *et al.*, 1999; Chen *et al.*, 2000). In the absence of unstable zero dynamics (*non-minimum phase or right-half plane (rhp) zeros*) and if the system is right invertible, perfect tracking of any reference signal is possible, that is, the $\mathcal{L}_2$ norm of the tracking error can be made arbitrarily small. However, in the presence of unstable zero dynamics, the tracking error increases as the signal frequencies approach those of the unstable zeros (Qiu and Davison, 1993), see also (Su *et al.*, 2003).

For step reference signals, Seron *et al.* (1999) generalized Qiu-Davison results for nonlinear systems of relative degree one. They showed that the best attainable $\mathcal{L}_2$ norm of the tracking error $e_T(t)$, denoted by J_T, is equal to the lowest $\mathcal{L}_2$ norm of control effort needed to stabilize the zero dynamics driven by $e_T(t)$. Extensions to non-right-invertible systems are given in (Woodyatt *et al.*, 2002; Braslavsky *et al.*, 2002).

For the problem of tracking any reference signal generated by a known exosystem, we show that the smallest achievable $\mathcal{L}_2$ norm of the tracking error is equal to the least amount of control energy needed to stabilize the zero-dynamics of the error system.

Path-following problems are concerned with the design of control laws that drive an object (robot arm, mobile robot, ship, aircraft, etc.) to reach and follow a geometric *path*. A secondary goal is to satisfy some additional dynamic specification such as to follow the path with some desired velocity. A common approach to the path-following problem is to parameterize the geometric path y_d by a *path variable* θ and then select a *timing law* for θ, (Hauser and Hindman, 1995; Al-Hiddabi and McClamroch, 2002; Skjetne *et al.*, 2004; Aguiar *et al.*, 2005; Aguiar *et al.*, 2004; Aguiar and Hespanha, 2004). Path-following as a method to avoid some limitations in reference-tracking was described in Aguiar *et al.* (2004). The key idea is to use θ as an

⋆ Supported by National Science Foundation under Grants No. ECS-9812346, No. ECS-0242798, and FCT-ISR/IST plurianual funding through the POS_C Program that includes FEDER funds. Corresponding author A. Pedro Aguiar. Tel. +1 805-893-7785. Fax +1 805-893-3262. Part of this research was done while the first author was with the Center for Control Engineering and Computation, University of California, Santa Barbara, CA, USA.

Email addresses: aguiar@ece.ucsb.edu (A. Pedro Aguiar), hespanha@ece.ucsb.edu (João P. Hespanha), petar@ece.ucsb.edu (Petar V. Kokotović).

additional control input to stabilize the unstable zero-dynamics while the original control variables keep the system on the path.

For a class of nonlinear systems we show that the fundamental performance limitations imposed on reference-tracking by unstable zero dynamics *do not apply* to the path-following problem. Furthermore, this is true even when an additional desired speed assignment is required to be satisfied asymptotically or in finite time.

In Section 2 we formulate the reference-tracking and path-following problems and briefly review our recent results for non-minimum phase linear systems, (Aguiar *et al.*, 2005). Section 3 presents the main results of the paper. An example in Section 4 illustrates the results. Concluding remarks are given in Section 5.

2 Reference-tracking versus path-following

2.1 Reference-tracking

For linear systems

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{1}$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^q$, and reference signals $r(t) \in \mathbb{R}^q$ generated by a known *exosystem*

$$\dot{w} = Sw, \quad r = Qw, \tag{2}$$

Davison (1976) and Francis (1977) show that it is possible to design a feedback controller such that the closed-loop system is asymptotically stable and the output $y(t)$ converges to $r(t)$, if and only if (A, B) is stabilizable, (C, A) is detectable, the number of inputs is at least as large as the number of outputs ($m \geq q$), and the zeros of (A, B, C, D) do not coincide with the eigenvalues of S. The *internal model approach*, (Francis and Wonham, 1976; Francis, 1977), designs the reference-tracking controller

$$u(t) = Kx(t) + (\Gamma - K\Pi)w(t),$$

where $A + BK$ is Hurwitz, and Π and Γ satisfy

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma, \\ 0 &= C\Pi + D\Gamma - Q. \end{aligned}$$

Then, the *tracking error* $e_T(t) := y(t) - r(t)$ converges to zero, and the transients

$$\tilde{x} := x - \Pi w, \quad \tilde{u} := u - \Gamma w \tag{3}$$

are governed by $\dot{\tilde{x}} = (A + BK)\tilde{x}$, $\tilde{u} = K\tilde{x}$.

Isidori and Byrnes (1990) show that the analogous nonlinear problem

$$\dot{x} = f(x, u), \qquad y = h(x, u), \tag{4}$$

$$\dot{w} = s(w), \qquad r = q(w), \tag{5}$$

where $f(0, 0) = 0$, $s(0) = 0$, $h(0, 0) = 0$, is solvable if and only if there exist smooth maps $\Pi(w)$ and $c(w)$, satisfying

$$\begin{aligned} &\frac{\partial \Pi}{\partial w} s(w) = f(\Pi(w), c(w)), \quad \Pi(0) = 0, \\ &h(\Pi(w), c(w)) - q(w) = 0, \quad c(0) = 0. \end{aligned} \tag{6}$$

Krener (1992) presents necessary and sufficient conditions for local solvability of (6).

The non-minimum phase (rhp) zeros impose a fundamental limitation on the attainable tracking performance. Kwakernaak and Sivan (1972) were the first to demonstrate that the cheap control problem

$$J_\epsilon := \min_{\tilde{u}} \int_0^\infty \left[\|e_T(t)\|^2 + \epsilon^2 \|\tilde{u}(t)\|^2 \right] dt \tag{7}$$

with $\tilde{u}$ defined in (3), in the presence of rhp zeros the limit $J_\epsilon \to J_T$ as $\epsilon \to 0$ is strictly positive.

Qiu and Davison (1993) showed that for $r(t) = \eta_1 \sin \omega t + \eta_2 \cos \omega t$, $\eta = \mathrm{col}(\eta_1, \eta_2)$, the rhp zeros $z_1, z_2, \ldots, z_p$ determine the value of J_T as follows:

$$J_T = \eta' M \eta, \quad \operatorname{trace} M = \sum_{i=1}^{p} \left(\frac{1}{z_i - j\omega} + \frac{1}{z_i + j\omega} \right).$$

For more general reference signals, Su *et al.* (2003) give explicit formulas which show the dependence of J_T on the rhp zeros and their frequency-dependent directional information.

2.2 Path-following

In path-following, the output $y(t)$ is required to reach and follow a geometric path $y_d(\theta)$ generated by the exosystem

$$\begin{aligned} &\frac{d}{d\theta} w(\theta) = s(w(\theta)), \quad w(\theta_0) = w_0 \\ &y_d(\theta) = q(w(\theta)), \end{aligned} \tag{8}$$

where $\theta \in \mathbb{R}$ is the path variable, $w \in \mathbb{R}^p$, $y_d \in \mathbb{R}^q$, and $\theta_0 := \theta(0)$. For a given *timing law* $\theta(t)$, the *path-following error* is defined as

$$e_P(t) := y(t) - y_d(\theta(t)). \tag{9}$$

We consider the following two path-following problems:

Geometric path-following: For a desired path $y_d(\theta)$, design a controller that achieves:

i) *boundedness:* the state $x(t)$ is uniformly bounded for all $t \geq 0$ and for every $(x(0), w(\theta(0))) = (x_0, w_0)$, in some neighborhood of $(0, 0)$,

ii) *error convergence:* the path-following error $e_P(t)$ converges to zero as $t \to \infty$, and

iii) *forward motion:* $\dot{\theta}(t) > c$ for all $t \geq 0$, where c is a positive constant.

Speed-assigned path-following: In addition to the geometric path-following task, a constant speed $v_d > 0$ is assigned and it is required that either $\dot{\theta}(t) \to v_d$ as $t \to \infty$, or $\dot{\theta}(t) = v_d$ for all $t \geq T$ and some $T \geq 0$.

Our main interest is to determine whether the freedom to select a timing law $\theta(t)$ can be used to achieve an arbitrarily small path-following error, that is, whether $\delta^\star > 0$ in

$$\int_0^\infty \|e_P(t)\|^2 \, dt \leq \delta^\star \tag{10}$$

can be made arbitrarily small.

In contrast to reference-tracking, the attainable performance for path-following is not limited by non-minimum phase zeros (Aguiar *et al.*, 2005). For the desired path generated by exosystem (8)

$$y_d(\theta) := \sum_{k=1}^{n_d} \left[a_k e^{j\omega_k \theta} + a_k^\star e^{-j\omega_k \theta} \right], \tag{11}$$

where the $\omega_k > 0$ are real numbers and the a_k are non-zero complex vectors, Aguiar *et al.* (2005) prove:

Theorem 1 (Aguiar *et al.* (2005)) *Consider the geometric path-following problem for* (1) *where* (A, B) *is stabilizable and* $x(0) = 0$. *Then for any given positive constant* $\delta^\star$ *there exist constant matrices* K *and* L, *and a timing law* $\theta(t)$ *such that the following feedback law achieves* (10)*:*

$$u(t) = Kx(t) + Lw(\theta(t)). \tag{12}$$

Theorem 2 (Aguiar *et al.* (2005)) *For the speed-assigned path-following problem, let* v_d *be specified so that the eigenvalues of* $v_d S$ *do not coincide with the zeros of* (1)*, and assume that* (A, B) *is stabilizable and* $x(0) = 0$. *Then,* (10) *can be satisfied for any* $\delta^\star > 0$ *with a timing law* $\theta(t)$ *and a controller of the form* (12) *but with time-varying piecewise-constant matrices* K *and* L.

Hence the stabilizability of (A, B) is both necessary and sufficient for the solvability of the geometric path-following problem. Furthermore, an arbitrarily small $\mathcal{L}_2$ norm of the path-following error is attainable even when the speed assignment v_d is specified beforehand.

3 Performance limitations for nonlinear systems

In the first part of this section, we present an internal model analog of the results in (Seron *et al.*, 1999; Braslavsky *et al.*, 2002) for the reference-tracking problem. In the second part, we present our main result for the path-following problem. We show that, in contrast to reference-tracking, the path-following problems can be solved with arbitrarily small $\mathcal{L}_2$ norm of the path-following error.

3.1 Reference-tracking

We consider the class of nonlinear systems which are locally diffeomorphic to systems in strict-feedback form (see for example Krstić *et al.* (1995, Appendix G)) [1] :

$$\begin{aligned} \dot{z} &= f_0(z) + g_0(z)\xi_1, && (13a) \\ \dot{\xi}_1 &= f_1(z, \xi_1) + g_1(z, \xi_1)\xi_2, \\ &\vdots \\ \dot{\xi}_{r_d} &= f_{r_d}(z, \xi_1, \ldots, \xi_{r_d}) + g_{r_d}(z, \xi_1, \ldots, \xi_{r_d})u, && (13b) \\ y &= \xi_1, && (13c) \end{aligned}$$

where $z \in \mathbb{R}^{n_z}$, $\xi := \mathrm{col}(\xi_1, \ldots, \xi_{r_d})$, $\xi_i \in \mathbb{R}^m$, $\forall i \in \{1, \ldots, r_d\}$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^m$. $f_i(\cdot)$ and $g_i(\cdot)$ are $\mathcal{C}^k$ functions of their arguments (for some large k), $f_i(0, \ldots, 0) = 0$, and the matrices $g_i(\cdot)$, $\forall i \in \{1, \ldots, r_d\}$ are always nonsingular. We assume that initially the system is at rest, $(z, \xi) = (0, 0)$.

When the reference-tracking problem is solvable, i.e., it is possible to design a continuous feedback law that drives the tracking error to zero, then there exist maps $\Pi = \mathrm{col}(\Pi_0, \ldots, \Pi_{r_d})$, $\Pi_0 : \mathbb{R}^p \to \mathbb{R}^{n_z}$, $\Pi_i : \mathbb{R}^p \to \mathbb{R}^m$, $\forall i \in \{1, \ldots, r_d\}$, and $c : \mathbb{R}^p \to \mathbb{R}^m$ that satisfy (6). The following locally diffeomorphic change of coordinates

$$\tilde{z} = z - \Pi_0(w), \tag{14}$$

$$\tilde{\xi} := \mathrm{col}(\tilde{\xi}_1, \ldots, \tilde{\xi}_{r_d}), \tag{15}$$

$$\tilde{\xi}_i = \xi_i - \Pi_i(w), \quad i = 1, \ldots, r_d \tag{16}$$

$$\tilde{u} = u - c(w), \tag{17}$$

transforms the system (13) into the *error system*

$$\dot{\tilde{z}} = \tilde{f}_0(\tilde{z}, w) + \tilde{g}_0(\tilde{z}, w)e_T, \tag{18a}$$

[1] When convenient we use the compact form (4) for (13). In that case, $f(\cdot)$ denotes the vector field described by the right-hand-side of (13a)–(13b), $h(\cdot)$ the output map described by (13c), and $x = \mathrm{col}(z, \xi_1, \ldots, \xi_{r_d})$.

$$\dot{\tilde{\xi}}_1 = \tilde{f}_1(\tilde{z}, \tilde{\xi}_1, w) + \tilde{g}_1(\tilde{z}, \tilde{\xi}_1, w)\tilde{\xi}_2,$$

$$\vdots \tag{18b}$$

$$\dot{\tilde{\xi}}_{r_d} = \tilde{f}_{r_d}(\tilde{z}, \tilde{\xi}_1, \ldots, \tilde{\xi}_{r_d}, w) + \tilde{g}_{r_d}(\tilde{z}, \tilde{\xi}_1, \ldots, \tilde{\xi}_{r_d}, w)\tilde{u},$$

$$e_T = \tilde{\xi}_1, \tag{18c}$$

where

$$\begin{aligned}\tilde{f}_0 &:= f_0(\tilde{z} + \Pi_0(w)) - f_0(\Pi_0(w)) \\ &\quad + \Big[g_0(\tilde{z} + \Pi_0(w)) - g_0(\Pi_0(w))\Big]q(w), \\ \tilde{g}_0 &:= g_0(\tilde{z} + \Pi_0(w)),\end{aligned}$$

$\tilde{f}_0(0, w) = 0$, $\tilde{g}_0(\tilde{z}, 0) = g_0(\tilde{z})$, and $\tilde{f}_i(\cdot)$, $\tilde{g}_i(\cdot)$, $\forall i \in \{1, \ldots, r_d\}$ are appropriately defined functions that satisfy $\tilde{f}_i(0, \ldots, 0, w) = 0$ and $\tilde{g}_i(\tilde{z}, \ldots, \tilde{\xi}_i, 0) = g_i(\tilde{z}, \ldots, \tilde{\xi}_i)$.

Our analysis makes use of the following two optimal control problems.

Cheap control problem: For the system consisting of the error system (18) and the exosystem (5) with initial condition $\big(\tilde{z}(0), \tilde{\xi}(0), w(0)\big) = \big(\tilde{z}_0, \tilde{\xi}_0, w_0\big)$, find the optimal feedback law $\tilde{u} = \alpha^{cc}_{\delta,\epsilon}(\tilde{z}, \tilde{\xi}, w)$ that minimizes the cost functional

$$\frac{1}{2}\int_0^\infty \left(\|e_T(t)\|^2 + \delta\|\tilde{z}(t)\|^2 + \epsilon^{2r_d}\|\tilde{u}(t)\|^2\right) dt \tag{19}$$

for $\delta > 0$, $\epsilon > 0$. We denote by $J^{cc}_{\delta,\epsilon}(\tilde{z}_0, \tilde{\xi}_0, w_0)$ the corresponding optimal value. The best-attainable cheap control performance for reference-tracking is then

$$J_T := \lim_{(\delta,\epsilon)\to 0} J^{cc}_{\delta,\epsilon}(\tilde{z}_0, \tilde{\xi}_0, w_0). \tag{20}$$

As shown by Krener (2001), in some neighborhood of $(0, 0, 0)$ and for every $\delta > 0$, $\epsilon > 0$, the value $J^{cc}_{\delta,\epsilon}(\cdot, \cdot, \cdot)$ is $\mathcal{C}^{k-2}$ under the following assumption:

Assumption 1 *The linearization around $(z, \xi) = (0, 0)$ of system (13) is stabilizable and detectable, and the linearization around $w = 0$ of the exosystem (5) is stable.*

Minimum-energy problem: For the system

$$\begin{aligned}\dot{\tilde{z}} &= \tilde{f}_0(\tilde{z}, w) + \tilde{g}_0(\tilde{z}, w)e_T, & \tilde{z}(0) &= z_0, & (21a)\\ \dot{w} &= s(w), & w(0) &= w_0, & (21b)\end{aligned}$$

with e_T viewed as the input, find the optimal feedback law $e_T = \alpha^{me}_\delta(\tilde{z}, w)$ that minimizes the cost

$$\frac{1}{2}\int_0^\infty \left(\delta\|\tilde{z}(t)\|^2 + \|e_T(t)\|^2\right) dt, \tag{22}$$

for $\delta > 0$. We denote by $J^{me}_\delta(\tilde{z}_0, w_0)$ the corresponding optimal value. Under Assumption 1, $J^{me}_\delta(\cdot, \cdot)$ is $\mathcal{C}^{k-2}$ in some neighborhood of $(0, 0)$.

Our analysis reveals that the best-attainable cheap control performance J_T is equal to the least control effort (as $\delta \to 0$) needed to stabilize the corresponding zero dynamics system (21) driven by the tracking error e_T.

Theorem 3 *Suppose that Assumption 1 holds and that (6) has a solution in some neighborhood of $w = 0$. Then, for any $(\tilde{z}(0), \tilde{\xi}(0), w(0)) = (\tilde{z}_0, \tilde{\xi}_0, w_0)$ in some neighborhood of $(0, 0, 0)$ there exists a solution to the cheap control problem and*

$$J_T = \lim_{\delta\to 0} J^{me}_\delta \tag{23}$$

Proof. Under Assumption 1 and from the formulations of the cheap control and minimum-energy problems, we conclude that for every $\delta > 0$, $\epsilon > 0$, and every initial condition $(\tilde{z}_0, \tilde{\xi}_0, w_0)$ in some neighborhood of $(0, 0, 0)$, the values $J^{me}_\delta(\tilde{z}_0, w_0)$ and $J^{cc}_{\delta,\epsilon}(\tilde{z}_0, \tilde{\xi}_0, w_0)$ exist and satisfy

$$J^{me}_\delta(\tilde{z}_0, w_0) \le J^{cc}_{\delta,\epsilon}(\tilde{z}_0, \tilde{\xi}_0, w_0). \tag{24}$$

On the other hand, from Lemma 6 in Appendix we have

$$J^{cc}_{\delta,\epsilon}(\tilde{z}_0, \tilde{\xi}_0, w_0) \le J^{me}_\delta(\tilde{z}_0, w_0) + O(\epsilon). \tag{25}$$

Therefore, from (24)–(25) we conclude that

$$J^{me}_\delta(\tilde{z}_0, w_0) \le J^{cc}_{\delta,\epsilon}(\tilde{z}_0, \tilde{\xi}_0, w_0) \le J^{me}_\delta(\tilde{z}_0, w_0) + O(\epsilon).$$

The result (23) follows from this and (20) as one makes $(\delta, \epsilon) \to 0$.

□

3.2 Path-following

For path-following, we define the corresponding cheap control problem by replacing e_T with e_P in (19). We then show that, in contrast to reference-tracking, the path-following problem can be solved with arbitrarily small $\mathcal{L}_2$ norm of e_P.

We let the vector field $s(w)$ and the output map $q(w)$ of the exosystem (8) be linear, $s(w) = Sw$, $q(w) = Qw$, such that all eigenvalues of $S \in \mathbb{R}^{p\times p}$ are non-zero and semisimple.

Theorem 4 *Assume that (6) has a solution when $s(w) = v_d Sw$, for v_d almost everywhere on $(0, \infty)$. Then, for every $w(\theta(0)) = w_0$ in a neighborhood around $w = 0$, there exist a timing law for $\theta(t)$ and a feedback law*

$$u = c(w) + \alpha_{\delta,\epsilon}(z, \xi, w) \tag{26}$$

which solve the geometric path-following and satisfy (10) *for every* $\delta^\star > 0$.

Proof. With the timing law

$$\dot{\theta}(t) = v_d, \qquad \theta(0) = 0, \tag{27}$$

and $v_d > 0$ a constant to be selected later, the path-following problem becomes the tracking problem of $r(t)$ generated by

$$\dot{w}(t) = v_d S w(t), \quad r(t) = Q w(t), \tag{28}$$

which, upon the substitution in (6), yields

$$\begin{aligned} \frac{\partial \Pi}{\partial w} v_d S w &= f(\Pi(w), c(w)), \\ h(\Pi(w), c(w)) - Q w &= 0. \end{aligned} \tag{29}$$

The function $c(w)$, used in the feedback law (26), solves (29), while $\alpha_{\delta,\epsilon}(z,\xi,w)$ minimizes (19) for the error system (18) together with the exosystem (28) and some small $\delta > 0$, $\epsilon > 0$. With the timing law (27), Theorem 3 allows us to conclude that as $(\epsilon, \delta) \to 0$ we have $J_P = \lim_{\delta \to 0} J_\delta^{me}$.

To prove that J_δ^{me} can be made arbitrarily small by selecting a sufficiently large v_d, we use Lemma 7 in Appendix. It shows that for the minimum-energy problem and every initial condition in some neighborhood of $(\tilde{z}, w) = (0, 0)$, there exist a sufficiently small $\delta > 0$ in (22) and a feedback law $e_T = \hat{\alpha}_\delta^{me}(\tilde{z}, w)$ for which $J_\delta^{me}(\tilde{z}_0, w_0)$ is bounded by

$$J_\delta^{me}(\tilde{z}_0, w_0) \le \frac{1}{2} \tilde{z}_0' P_0 \tilde{z}_0,$$

where $P_0 > 0$ does not depend on v_d. Observing that $\tilde{z}_0 = \Pi_0(w_0)$, since $z(0) = 0$, the proof is completed using Lemma 8 in Appendix which establishes that $\|\Pi_0(w_0)\|$ can be made arbitrarily small by choosing a sufficiently large v_d.

□

Next we show that an arbitrarily small $\mathcal{L}_2$ norm of the path-following error is attainable even when the speed v_d is specified beforehand.

Theorem 5 *Consider the speed-assigned path-following problem with* v_d *specified so that* (29) *has a solution in some neighborhood of* $w = 0$. *Then,* (10) *can be satisfied for any* $\delta^\star > 0$ *with a suitable timing law* $\theta(t)$ *and a controller of the form* (26) *with time-varying piecewise-continuous maps* $c(w)$ *and* $\alpha(z, \xi, w)$.

Proof. To construct a path-following controller that satisfies (10) we start with

$$u = c_\sigma(w) + \alpha_\sigma(z, \xi, w), \tag{30a}$$

$$\dot{\theta} = v_\sigma, \tag{30b}$$

where for each positive constant v_ℓ, $\ell \in \mathcal{I} := \{0, 1, 2, \dots N\}$, the maps $\Pi_\ell := col(\Pi_{\ell_0}, \Pi_{\ell_\xi})$, $\Pi_{\ell_\xi} := col(\Pi_{\ell_{\xi_1}}, \dots, \Pi_{\ell_{\xi_{r_d}}})$, $\Pi_{\ell_0} : \mathbb{R}^p \to R^{n_z}$, $\Pi_{\ell_i} : \mathbb{R}^p \to R^m$, $\forall i \in \{1, \dots, r_d\}$, and $c_\ell : \mathbb{R}^p \to \mathbb{R}^m$ satisfy

$$\begin{aligned} \frac{\partial \Pi_\ell}{\partial w} v_\ell S w &= f(\Pi_\ell(w), c_\ell(w)), \\ h(\Pi_\ell(w), c_\ell(w)) - Q w &= 0, \end{aligned} \tag{31}$$

and $\sigma(t) : [t_0 := 0, \infty) \to \mathcal{I}$, is the piecewise constant switching signal

$$\sigma(t) = \begin{cases} i, & t_i \le t < t_{i+1}, \ i = 0, \dots, N-1 \\ N, & t \ge t_N \end{cases}$$

Each $\alpha_\ell(z, \xi, w)$ is the optimal feedback-law that minimizes

$$\int_0^\infty \left(\|e_P\|^2 + \delta \|z - \Pi_{\ell_0}(w)\|^2 + \epsilon^{2 r_d} \|u - c_\ell(w)\|^2 \right) dt,$$

for some small $\delta > 0$, $\epsilon > 0$. Note that (30) is a speed-assignment path-following controller for which $\dot{\theta}(t)$ converges to $v_N = v_d$ in finite time.

We now prove that for any $\delta^\star > 0$, (10) can be satisfied by appropriate selection of a finite sequence $t_0, t_1, \dots, t_N$ together with $(v_0, \Pi_0, \alpha_0, c_0), (v_1, \Pi_1, \alpha_1, c_1),$ $\dots, (v_N, \Pi_N, \alpha_N, c_N)$ used in the feedback controller (30). To this end, we show in Lemma 9 in Appendix that J_P is bounded by

$$\begin{aligned} J_P \le \frac{1}{2} \tilde{z}_0' P_0 \tilde{z}_0 + \gamma \frac{\lambda_{max}(P_0)}{2} \sum_{\ell=1}^{N} (v_{\ell-1} - v_\ell)^2 \\ + \lambda_{max}(P_0) \sum_{\ell=1}^{N} \tilde{z}_{\ell-1}(t_\ell)' \left[\bar{z}_{\ell-1}(t_\ell) - \bar{z}_\ell(t_\ell) \right] \\ + \frac{\lambda_{max}(P_0)}{2} \sum_{\ell=1}^{N} \|\tilde{z}_{\ell-1}(t_\ell)\|^2, \end{aligned} \tag{32}$$

where $\lambda_{max}(P_0)$ denotes the maximum eigenvalue of $P_0 > 0$, γ is a positive constant, $\tilde{z}_0 := \tilde{z}(0)$, $\bar{z}_\ell := \Pi_{\ell_0}(w)$, and the transient $\tilde{z}_\ell := z - \Pi_{\ell_0}(w)$ converges to zero as $t \to \infty$.

We show that each term of (32) is upper-bounded by $\frac{\delta^\star}{4}$ so that $J_P \le \delta^\star$. Applying the same arguments as in

Theorem 4, the first term in (32) is bounded by $\frac{\delta^\star}{4}$ using a sufficiently large v_0. To prove that the second term in (32) is smaller than $\frac{\delta^\star}{4}$, we select the parameters v_ℓ, $\ell \in \mathcal{I}$ to satisfy

$$v_{\ell-1} - v_\ell = \mu, \quad v_N = v_d, \quad \ell = 1, 2, \ldots, N \qquad (33)$$

where $\mu := \frac{2\delta^\star}{\gamma\lambda_{max}(P_0)(v_0 - v_N)}$, and $N := \frac{v_0 - v_N}{\mu}$. Then

$$\gamma \frac{\lambda_{max}(P_0)}{2} \sum_{\ell=1}^{N} (v_{\ell-1} - v_\ell)^2 \leq \gamma \frac{\lambda_{max}(P_0)}{2} N\mu^2$$
$$= \gamma \frac{\lambda_{max}(P_0)}{2}(v_0 - v_N)\mu = \frac{\delta^\star}{4}.$$

The above selection for the v_ℓ, $\ell \in \mathcal{I}$, is made under the constraint that the reference-tracking problem for the signal $r(t)$ generated by (28) with v_d replaced by v_ℓ is solvable. This can always be satisfied by appropriately adjusting v_0. Finally, for any given N, each of the last two terms in (32) can be made smaller than $\frac{\delta}{4}$ by choosing t_ℓ, $\ell = 1, 2, \ldots, N$ sufficiently large.

□

4 Illustrative Example

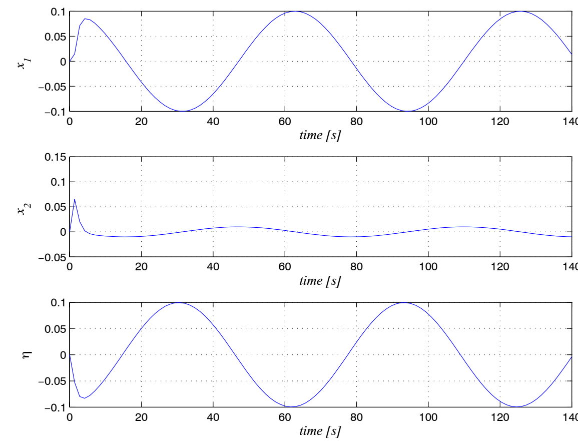
To illustrate the results of the paper we consider the system described in (Isidori, 1999, Example 8.3.5)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ \dot{\eta} &= \eta + x_1 + x_2^2 \\ y &= x_1 \end{aligned}$$

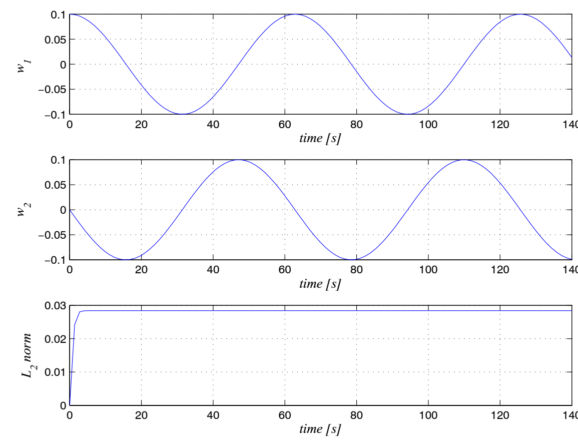
$x := (x_1, x_2, \eta)'$, which is already in normal form. This system is not exactly linearizable via feedback and the zero-dynamics are unstable. Suppose that the reference-tracking task is to asymptotically track any reference output of the form $r(t) = M \sin(at + \phi)$, where a is a fixed positive number, and M, ϕ arbitrary parameters. In this case the exosystem (5) is $s(\omega) = (a\omega_2, -a\omega_1)'$, $q(\omega) = \omega_1$. Following the procedure illustrated in Isidori (1999, Example 8.3.5) we determine a feedback law of the form $u = c(\omega) + K(x - \Pi(\omega))$, where $K = (k_1, k_2, k_3)'$ is any matrix which places the eigenvalues of $A + BK$,

$$A := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

in the left-half complex plane. Fig. 1 displays the simulation results obtained with $a = 0.1\,rad/s$ and K computed by solving the linear quadratic regulator problem ($\min_{\tilde{u}} \int_0^\infty [e(t)'e(t) + \epsilon^2 \tilde{u}(t)\tilde{u}(t)]\,dt$) with $\epsilon = 0.001$. The initial conditions are $x(0) = 0, \omega(0) = (0.1, 0)'$. The convergence to the desired reference signal is achieved with transient error $J_T \simeq 28 \times 10^{-3}$.



(a)



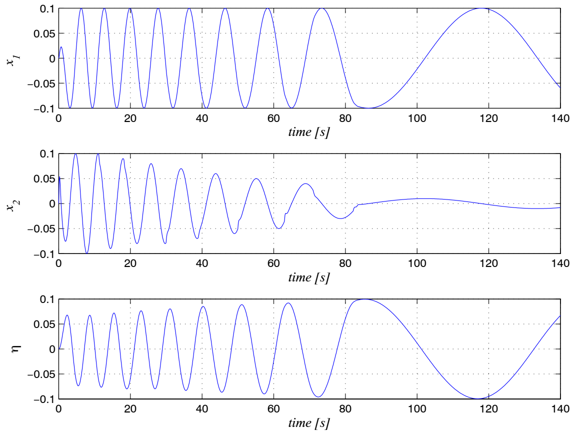
(b)

Fig. 1. Reference-tracking: Time evolution of 1(a) the state (x_1, x_2, η); and 1(b) the exogenous state (ω_1, ω_2) and the transient tracking error $\int_0^t \|e_T(\tau)\|^2\, d\tau$.

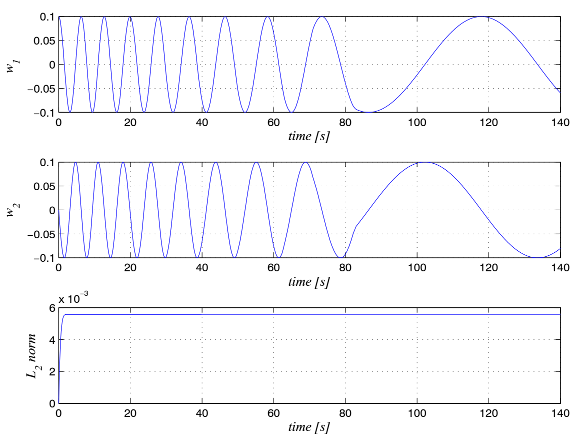
In contrast, Fig. 2 shows the simulation results obtained with the path-following controller (30) with $\alpha_\sigma = K(x - \Pi_\sigma(\omega))$ and K with the same value as the previous case. Starting with $v_0 = 1$, the values of v_σ were selected to decrease by 0.1 successively until $v_N = a = 0.1 (N = 10)$. As it can be seen, the path-following controller manipulates the evolution of the states of the exosystem in such a way that the transient error is reduced by a factor of 5 to $J_P \simeq 5.6 \times 10^{-3}$. Different sequences of v_σ could be used to reduce it even further. This illustrates how the tracking limit is removed by the path-following.
Other examples that use the freedom of manipulating the path-variable can be found in, *e.g.*, (Aguiar *et al.*, 2005; Aguiar *et al.*, 2004; Aguiar and Hespanha, 2004; Skjetne *et al.*, 2004; Dačić and Kokotović, 2006).

5 Conclusions

This paper demonstrates that the task of following a geometric path $y_d(\theta)$ is less restrictive than the task of track-



(a)



(b)

Fig. 2. Path-following: Time evolution of 2(a) the state (x_1, x_2, η); and 2(b) the exogenous state $(\omega_1(\theta(t)), \omega_2(\theta(t)))$ and the transient path-following error $\int_0^t \|e_P(\tau)\|^2\, d\tau$.

ing a reference signal $r(t)$. The reference-tracking problem is subject to the limitations imposed by the unstable zero-dynamics, a nonlinear analog of the Bode's limitations caused by non-minimum phase zeros. Our analysis revealed that the limitation is due to the need to stabilize the zero-dynamics by the tracking error, which therefore prevents the output $y(t)$ from achieving perfect tracking. In path-following one has available an additional degree of freedom to select a timing law $\theta(t)$ with which a prescribed path $y_d(\theta)$ will be followed. In Theorems 4 and 5, we prove that with an appropriate choice of $\theta(t)$ the $\mathcal{L}_2$ norm of the path-following error can be made arbitrarily small, that is, the path-following problem is not subject to the limitations of reference-tracking. The results of this paper are structural in the sense that they hold for all nonlinear systems which can be transformed into (13a)–(13c) with a diffeomorphism and a feedback transformation. They may also be of practical significance, because the path-following formulation is convenient for many applications. Design of path-following controllers for non-minimum phase systems is a topic of current research, (Dačić and Kokotović, 2006; Dačić *et al.*, 2004).

A Appendix

Lemma 6 *Suppose that Assumption 1 holds. For every initial condition* $(\tilde{z}_0, \tilde{\xi}_0, w_0)$ *for the error system* (18) *and the exosystem* (5) *in some neighborhood of* $(0,0,0)$, *and every* $\delta > 0$, *there exist a sufficiently small* $\epsilon > 0$ *and feedback law* $\tilde{u} = \hat{\alpha}^{cc}_{\delta,\epsilon}(\tilde{z}, \tilde{\xi}, w)$ *for which the value of* (19) *does not exceed*

$$J^{me}_{\delta}(\tilde{z}_0, w_0) + O(\epsilon). \tag{A.1}$$

Proof. The optimal control law for the minimum-energy problem formulated in Section 3 is

$$\alpha^{me}_{\delta}(\tilde{z}, w) = -\tilde{g}_0' \frac{\partial' J^{me}_{\delta}}{\partial \tilde{z}}. \tag{A.2}$$

Its existence is ensured by Assumption 1. As in (Braslavsky *et al.*, 2002), consider the change of coordinates

$$\eta_1 = \tilde{\xi}_1 - \alpha^{me}_{\delta}, \quad \eta_2 = \dot{\tilde{\xi}}_1 - \dot{\alpha}^{me}_{\delta}, \ldots \\ \eta_{r_d} = \tilde{\xi}_1^{(r_d-1)} - \alpha_{\delta}^{(r_d-1)} \tag{A.3}$$

that takes (18) to the form

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{f}_0(\tilde{z}, w) + \tilde{g}_0(\tilde{z}, w)\alpha^{me}_{\delta}(\tilde{z}, w) + \tilde{g}_0(\tilde{z}, w)C\eta, \\ \dot{\eta} &= A\eta + B\tilde{f}_{\eta}(\tilde{z}, \eta, w) + Bv, \\ \eta_1 &= C\eta, \end{aligned} \tag{A.4}$$

where $\eta := \mathrm{col}(\eta_1, \eta_2, \ldots, \eta_{r_d})$, $v := \tilde{g}\tilde{u}$, $\tilde{g} := \Pi_{i=1}^{r_d} \tilde{g}_i$,

$$A := \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \; B := \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ I \end{bmatrix}, \; C := \begin{bmatrix} I & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{A.5}$$

and $\tilde{f}_{\eta}(\cdot)$ is a suitably defined function. We first prove that for every $\delta > 0$ and every initial condition $(\tilde{z}_0, \eta_0, w_0)$ in some neighborhood of $(0,0,0)$ for system (A.4) together with the exosystem (5), there exist $\epsilon > 0$ and $v = \hat{\alpha}_{\delta,\epsilon}(\tilde{z}, \eta, w)$ such that

$$V_{\delta,\epsilon}(\tilde{z}_0, \eta_0, w_0) := \frac{1}{2}\int_0^{\infty} \left(\|\eta_1 + \alpha^{me}_{\delta}(\tilde{z}, w)\|^2 \right. \\ \left. + \delta\|\tilde{z}\|^2 + \epsilon^{2r_d}\|v\|^2\right) dt \tag{A.6}$$

does not exceed (A.1). Then, we derive a feedback law for $\tilde{u}$ which ensures that (19) does not exceed (A.1).

Consider $P \in \mathbb{R}^{mr_d \times mr_d}$, the positive-definite solution of

$$PA + A'P - PBB'P = -C'C, \tag{A.7}$$

and the feedback law

$$v = -\frac{1}{\epsilon^{r_d}} B'PE\eta, \tag{A.8}$$

with A, B, C given by (A.5) and

$$E = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & \epsilon I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \epsilon^{r_d - 1} I \end{bmatrix}.$$

We now show that (A.8) achieves asymptotic stability of the closed-loop system

$$\begin{aligned} \dot{\tilde{z}} &= \tilde{f}_0(\tilde{z}, w) + \tilde{g}_0(\tilde{z}, w)\alpha_\delta^{me}(\tilde{z}, w) + \tilde{g}_0(\tilde{z}, w)CE\eta, \\ \epsilon E\dot{\eta} &= (A - BB'P)E\eta + \epsilon^{r_d} B\tilde{f}_\eta(\tilde{z}, \eta, w). \end{aligned} \tag{A.9}$$

obtained by substituting (A.8) into (A.4). With $Q \in \mathbb{R}^{mr_d \times mr_d}$ satisfying

$$Q(A - BB'P) + (A' - PBB')Q = -C'C - I,$$

where $(A - BB'P)$ is Hurwitz, we select

$$W = J_\delta^{me}(\tilde{z}, w) + \frac{\epsilon}{2}\eta' EQE\eta \tag{A.10}$$

as a Lyapunov function. Computing $\dot{W}$ along (A.9), we obtain

$$\begin{aligned} \dot{W} &= \frac{\partial J_\delta^{me}}{\partial \tilde{z}}(\tilde{f}_0 + \tilde{g}_0\alpha_\delta^{me} + \tilde{g}_0 CE\eta) + \frac{\partial J_\delta^{me}}{\partial w}s(w) \\ &\quad + \frac{1}{2}\eta' E[Q(A - BB'P) + (A' - PBB')Q]E\eta \\ &\quad + \epsilon^{r_d}\eta' EQB\tilde{f}_\eta \\ &= -\frac{1}{2}\delta\|\tilde{z}\|^2 - \frac{1}{2}\|\alpha_\delta^{me}\|^2 - \alpha_\delta^{me\prime} C\eta - \frac{1}{2}\|CE\eta\|^2 \\ &\quad - \frac{1}{2}\|E\eta\|^2 + \epsilon^{r_d}\eta' EQB\tilde{f}_\eta, \end{aligned}$$

where we have used the fact that $J_\delta^{me}(\tilde{z}, w)$ satisfies

$$\frac{\partial J_\delta^{me}}{\partial \tilde{z}}\tilde{f}_0(\tilde{z}, w) + \frac{\partial J_\delta^{me}}{\partial w}s(w) + \frac{1}{2}\delta\|\tilde{z}\|^2 - \frac{1}{2}\left\|\tilde{g}_0' \frac{\partial' J_\delta^{me}}{\partial \tilde{z}}\right\|^2 = 0. \tag{A.11}$$

For any $\rho_{\tilde{z}}$, ρ_η, ρ_w such that $\|\tilde{z}\| \le \rho_{\tilde{z}}$, $\|\eta\| \le \rho_\eta$, $\|w\| \le \rho_w$, there exist $\gamma_{\tilde{z}}$, γ_η satisfying

$$\|\tilde{f}_\eta(\tilde{z}, \eta, w)\| \le \gamma_{\tilde{z}}\|\tilde{z}\| + \gamma_\eta\|\eta\|,$$

and, hence,

$$\begin{aligned} \dot{W} &\le -\frac{1}{2}\delta\|\tilde{z}\|^2 - \frac{1}{2}\|C\eta + \alpha_\delta^{me}\|^2 - \frac{1}{2}\|E\eta\|^2 \\ &\quad + \epsilon^{r_d}\left(\gamma_1\|\tilde{z}\| + \gamma_2\|\eta\|\right)\|E\eta\| \\ &\le -\frac{1}{2}(\delta - \gamma_1\epsilon^{r_d})\|\tilde{z}\|^2 - \frac{1}{2}(1 - \gamma_1\epsilon^{r_d} - 2\gamma_2\epsilon)\|E\eta\|^2, \end{aligned}$$

where $\gamma_1 := \gamma_{\tilde{z}}\|QB\|$, $\gamma_2 := \gamma_\eta\|QB\|$, and $\epsilon^{r_d}\|\eta\| \le \epsilon\|E\eta\|$ for $\epsilon < 1$. Consequently, for any $\delta > 0$, there exists $\epsilon > 0$ such that $\dot{W} \le 0$. Asymptotic stability of $(\tilde{z}, \eta) = (0, 0)$ thus follows from LaSalle's theorem. From this and (A.10) we can also conclude that W converges to zero as $t \to \infty$ because $J_\delta^{me}(0, w) = 0$.

To prove that

$$V_{\delta,\epsilon}(\tilde{z}_0, \eta_0, w_0) \le J_\delta^{me}(\tilde{z}_0, w_0) + O(\epsilon), \tag{A.12}$$

we define the positive-definite function

$$\hat{V} := J_\delta^{me}(\tilde{z}, w) + \epsilon\hat{V}_\epsilon(\eta), \quad \hat{V}_\epsilon := \frac{1}{2}\eta' EPE\eta.$$

The time-derivatives of J_δ^{me} and $\hat{V}_\epsilon$ along (A.9) satisfy

$$\begin{aligned} \dot{J}_\delta^{me} &= -\frac{1}{2}\delta\|\tilde{z}\|^2 - \frac{1}{2}\|\alpha_\delta^{me}\|^2 - \alpha_\delta^{me\prime} C\eta, \\ \dot{\hat{V}}_\epsilon &= -\frac{1}{2\epsilon}\|CE\eta\|^2 - \frac{1}{2\epsilon}\|B'PE\eta\|^2 + \epsilon^{r_d - 1}\eta' EPB\tilde{f}_\eta. \end{aligned}$$

and

$$\begin{aligned} \dot{\hat{V}} &= -\frac{1}{2}\delta\|\tilde{z}\|^2 - \frac{1}{2}\|C\eta + \alpha_\delta^{me}\|^2 \\ &\quad - \frac{1}{2}\|B'PE\eta\|^2 + \epsilon^{r_d}\eta' EPB\tilde{f}_\eta. \end{aligned} \tag{A.13}$$

From the fact that W converges to zero as $t \to \infty$, we conclude the same for $\hat{V}$. Noticing that $V_{\delta,\epsilon}$ defined in (A.6) satisfies

$$V_{\delta,\epsilon} = \frac{1}{2}\int_0^\infty \left(\|C\eta + \alpha_\delta^{me}\|^2 + \delta\|\tilde{z}\|^2 + \|B'PE\eta\|^2\right) dt,$$

and integrating (A.13) we get

$$V_{\delta,\epsilon} = \hat{V}(\tilde{z}_0, \eta_0, w_0) + \int_0^\infty \epsilon^{r_d}\eta' EPB\tilde{f}_\eta \, dt.$$

Now (A.12) follows from the fact that $\int_0^\infty \epsilon^{r_d - 1}\eta' EPB\tilde{f}_\eta \, dt$ is bounded by $\|\int_0^\infty \epsilon^{r_d - 1}\eta' EPB\tilde{f}_\eta \, dt\| \le \hat{V}_\epsilon(\eta_0)$.

We next prove that for every $\delta > 0$ and every initial condition in some neighborhood of $(z, \xi, w) = (0, 0, 0)$, there exists $\epsilon^\star > 0$ for which the feedback law

$$\tilde{u} = \tilde{g}^{-1}\hat{\alpha}_{\delta,\epsilon^\star}(\tilde{z}, \eta, w) = -\frac{1}{\epsilon^{\star r_d}}\tilde{g}^{-1}B'PE\eta$$

ensures that (19) does no exceed (A.1).

Let γ be the lowest value of the smallest singular value of

$\tilde{g}$ in a compact set containing the trajectories generated by $\tilde{u}$. Note that $\gamma > 0$ because $\tilde{g}_i^{-1}$, $\forall i \in \{1, \ldots, r_d\}$ are nonsingular. Consider $\epsilon^\star = \epsilon/\gamma^{\frac{1}{r_d}}$, then,

$$\begin{aligned}\frac{1}{2}\int_0^\infty &\left(\|e_T\|^2 + \delta\|\tilde{z}\|^2 + \epsilon^{2r_d}\|\tilde{u}\|^2\right) dt \\ &\leq \frac{1}{2}\int_0^\infty \left(\|e_T\|^2 + \delta\|\tilde{z}\|^2 + \frac{\epsilon^{2r_d}}{\gamma^2}\|\tilde{g}\tilde{u}\|^2\right) dt \\ &= V_{\delta,\epsilon^\star} \leq J_\delta^{me}(\tilde{z}_0, w_0) + O(\epsilon^\star).\end{aligned}$$

□

Lemma 7 *Consider the minimum-energy problem formulated in Section 3. For every initial condition $(\tilde{z}(0), w(0)) = (\tilde{z}_0, w_0)$ for (21) in some neighborhood of $(0,0)$, there exist $\delta > 0$ in (22) and a feedback law $e_T = \hat{\alpha}_\delta^{me}(\tilde{z}, w)$ for which (22) does not exceed*

$$\frac{1}{2}\tilde{z}_0' P_0 \tilde{z}_0,$$

where $P_0 > 0$ does not depend on v_d.

Proof. Let

$$\dot{z} = F_0 z + G_0 \xi_1$$

be the linearization of (13a) around $z = 0$. Then (18a) can be written as

$$\dot{\tilde{z}} = F_0\tilde{z} + G_0 e_T + h_{\tilde{f}_0}(\tilde{z}, w) + h_{\tilde{g}_0}(\tilde{z}, w)e_T, \quad \text{(A.14)}$$

where for all $\|\tilde{z}\| \leq \rho_{\tilde{z}}$, $\|w\| \leq \rho_w$ and some γ_i, $i = 1, \ldots, 4$, the maps $h_{\tilde{f}_0}(\tilde{z}, w) := \tilde{f}_0(\tilde{z}, w) - F_0\tilde{z}$ and $h_{\tilde{g}_0}(\tilde{z}, w) := \tilde{g}_0(\tilde{z}, w) - G_0$ satisfy

$$\begin{aligned}\|h_{\tilde{f}_0}(\tilde{z}, w)\| &\leq \gamma_1\|\tilde{z}\|^2 + \gamma_2\|\tilde{z}\|\,\|\Pi_0(w)\|, \\ \|h_{\tilde{g}_0}(\tilde{z}, w)\| &\leq \gamma_3\|\tilde{z}\|^2 + \gamma_4\|\Pi_0(w)\|.\end{aligned}$$

Consider (A.14) in closed-loop with the feedback law

$$e_T = \hat{\alpha}_\delta^{me}(\tilde{z}, w) := -G_0' P_0 \tilde{z}, \quad \text{(A.15)}$$

where $P_0 > 0$ satisfies

$$F_0' P_0 + P_0 F_0 + I - P_0 G_0 G_0' P_0 = 0. \quad \text{(A.16)}$$

Computing the time-derivative of $\hat{V}_\delta^{me} := \frac{1}{2}\tilde{z}' P_0 \tilde{z}$ along the trajectories of (A.14), (A.15), we get

$$\begin{aligned}\dot{\hat{V}}_\delta^{me} &= \frac{1}{2}\tilde{z}'[P_0F_0 + F_0'P_0 - 2P_0G_0G_0'P_0]\tilde{z} \\ &\quad + \frac{1}{2}\tilde{z}'[Ph_{\tilde{f}_0} + h_{\tilde{f}_0}'P - 2P_0h_{\tilde{g}_0}G_0'P_0]\tilde{z} \\ &\leq -\frac{1}{2}\|e_T\|^2 - \frac{1}{2}\Big[1 - 2\lambda_{max}(P_0)\big(\gamma_1\|\tilde{z}\|^2 \\ &\quad + \gamma_2\|\tilde{z}\|\,\|\Pi_0(w)\| - \lambda_{max}(P_0)(\gamma_3\|\tilde{z}\| \\ &\quad + \gamma_4\|\Pi_0(w)\|)\big)\Big]\|\tilde{z}\|^2,\end{aligned}$$

which proves that there exist $\rho_{\tilde{z}}$, ρ_w, $\delta > 0$ such that for all $\|\tilde{z}\| \leq \rho_{\tilde{z}}$, $\|w\| \leq \rho_w$, $\dot{\hat{V}}_\delta^{me}$ satisfies

$$\dot{\hat{V}}_\delta^{me} \leq -\frac{1}{2}\|e_T\|^2 - \frac{\delta}{2}\|\tilde{z}\|^2, \quad \text{(A.17)}$$

and, hence, $\hat{V}_\delta^{me} \to 0$ as $t \to \infty$. Integrating (A.17), it follows that

$$\frac{1}{2}\int_0^\infty (\|e_T\|^2 + \delta\|\tilde{z}\|^2)\, dt \leq \hat{V}_\delta^{me}(\tilde{z}_0) = \frac{1}{2}\tilde{z}_0' P_0 \tilde{z}_0.$$

□

Lemma 8 *In the reference-tracking problem for the nonlinear system (13) let the vector field $s(w)$ and the output map $q(w)$ of the exosystem (5) be $s(w) = v_d S w$, $q(w) = Qw$. Suppose that the eigenvalues of $S \in \mathbb{R}^{p\times p}$ are non-zero and semisimple, and that for some $v_d > 0$, (6) has a solution in some neighborhood of $w = 0$. Then, for any $\rho > 0$, there exists $v_d^\star > 0$ such that the map $\Pi_0 : \mathbb{R}^p \to \mathbb{R}^{n_z}$ satisfying*

$$\frac{\partial \Pi_0(w)}{\partial w} S w = \mu\big[f_0(\Pi_0(w)) + g_0(\Pi_0(w))Qw\big], \quad \text{(A.18)}$$

$\mu := \frac{1}{v_d^\star}$, is bounded by

$$\|\Pi_0(w)\| \leq \rho. \quad \text{(A.19)}$$

Proof. The proof is based on a term-by-term examination of the Taylor expansions for $\Pi_0(w)$, $f_0(z)$, and $g_0(z)$. For this we adopt the methodology of Krener (1992). We will carry out the proof in the complex field, which allow us to assume without loss of generality that S is a diagonal matrix. For a given neighborhood $\{w \in \mathbb{C}^p : \|w\| \leq \varepsilon\}$ and ρ, we expand $\Pi_0(w)$ in Taylor series

$$\Pi_0(w) = \Pi_0^{[1]}(w) + \frac{1}{2}\Pi_0^{[2]}(w) + \cdots + \frac{1}{k!}\Pi_0^{[k]}(w) + O(w)^{k+1} \quad \text{(A.20)}$$

where the superscript $[k]$ denotes terms composed of homogeneous polynomials of degree k, i.e.,

$$\Pi_0^{[k]}(w) = \sum_{1\leq i_1 \leq \cdots \leq i_k \leq p} \Pi_{0_{i_1\ldots i_k}} w_{i_1}\ldots w_{i_k}, \quad \text{(A.21)}$$

$\Pi_{0_{i_1\ldots i_k}} \in \mathbb{C}^{n_z}$. We pick a sufficiently large N such that

$$\|\Pi_0(w) - \sum_{k=1}^N \frac{1}{k!}\Pi_0^{[k]}(w)\| \leq \frac{\rho}{2}, \quad \forall_{\|w\|\leq\varepsilon}, \quad \text{(A.22)}$$

and note that

$$\|\Pi_0(w)\| \leq \|\Pi_0(w) - \sum_{k=1}^{N} \frac{1}{k!}\Pi_0^{[k]}(w)\| + \|\sum_{k=1}^{N} \frac{1}{k!}\Pi_0^{[k]}(w)\| \\ \leq \frac{\rho}{2} + \sum_{k=1}^{\infty} \frac{1}{k!}\|\Pi_0^{[k]}(w)\|.$$

If there exists $\mu > 0$ such that every term $\Pi_{0_{i_1\ldots i_k}}$ in (A.21) can be bounded by

$$\|\Pi_{0_{i_1\ldots i_k}}\| \leq \delta, \quad \delta := \frac{\rho}{2(e^{p\varepsilon}-1)}, \qquad \text{(A.23)}$$

then every coefficient in the series (A.20) satisfies $\|\Pi_0^{[k]}(w)\| \leq \delta p^k \|w\|^k$, and, hence,

$$\|\Pi_0(w)\| \leq \frac{\rho}{2} + \delta \sum_{k=1}^{\infty} \frac{1}{k!}(p\varepsilon)^k = \frac{\rho}{2} + \delta(e^{p\varepsilon}-1) = \rho.$$

To prove (A.23), we substitute

$$f_0(z) = F_0 z + \frac{1}{2} f_0^{[2]}(z) + \cdots + \frac{1}{k!} f_0^{[k]}(z) + O(z)^{k+1}, \\ g_0(z) = G_0 + g_0^{[1]}(z) + \cdots + \frac{1}{k!} g_0^{[k-1]}(z) + O(z)^k,$$

together with the expansion of $\Pi_0(w)$ into (A.18). Matching the first order terms, we get

$$\frac{\partial \Pi_0^{[1]}}{\partial w}(w) S w = \mu \big[F_0 \Pi_0^{[1]}(w) + G_0 Q w\big]. \qquad \text{(A.24)}$$

Substituting $\Pi_0^{[1]}(w) = \sum_{i=1}^{p} \Pi_i w_i$ in (A.24) and matching the term in w_i, we obtain for all $i = 1, \ldots, p$

$$\lambda_i \Pi_i w_i = \mu \big[F_0 \Pi_i + G_0 Q \mathrm{e}_i\big] w_i, \qquad \text{(A.25)}$$

where λ_i denotes the i^{th} eigenvalue of S, and $\mathrm{e}_i := (0, \ldots, 1, \ldots 0)'$. With $\bar{f}_0 := \|F_0\|$ and $a_{1_i} := \|G_0 Q \mathrm{e}_i\|$ we get

$$\|\Pi_i\| \leq \mu \frac{\bar{f}_0 \|\Pi_i\| + a_{1_i}}{|\lambda_i|},$$

and conclude that $\|\Pi_i\| \leq \delta$ if

$$\mu \leq \min_{1 \leq i \leq p} \frac{\delta |\lambda_i|}{\bar{f}_0 \delta + a_{1_i}} \ \text{ and } \ \mu < \frac{|\lambda_i|}{\bar{f}_0}.$$

For (A.25) to have a solution for all Π_i, as shown in (Krener, 1992), it is also required that $\mu z_i \neq \lambda_j, \quad i = 1, \ldots, n_z;\ j = 1, \ldots, p$ where z_i, the eigenvalues of F_0, are the zeros of the linearization of (13).
The degree two term of (A.18) satisfies

$$\frac{\partial \Pi_0^{[2]}}{\partial w}(w) S w = \mu \big[F_0 \Pi_0^{[2]}(w) \\ + f_0^{[2]}(\Pi_0^{[1]}(w)) + \frac{1}{2} g_0^{[1]}(\Pi_0^{[1]}(w)) Q w\big]. \qquad \text{(A.26)}$$

We can expand $\Pi_0^{[2]}$, $f_0^{[2]}$, and $g_0^{[1]} Q w$ as

$$\Pi_0^{[2]}(w) = \sum \Pi_{0_{ij}} w_i w_j, \\ f_0^{[2]}(\Pi_0^{[1]}(w)) = \sum f_{0_{ij}} w_i w_j, \\ g_0^{[1]}(\Pi_0^{[1]}(w)) Q w = \sum g_{0_{ij}} w_i w_j,$$

where $\Pi_{0_{ij}} \in \mathbb{C}^{n_z}$, $f_{0_{ij}} \in \mathbb{C}^{n_z}$, $g_{0_{ij}} \in \mathbb{C}^{n_z}$, and the sums range over $1 \leq i \leq j \leq p$. Substituting these series into (A.26), matching the terms $w_i w_j$ and noticing that

$$\frac{\partial \Pi_0^{[2]}}{\partial w}(w) S w = \sum \Pi_{0_{ij}} (\lambda_i + \lambda_j) w_i w_j, \quad \forall_{i,j}$$

we obtain

$$(\lambda_i + \lambda_j) \Pi_{0_{ij}} w_i w_j = \mu \big[F_0 \Pi_{0_{ij}} + f_{0_{ij}} + \frac{1}{2} g_{0_{ij}}\big] w_i w_j,$$

which, using, $a_{2_{ij}} := \|f_{0_{ij}} + \frac{1}{2} g_{0_{ij}}\|$, yields

$$\|\Pi_{0_{ij}}\| \leq \mu \frac{\bar{f}_0 \|\Pi_{0_{ij}}\| + a_{2_{ij}}}{|\lambda_i + \lambda_j|}.$$

Therefore, we conclude that $\|\Pi_{0_{ij}}\| \leq \delta$ provided that

$$\mu \leq \min_{1 \leq i \leq j \leq p} \frac{\delta |\lambda_i + \lambda_j|}{\bar{f}_0 \delta + a_{2_{ij}}}, \quad \mu < \frac{|\lambda_i + \lambda_j|}{\bar{f}_0},$$

and $\mu z_i \neq \lambda_{j_1} + \lambda_{j_2}, \quad 1 \leq j_1 \leq j_2 \leq p$.
For the degree k term, we have

$$\frac{\partial \Pi_0^{[k]}}{\partial w}(w) S w = \mu \big[F_0 \Pi_0^{[k]}(w) \\ + f_0^{[k]}(w) + \frac{1}{k} g_0^{[k-1]}(w) Q w\big], \qquad \text{(A.27)}$$

where $f_0^{[k]}(w)$ and $g_0^{[k-1]}(w)$ are the degree k and $k-1$ terms of the composition of $f_0(z)$ and $g_0(z)$ with the expansion of $\Pi_0(w)$ up to degree $k-1$. As before, expanding $\Pi_0^{[k]}$, $f_0^{[k]}$, and $g_0^{[k-1]} Q w$ in terms of $w_{i_1} w_{i_2} \ldots w_{i_k}$ with $i_1 \leq i_2 \leq \cdots \leq i_k \leq p$, and substituting these expansions in (A.27) yields

$$(\lambda_{i_1} + \cdots + \lambda_{i_k}) \Pi_{0_{i_1\ldots i_k}} w_{i_1} \ldots w_{i_k} \\ = \mu \big[F_0 \Pi_{0_{i_1\ldots i_k}} + f_{0_{i_1\ldots i_k}} + \frac{1}{k} g_{0_{i_1\ldots i_k}}\big] w_{i_1} \ldots w_{i_k},$$

$$\|\Pi_{0_{i_1\ldots i_k}}\| \leq \mu \frac{\bar{f}_0 \|\Pi_{0_{i_1\ldots i_k}}\| + a_{k_{i_1\ldots i_k}}}{|\lambda_{i_1} + \cdots + \lambda_{i_k}|},$$

where $a_{k_{i_1\dots i_k}} := \|f_{0_{i_1\dots i_k}} + \frac{1}{k}g_{0_{i_1\dots i_k}}\|$. Thus, it follows that $\|\Pi_{0_{i_1\dots i_k}}\| \leq \delta$ provided that

$$\mu \leq \min_{1\leq i_1\leq\dots\leq i_k\leq p} \frac{\delta|\lambda_{i_1}+\dots+\lambda_{i_k}|}{\bar{f}_0\delta + a_{k_{i_1\dots i_k}}}, \quad \mu < \frac{|\lambda_{i_1}+\dots+\lambda_{i_k}|}{\bar{f}_0},$$

and $\mu z_i \neq \lambda_{j_1}+\dots+\lambda_{j_k}, \quad 1\leq j_1\leq j_2\leq\dots\leq j_k\leq p$. This completes the proof. □

Lemma 9 *Under the conditions of Theorem 5, the path-following controller* (30) *ensures that there exists $\gamma > 0$ such that J_P satisfies* (32).

Proof. We first compute

$$J_\ell := \int_{t_\ell}^{\infty} \|e_P(t)\|^2\, dt, \quad \ell \in \mathcal{I}$$

with $\sigma(t) = \ell$ for all $t \geq t_\ell$ and note that

$$J_P \leq \sum_{\ell=0}^{N} J_\ell. \tag{A.28}$$

As in the proof of Theorem 4, we get

$$J_\ell \leq \frac{1}{2}\tilde{z}_\ell(t_\ell)' P_0 \tilde{z}_\ell(t_\ell), \tag{A.29}$$

where $P_0 > 0$ satisfies (A.16), $\tilde{z}_\ell := z - \bar{z}_\ell$ and $\bar{z}_\ell = \Pi_{\ell_0}(w)$, that is, $\bar{z}_\ell$ is the steady-state of z when $\sigma(t) = \ell$ for all $t \geq t_\ell$. For $\ell = 1, 2, \dots, N$ we substitute $z(t_\ell) = \tilde{z}_{\ell-1}(t_\ell) + \bar{z}_{\ell-1}(t_\ell)$ in (A.29) and get

$$\begin{aligned} J_\ell &\leq \frac{1}{2}\big(\tilde{z}_{\ell-1}(t_\ell) + \bar{z}_{\ell-1}(t_\ell) - \bar{z}_\ell(t_\ell)\big)' \\ &\quad P_0\big(\tilde{z}_{\ell-1}(t_\ell) + \bar{z}_{\ell-1}(t_\ell) - \bar{z}_\ell(t_\ell)\big) \\ &\leq \frac{\lambda_{max}(P_0)}{2}\Big(\|\bar{z}_{\ell-1}(t_\ell) - \bar{z}_\ell(t_\ell)\|^2 + \|\tilde{z}_{\ell-1}(t_\ell)\|^2 \\ &\quad + 2\tilde{z}'_{\ell-1}(t_\ell)\big[\bar{z}_{\ell-1}(t_\ell) - \bar{z}_\ell(t_\ell)\big]\Big). \end{aligned} \tag{A.30}$$

We now prove that $\tilde{\Pi}_{\ell_0}(w) := \bar{z}_{\ell-1} - \bar{z}_\ell = \Pi_{\ell-1_0}(w) - \Pi_{\ell_0}(w)$ can be written as

$$\tilde{\Pi}_{\ell_0}(w) = \alpha_\ell(w)\tilde{\mu}_l, \tag{A.31}$$

where $\tilde{\mu}_\ell := \mu_{\ell-1} - \mu_\ell$ and $\alpha_\ell(w)$ is a bounded continuous function. Consider (A.18) for each $\Pi_{\ell-1_0}(w)$, $\Pi_{\ell_0}(w)$, that is,

$$\begin{aligned} \frac{\partial \Pi_{\ell-1_0}(w)}{\partial w} Sw &= \mu_{\ell-1}\big[f_0(\Pi_{\ell-1_0}(w)) + g_0(\Pi_{\ell-1_0}(w))Qw\big] \\ \frac{\partial \Pi_{\ell_0}(w)}{\partial w} Sw &= \mu_\ell\big[f_0(\Pi_{\ell_0}(w)) + g_0(\Pi_{\ell_0}(w))Qw\big] \end{aligned}$$

Subtracting the equations, yields

$$\begin{aligned} \frac{\partial \tilde{\Pi}_{\ell_0}(w)}{\partial w} Sw &= \tilde{\mu}_\ell\big[f_0(\Pi_{\ell-1_0}(w)) + g_0(\Pi_{\ell-1_0}(w))Qw\big] \\ &\quad + \mu_\ell\big[f_0(\Pi_{\ell-1_0}(w)) - f_0(\Pi_{\ell_0}(w)) \\ &\quad + \big(g_0(\Pi_{\ell-1_0}(w)) - g_0(\Pi_{\ell_0}(w))\big)Qw\big]. \end{aligned} \tag{A.32}$$

By the mean value theorem there exist functions $\beta_f(\cdot)$, $\beta_g(\cdot)$ such that

$$\begin{aligned} f_0(\Pi_{\ell-1_0}(w)) - f_0(\Pi_{\ell_0}(w)) &= \beta_f(\Pi_{\ell-1_0}(w), \Pi_{\ell_0}(w))\tilde{\Pi}_{\ell_0}(w), \\ g_0(\Pi_{\ell-1_0}(w)) - g_0(\Pi_{\ell_0}(w)) &= \beta_g(\Pi_{\ell-1_0}(w), \Pi_{\ell_0}(w))\tilde{\Pi}_{\ell_0}(w). \end{aligned}$$

Substituting these expressions in (A.32), we verify (A.31) with α_ℓ which satisfies

$$\begin{aligned} \frac{\partial \alpha_\ell}{\partial w}(w) Sw &= f_0(\Pi_{\ell-1_0}(w)) + g_0(\Pi_{\ell-1_0}(w))Qw \\ &\quad + \mu_\ell\big[\beta_f(\Pi_{\ell-1_0}(w), \Pi_{\ell_0}(w))\alpha_\ell(w) \\ &\quad + \beta_g(\Pi_{\ell-1_0}(w), \Pi_{\ell_0}(w))\alpha_\ell(w)Qw\big]. \end{aligned} \tag{A.33}$$

Assuming that $\Pi_{\ell-1_0}(w)$ and $\Pi_{\ell_0}(w)$ exist, then $\alpha_\ell(w)$ is bounded and there exists $\gamma > 0$ such that $\|\tilde{\Pi}_{\ell_0}(w)\|^2 \leq \gamma\tilde{\mu}_\ell^2$. The bound (32) follows from this fact, together with (A.28) and (A.30).

□

References

Aguiar, A. P. and J. P. Hespanha (2004). Logic-based switching control for trajectory-tracking and path-following of underactuated autonomous vehicles with parametric modeling uncertainty. In: *Proc. of the 2004 Amer. Contr. Conf.*. Boston, MA, USA.

Aguiar, A. P., D. B. Dačić, J. P. Hespanha and P. Kokotović (2004). Path-following or reference-tracking? An answer relaxing the limits to performance. In: *Proc. of IAV2004 - 5th IFAC/EURON Symp. on Intel. Auton. Vehicles*. Lisbon, Portugal.

Aguiar, A. P., J. P. Hespanha and P. Kokotović (2005). Path-following for non-minimum phase systems removes performance limitations. *IEEE Trans. on Automat. Contr.* **50**(2), 234–239.

Al-Hiddabi, S. A. and N. H. McClamroch (2002). Tracking and maneuver regulation control for nonlinear nonminimum phase systems: application to flight control. *IEEE Trans. on Contr. Systems Tech.* **10**(6), 780–792.

Braslavsky, J. H., R. H. Middleton and J. S. Freudenberg (2002). Cheap control performance of a class of nonright-invertible nonlinear systems. *IEEE Trans. on Automat. Contr.* **47**(8), 1314–1319.

Chen, Jie, Li Qiu and O. Toker (2000). Limitations on maximal tracking accuracy. *IEEE Trans. on Automat. Contr.* **45**(2), 326–331.

Dačić, D.B. and P.V. Kokotović (2006). Path-following for linear systems with unstable zero dynamics. *Automatica* **42**(10), 1673–1683.

Dačić, D.B., M.V. Subbotin and P.V. Kokotović (2004). Path-following for a class of nonlinear systems with unstable zero dynamics. In: *Proc. of the 43rd Conf. on Decision and Contr.*. Paradise Island, Bahamas.

Davison, Edward J. (1976). The robust control of a servomechanism problem for linear time-invariant multivariable systems. *IEEE Trans. on Automat. Contr.* **21**(1), 25–34.

Francis, B.A. (1977). The linear multivariable regulator problem. *SIAM J. Contr. Optimization* **15**(3), 486–505.

Francis, B.A. and W.M. Wonham (1976). The internal model principle of control theory. *Automatica* **12**(5), 457–465.

Hauser, J. and R. Hindman (1995). Maneuver regulation from trajectory tracking: Feedback linearizable systems. In: *Proc. IFAC Symp. Nonlinear Control Systems Design*. Lake Tahoe, CA, USA. pp. 595–600.

Isidori, A. (1999). *Nonlinear Control Systems*. Vol. II. Springer. London.

Isidori, Alberto and Christopher I. Byrnes (1990). Output regulation of nonlinear systems. *IEEE Trans. on Automat. Contr.* **35**(2), 131–140.

Krener, A.J. (1992). *The construction of optimal linear and nonlinear regulators*. pp. 301–322. Systems, Models and Feedback: Theory and Applications. A. Isidori and T. J. Tarn. Birkhauser, Boston.

Krener, A.J. (2001). The local solvability of a Hamilton - Jacobi - Bellman PDE around a nonhyperbolic critical point. *SIAM J. Contr. Optimization* **39**(5), 1461–1484.

Krstić, M., I. Kanellakopoulos and P. Kokotović (1995). *Nonlinear and Adaptive Control Design*. John Wiley & Sons, Inc.. New York, USA.

Kwakernaak, H. and R. Sivan (1972). The maximal achievable accuracy of linear optimal regulators and linear optimal filters. *IEEE Trans. on Automat. Contr.* **17**(1), 79–86.

Middleton, R. H. (1991). Trade-offs in linear control systems design. *Automatica* **27**(2), 281–292.

Qiu, L. and E. J. Davison (1993). Performance limitations of nonminimum phase systems in the servomechanism problem. *Automatica* **29**, 337–349.

Seron, M. M., J. H. Braslavsky, Petar V. Kokotović and David Q. Mayne (1999). Feedback limitations in nonlinear systems: From bode integrals to cheap control. *IEEE Trans. on Automat. Contr.* **44**(4), 829–833.

Skjetne, R., T. I. Fossen and P. Kokotović (2004). Robust output maneuvering for a class of nonlinear systems. *Automatica* **40**(3), 373–383.

Su, Weizhou, Li Qiu and Jie Chen (2003). Fundamental performance limitations in tracking sinusoidal signals. *IEEE Trans. on Automat. Contr.* **48**(8), 1371–1380.

Woodyatt, A.R., M.M. Seron, J.S. Freudenberg and R.H. Middleton (2002). Cheap control tracking performance for non-right-invertible systems. *Int. J. of Robust and Nonlinear Control* **12**(15), 1253–1273.