The Entropy Penalized Minimum Energy Estimator

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***Abstract*— This paper addresses the state estimation problem of nonlinear systems. We formulate the problem using a minimum energy estimator (MEE) approach and propose an entropy penalized scheme to approximate the viscosity solution of the Hamilton-Jacobi equation that follows from the MEE formulation. We derive an explicit observer algorithm that is iterative and filtering-like, which continuously improves the state estimation as more measurements arise. In addition, we propose a computationally efficient procedure to estimate the state by performing an approximation of the nonlinear system along the trajectory of the estimate. In this case, for the first and second order approximations of the state equation, we derive a closed-form (iterative) solution for the Hessian of the entropy-like version of the optimal cost function of the MEE. We illustrate and contrast the performance of our algorithms with the extended Kalman filter (EKF) using specific nonlinear examples with the feature that the EKF do not converge to the correct value.**

I. INTRODUCTION

The determination in real time of an estimate of the state of a given nonlinear system from partial and noisy measurements of the inputs and outputs and inexact knowledge of the initial condition has been a fundamental and a challenging problem in theory and applications of control systems [1], [2]. By far, the extended Kalman filter (EKF) is the most widely used method for estimating the state. It is obtained by linearizing the nonlinear dynamics and the observation along the trajectory of the estimate. However, since it is only a local method, it may fail to converge. Several nonlinear observers using deterministic and stochastic approaches can be found in the literature[1-12]: Lyapunov-like, Luenberger-like, high-gain observers, sliding-mode observers, optimization-based, etc. Particular interesting classes of optimal nonlinear observers are the minimum energy estimator (MEE) and the closely related H_∞ estimator [7]. The MEE were first proposed by Mortersen [8] and further improved by O. Hijab [9]. In [10] the convergence of the MEE is proven, provided that the system is uniformly observable for every input. In the MEE approach one tries to find an observer that computes an estimate $\hat{x}(t)$ of the state $x(t)$ that is compatible with the system's dynamics and measured outputs and minimizes the energy of the noise and disturbances. One important feature of this approach is that if the system is linear, then one would obtain precisely the Kalman-Bucy filter. However, in general, both minimum-energy and H_∞ state estimators for nonlinear systems lead to infinite-dimensional observers whose state evolves according to a first-order nonlinear PDE of Hamilton-Jacobi (HJ) type, driven by the observations. To the best of our knowledge, besides linear systems, the only class of systems reported in the literature where one can have a closed-form solution that is filtering-like and iterative are state-affine systems with implicit outputs [11], [12].

Motivated by the above considerations, this paper addresses the problem of computing the viscosity solution $J(x, t)$ of the HJ PDE that arises in the minimum energy estimator formulation. We propose an alternative strategy (compared to the traditional ones of computing directly $J(x, t)$ via finite elements) that consists in the discretization of the estimation problem and in an addition of a suitable entropy functional. This functional regularizes the evolution of the discrete value function by acting as a viscosity term. This strategy follows from recent results on entropy penalization methods for HJ equations described in [13].

The main contributions of this paper are twofold. First, we derive an explicit observer algorithm, called *entropy penalized minimum energy estimator* that is iterative and filtering-like, which continuously improves the state estimation as more measurements arise. Second, we propose a computationally efficient procedure to estimate the state. In this case, for the first and second order approximations of the state equation and linear (or linearized) output equation, we derive a closed-form (iterative) solution for the Hessian of the entropy-like version of the optimal cost function of the MEE. We illustrate and contrast the performance of the proposed algorithms with the extended Kalman filter (EKF) using specific nonlinear examples with the feature that the EKF do not converge to the correct value.

The paper is organized as follows. Section II introduces the entropy penalized the minimum energy estimator (MEE). In Section 3, we derive the explicit iterative formulas of the entropy penalized MEE when the output equation of the process is linearized and the state equation is replaced by a first or second order approximation. Section 4 illustrates the performance of the entropy penalized MEE through computer simulations using highly nonlinear systems. Conclusions and suggestions for further research are presented in Section 5.

II. THE ENTROPY PENALIZED MINIMUM ENERGY ESTIMATOR

Consider the problem of estimating the current state $x \in \mathbb{R}^n$ of a nonlinear system

$$\dot{x} = f(x) + g(x)w, \quad x(0) = x_0 \tag{1a}$$

$$y = h(x) + v, \tag{1b}$$

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where $f:\mathbb{R}^n \to \mathbb{R}^n$, $g:\mathbb{R}^n \to \mathbb{R}^{n\times m}$, $h:\mathbb{R}^n \to \mathbb{R}^p$, $y \in \mathbb{R}^p$ denotes the measured output, $w \in \mathbb{R}^m$ an input disturbance that cannot be measured and $v \in \mathbb{R}^p$ the measurement noise affecting the output. The initial condition x_0 and the signals w and v are assumed deterministic but unknown. For simplicity, we have considered a system without control signal since for estimator design purposes this one is assumed to be known. The minimum energy estimator produces an estimate for the state x that is compatible with the system's dynamics and measured outputs for noise v and disturbances w with lowest integral-square-norm [10], [11]. The optimal state estimate $\hat{x}$ at time t is defined as

$$\hat{x}(t) := \arg\min_{z\in\mathbb{R}^n} J(z,t), \tag{2}$$

where the cost functional is given by

$$J(x,t) := \|x_0-\hat{x}_0\|^2_{Q_0} + \min_{w:[0,t]} \frac{1}{2}\int_0^t \|w(\tau)\|^2 + \|y(\tau)-h(x(\tau))\|^2 d\tau, \tag{3}$$

$\|x_0-\hat{x}_0\|^2_{Q_0} := (x_0-\hat{x}_0)^T Q_0 (x_0-\hat{x}_0)$, $Q_0 > 0$, and $\hat{x}_0$ encodes a priori information about the state. Under reasonable assumptions in (1) (see details in [10], [14]), $J(x,t)$ is locally Lipschitz continuous and it is a viscosity solution of the HJ PDE

$$J_t(x,t) + J_x(x,t)f(x) + \frac{1}{2}\|J_x^T(x,t)\|^2_\Gamma - \frac{1}{2}\|y(t)-h(x)\|^2 = 0, \tag{4}$$

where $\Gamma(x) := g(x)g^T(x)$, and J_t and J_x denote the partial derivatives of J w.r.t. t and x, respectively. Note that (2) and (4) define an infinite dimensional observer with state $J(\cdot,t)$. To rewrite it in a filtering like form (similar to a Kalman filter) we follow the steps proposed in [10], [14] by supposing that $J(x,t)$ is a smooth solution to (4). In this case, since $\hat{x}(t)$ is a minimum of $J(x,t)$ then

$$J_x(\hat{x}(t),t) = 0, \tag{5a}$$

$$J_{xx}(\hat{x}(t),t)\dot{\hat{x}}(t) + J_{xt}(\hat{x}(t),t) = 0. \tag{5b}$$

Taking partial derivatives of (4) w.r.t. x, evaluating at $(\hat{x}(t),t)$, and using (5a) it follows that

$$J_{tx}(\hat{x}(t),t) + J_{xx}(\hat{x}(t),t)f(\hat{x}) + h_x^T(\hat{x}(t))\big(y-h(\hat{x}(t))\big) = 0. \tag{6}$$

From these two last equations we finally arrive to the dynamic observer equation

$$\dot{\hat{x}}(t) = f(\hat{x}) + J_{xx}^{-1}(\hat{x},t)h_x^T(\hat{x})\big(y-h(\hat{x})\big). \tag{7}$$

It is important to stress that we still have the problem of computing $J(x,t)$ (or more precisely the Hessian of $J(x,t)$). To solve this problem, we propose an entropy penalized scheme to approximate the viscosity solution [15] of the HJ PDE (4) following the procedure in [13]. To this effect, we discretize in time, with time step $\Delta_t = t_{k+1}-t_k$, the cost functional (3) and use the Lax-Hopf formula ([16]) to obtain

$$J_{\Delta_t}(x,i+1) = \min_{w\in\mathbb{R}^m}\left[J_{\Delta_t}(x+\Delta_t\dot{x},i) + \Delta_t L(x,w)\right], \tag{8}$$

where J_{Δ_t} denotes the time discretization of J, i the iterative step, and L the Lagrangian given by

$$L(x,w) := \frac{1}{2}\|w\|^2 + \frac{1}{2}\|y-h(x))\|^2. \tag{9}$$

As in [13], equation (8) can be written as

$$J_{\Delta_t}(x,i+1) = \min_{\mu\in\wp}\int_{\mathbb{R}^m}\left[J_{\Delta_t}(x+\Delta_t\dot{x},i) + \Delta_t L(x,w)\right]d\mu(w) \tag{10}$$

where $\wp$ denotes the space of the probability measures on $\mathbb{R}^m$. We now add an entropy functional to regularize the evolution of the discrete value function which acts as a viscosity term. For this purpose, let $\varepsilon > 0$ be a small parameter and $\mathcal{D}$ the set of probability densities on $\mathbb{R}^m$, i.e.,

$$D = \left\{\gamma \in L^1(\mathbb{R}^m)|\gamma(w)\geq 0 \quad a.e., \int_{\mathbb{R}^m}\gamma(w)dw = 1\right\}.$$

Then, applying the entropy penalized scheme to (10) we obtain an approximation to J_{Δ_t} that, to simplify notation we still denote by J_{Δ_t} and is given by

$$J_{\Delta_t}(x,i+1) = \min_{\gamma\in\mathcal{D}}\int_{\mathbb{R}^m}\big[J_{\Delta_t}(x+\Delta_t\dot{x},i) + \Delta_t L(x,w) + \varepsilon\ln\gamma(w)\big]\gamma(w)dw. \tag{11}$$

Using the results in [13], we can then conclude that (11) is equivalent to the following explicit iteration scheme

$$J_{\Delta_t}(x,i+1) = -\varepsilon\ln\left(\int_{\mathbb{R}^m} e^{-\frac{J_{\Delta_t}(x+\Delta_t\dot{x},i)+\Delta_t L(x,w)}{\varepsilon}}dw\right). \tag{12}$$

We call the system composed by (7), or a discretization of it, e.g.,

$$\hat{x}_{i+1} = \hat{x}_i + \Delta_t\left[f(\hat{x}_i) + (J_{\Delta_t})^{-1}_{xx}(\hat{x},i)h_x^T(\hat{x}_i)\left[y-h(\hat{x}_i)\right]\right]$$

together with the scheme (12) to approximate the value function $J(x,t)$ as *the entropy penalized minimum energy estimator*.

In the next section, for the first and second order approximations of the state equation and the linearized output equation, we present explicitly iterative formulas to compute $(J_{\Delta_t})_{xx}(x,i)$.

III. The Entropy Penalized MEE for Systems with Linearized Outputs

In this section, we restrict system (1) to

$$\dot{x} = f(x) + Gw, \quad x(0) = x_0, \tag{13a}$$

$$y = h_x(\hat{x})x + v, \tag{13b}$$

where $G \in \mathbb{R}^{n\times m}$ is assumed to be a Hermitian matrix, and h_x is the Jacobian matrix of h around the estimated state

$\hat{x}$. In the following, we derive iterative formulas using the entropy penalized scheme to compute the approximation of the viscosity solution of the HJ PDE (4). In Section III-A, we consider a first order approximation of the state equation in the estimate point $\hat{x}$ and compute an approximation of the cost function valid in a neighborhood of that point. In Section III-B, we further improve the results by using a second order approximation.

A. First order approximation

Theorem 1: Consider system (13) with f replaced by a first order approximation

$$f(x) = f(\hat{x}_i) + f_x(\hat{x}_i)(x - \hat{x}_i) \tag{14}$$

and set the initial condition of the value function $J_{\Delta_t}(x, 0)$ as

$$J_{\Delta_t}(x, 0) = \|x - \mu_0\|^2_{Q_0},$$

where $\mu_0 = \hat{x}_0$ and Q_0 is a positive definite matrix.

Then, the cost function $J_{\Delta_t}(x, i)$ at each time step $i \geq 1$ is quadratic and its Hessian $(J_{\Delta_t})_{xx}(x, i)$ can be written as

$$(J_{\Delta_t})_{xx}(x, i) = \left(\|x - \mu_i\|^2_{Q_i}\right)_{xx},$$

where

$$\mu_i = -\frac{1}{2} Q_{i-1}^{-1} \big(2\Theta_{2_{i-1}}^T \Lambda_{i-1} \Theta_{1_{i-1}} - \Delta_t y^T h_x(\hat{x}_i)\big)^T, \tag{15}$$

$$Q_i = \Theta_{1_i}^T \Lambda_{i-1} \Theta_{1_i} + \left(\frac{1}{2}\Delta_t h_x^T(\hat{x}_i) h_x(\hat{x}_i)\right), \tag{16}$$

$$\Theta_{1_i} = \left(I + \Delta_t f_x(\hat{x}_i)^T\right), \tag{17}$$

$$\Theta_{2_i} = \left(\Delta_t f(\hat{x}_i) - \Delta_t f_x(\hat{x}_i)^T \hat{x}_i\right) - \mu_i, \tag{18}$$

$$\Lambda_i = \left(Q_i - Q_i G U_i^{-1} \Delta_t^2 G^T Q_i^T\right), \tag{19}$$

$$U_i = \left(\Delta_t^2 G^T Q_i G + \frac{1}{2}\Delta_t I\right). \tag{20}$$

Proof: The proof follows by induction. For $i = 0$ the result is trivial since the initial condition is quadratic. For the induction step, assume that the hypothesis holds for $t = t_i = \Delta_t i$. We will show that it also holds at $t + \Delta_t = (i+1)\Delta_t$. Recall (12) and consider first the argument of the exponential power. Using the fact that $J_{\Delta_t}(x, i)$ can be written by a quadratic term of the form $J_{\Delta_t}(x, i) = \|x - \mu_i\|^2_{Q_i} + cte$ due to the induction hypothesis, it follows that

$$\begin{aligned}
J_{\Delta_t}(x + \Delta_t \dot{x}, i) &= \|(x + \Delta_t \dot{x} - \mu_i)\|^2_{Q_i} + cte \\
&= \|x + \Delta_t \left(f(\hat{x}_i) + f_x(\hat{x}_i)(x - \hat{x}_i)\right) - \mu_i\|^2_{Q_i} + cte \\
&= \|(\alpha + \Delta_t G w)\|^2_{Q_i} + cte,
\end{aligned}$$

where cte denotes a constant term that we will neglect since we are interesting in the Hessian and α is given by

$$\alpha = (I + \Delta_t f_x(\hat{x}_i)) x + (\Delta_t f(\hat{x}_i) - \Delta_t f_x(\hat{x}_i)\hat{x}_i) - \mu_i. \tag{21}$$

The Lagrangian (9) can be re-wrriten as

$$\Delta_t L(x, w) = \|w\|^2_{\frac{1}{2}\Delta_t I} + \|x\|^2_{\frac{1}{2}\Delta_t h_x^T(\hat{x}_i) h_x(\hat{x}_i)} - \Delta_t y^T h_x(\hat{x}_i) x + \|y\|^2_{\frac{1}{2}\Delta_t I}.$$

where $\|y\|^2_{\frac{1}{2}\Delta_t I}$ is a constant term that can be neglected. Thus,

$$\begin{aligned}
J_{\Delta_t}(x + \Delta_t \dot{x}, i) + \Delta_t L(x, w) = \\
&\|(\alpha + \Delta_t G w)\|^2_{Q_i} + \|w\|^2_{\frac{1}{2}\Delta_t I} \\
&+ \|x\|^2_{\frac{1}{2}\Delta_t h_x^T(\hat{x}_i) h_x(\hat{x}_i)} - \Delta_t y^T h_x(\hat{x}_i) x + cte.
\end{aligned}$$

Completing the squares we obtain

$$\begin{aligned}
J_{\Delta_t}(x + \Delta_t \dot{x}, i) + \Delta_t L(x, w) &= (L^* w)^T L^* w \\
&+ 2\left(\Delta_t \alpha^T Q_i G (L^*)^{-1}\right) L^* w \\
&+ \left(\Delta_t \alpha^T Q_i G\right) U_i^{-1} \left(\Delta_t \alpha^T Q_i G\right)^T \\
&- \left(\Delta_t \alpha^T Q_i G\right) U_i^{-1} \left(\Delta_t \alpha^T Q_i G\right)^T \\
&+ \alpha^T Q_i \alpha + \|x\|^2_{\frac{1}{2}\Delta_t h_x^T(\hat{x}_i) h_x(\hat{x}_i)} \\
&- \Delta_t y^T h_x(\hat{x}_i) x + cte \\
= \left\|L^* w + \Delta_t L^{-1} G^T Q_i^T \alpha\right\|^2 \\
&+ \alpha^T \left(Q_i - Q_i G U_i^{-1} \Delta_t^2 G^T Q_i^T\right) \alpha \\
&+ \|x\|^2_{\frac{1}{2}\Delta_t h_x^T(\hat{x}_i) h_x(\hat{x}_i)} - \Delta_t y^T h_x(\hat{x}_i) x + cte.
\end{aligned} \tag{22}$$

where

$$U_i = \left(\Delta_t^2 G^T Q_i G + \frac{1}{2}\Delta_t I\right) = L L^*,$$

is the Cholesky decomposition. Using the notation in (17)–(20), noticing that $\alpha = \Theta_{1_i} x + \Theta_{2_i}$, and replacing (22) in (12) we obtain

$$\begin{aligned}
J_{\Delta_t}(x, i+1) = \int_{\mathbb{R}^m} & e^{-\frac{\left\|L^* w + \Delta_t L^{-1} G^T Q_i^T \alpha\right\|^2}{\varepsilon}} dw \\
&+ \alpha^T \left(Q_i - Q_i G U_i^{-1} \Delta_t^2 G^T Q_i^T\right) \alpha \\
&+ \|x\|^2_{\frac{1}{2}\Delta_t h_x^T(\hat{x}_i) h_x(\hat{x}_i)} - \Delta_t y^T h_x(\hat{x}_i) x + cte.
\end{aligned}$$

where the integrant term represents a Gaussian distribution with suitable parameters, and therefore the integral assumes a constant value. Rearranging the terms to provide symmetric form yields

$$\begin{aligned}
J_{\Delta_t}(x, i{+}1) = &\left\|x + \frac{1}{2} Q_i^{-1} \left(2\Theta_{2_i}^T \Lambda_i \Theta_{1_i} - \Delta_t y^T h_x(\hat{x}_i)\right)^T\right\|^2_{Q_i} \\
&- \frac{1}{4}\left(2\Theta_{2_i}^T \Lambda \Theta_{1_i} - \Delta_t y^T h_x(\hat{x}_i)\right) \\
&\quad Q_i^{-1} \left(2\Theta_{2_i}^T \Lambda_i \Theta_{1_i} - \Delta_t y^T h_x(\hat{x}_i)\right)^T \\
&+ \Theta_{2_i}^T \Lambda_i \Theta_{2_i} + cte.
\end{aligned}$$

Differentiating twice with respect to x we arrive at

$$(J_{\Delta_t})_{xx} = \left(\left\|x + \frac{1}{2} Q_i^{-1} \left(2\Theta_{2_i}^T \Lambda_i \Theta_{1_i} - \Delta_t y^T h_x(\hat{x}_i)\right)^T\right\|^2_{Q_i}\right)_{xx}$$

which ends the proof. ∎

B. Second order approximation

In this section we improve our method by performing a second order approximation. In this case $f(x)$ is replaced by

$$f(x) = f(\hat{x}_i)+f_x(\hat{x}_i)(x-\hat{x}_i)+\frac{1}{2}(x-\hat{x}_i)^T f_{xx}(\hat{x}_i)(x-\hat{x}_i)$$

Before we proceed, the following elementary lemma is needed.

Lemma 1: The integral

$$\int\limits_{\mathbb{R}^m} e^{-\frac{\mathrm{M}+\mathrm{T}}{\varepsilon}} dw_1 ... dw_m, \tag{23}$$

where

$$\begin{aligned} \mathrm{M} &= (w+c_2)^T P (w+c_2), \\ \mathrm{T} &= \Lambda^T W \Lambda, \\ \Lambda &= (w+c_1)^T f_{xx}(\hat{x})(w+c_1), \end{aligned}$$

$\varepsilon > 0$, $W, P \in \mathbb{R}^{m\times m}$ are positive definite matrices, and c_j are suitable constant vectors, is absolutely convergent.

Proof: Since W is positive definite, T is bounded from below. Since P is positive definite, the term M has quadratic growth as $|w| \to \infty$, which ensures convergence. ■

Theorem 2: Consider system (13) with f replaced by the second order approximation and set the initial condition of the value function $J_{\Delta_t}(x, 0)$ as

$$J_{\Delta_t}(x,0) = \|x-\mu_0\|^2_{Q_0},$$

where $\mu_0 = \hat{x}_0$ and Q_0 is a positive definite matrix. Then, the cost function $J_{\Delta_t}(x, i)$ at each time step $i \geq 1$ is of order four and its Hessian $(J_{\Delta_t})_{xx}(x, i)$ can be computed using the following iterative method. For $i = 1$,

$$\begin{aligned} (J_{\Delta_t})_{xx}(x,1) = \Big(&\alpha_i^T \left(Q_0 - \Delta_t^2 Q_0 G U^{-1} G^T Q_0^T\right)\alpha_i \\ &+ \frac{1}{2}\Delta_t x^T h_x^T(\hat{x}_i) h_x(\hat{x}_i) x\Big)_{xx}, \end{aligned}$$

where

$$U = \left(\Delta_t^2 G^T Q_0 G + \frac{1}{2}\Delta_t I\right),$$

and α_i is given by

$$\begin{aligned} \alpha_i = x &+ \Delta_t \left(f(\hat{x}_i) + f_x(\hat{x}_i)^T (x-\hat{x}_i)\right) \\ &+ \Delta_t \left(\frac{1}{2}(x-\hat{x}_i)^T f_{xx}(\hat{x}_i)(x-\hat{x}_i)\right). \end{aligned}$$

For $i \geq 2$,

$$(J_{\Delta_t})_{xx}(x,i) = \left(\Xi_i^T \Omega_i \Xi_i + \frac{1}{2}\Delta_t x^T h_x^T(\hat{x}_i) h_x(\hat{x}_i) x\right)_{xx},$$

where

$$\begin{aligned} \Xi_i &= \alpha_i + \\ &\left(\frac{1}{2}\Omega_i^{-1}\left(\Delta_t y^T h_x(\hat{x}_i) - \frac{1}{2}\Delta_t^4 y^T h_x(\hat{x}_i) G G^T C^T h_x(\hat{x}_i)\right)\right)^T \end{aligned}$$

$$\begin{aligned} \Omega_i = &\frac{1}{2}\Delta_t h_x^T(\hat{x}_i) h_x(\hat{x}_i) \\ &- \frac{1}{4}\Delta_t^4 h_x^T(\hat{x}_i) h_x(\hat{x}_i) G G^T h_x^T(\hat{x}_i) h_x(\hat{x}_i). \end{aligned}$$

Proof: [outline] The proof follows by induction, using the same techniques as in the Theorem 1. However, here we do not identify the distribution as a Gaussian, but apply Lemma 1 to obtain the constant term. See [17] for a detailed proof. ■

IV. Simulation Results

The performance of the entropy penalized minimum energy estimator (EPMEE) is now evaluated through two examples similar to those presented in [14]. The first example is an one dimensional system and the second one is a two dimensional system. In both cases we illustrate the first and second order approximation and compare the performance with the EKF. In the following, $e_i = \hat{x}_i - x_i$ denotes the estimation error in the coordinate i and $|e|$ the absolute error.

A. One dimensional example

Consider the system

$$\begin{aligned} \dot{x} &= 0.1x^2 - x \\ y &= x \end{aligned}$$

where we set the initial conditions $x_0 = 2$, $\hat{x}_0 = 0.1$, $Q_0 = 1$, and a time step of $\Delta_t = 0.1$. For the EKF, we used the same initial condition, $P_0 = 0.1419$ (generated randomly) and $Q = R = 10$.

Figure 1 shows the time evolution of the estimation errors of the EPMEE using a first order approximation and the EKF. Note that contrary to the EPMEE estimator, the state estimate of the EKF do not converge to the correct value. Figure 2 illustrates the case for the EPMEE using a second order approximation. The absolute value of e is displayed in Fig. 3 and 4. From these figures, one can see that the convergence of the second-order EPMEE is faster.

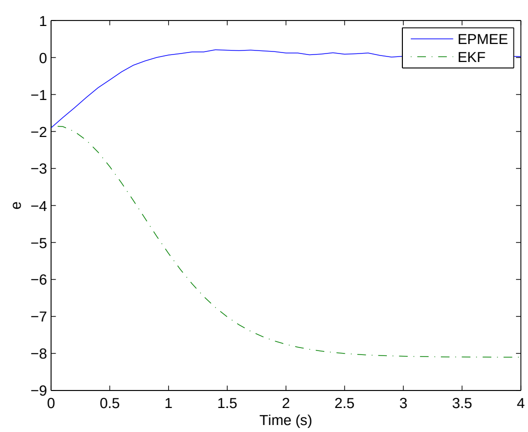

Fig. 1. 1D example - First Order Approximation Error

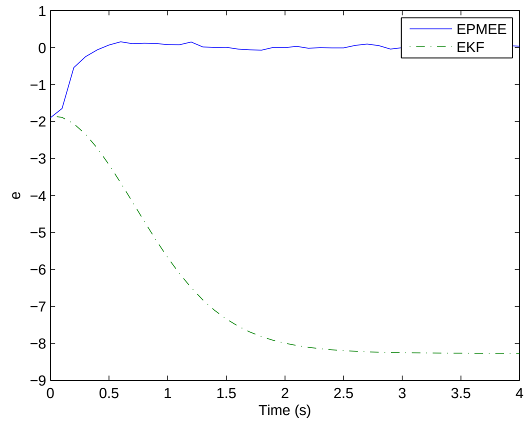

Fig. 2. 1D example - Second Order Approximation Error

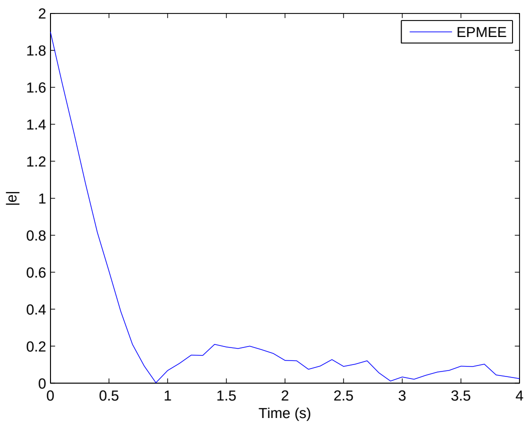

Fig. 3. 1D example - Absolute EPMEE First Approximation Error

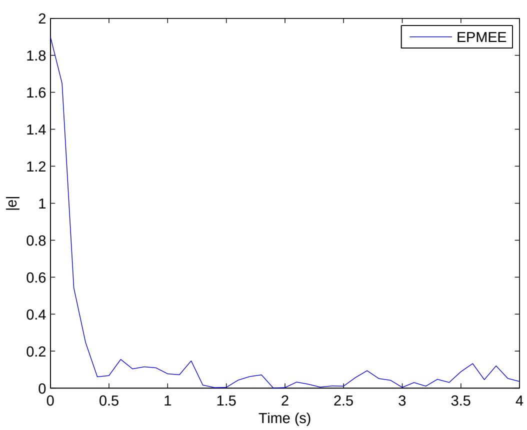

Fig. 4. 1D example - Absolute EPMEE Second Approximation Error

B. Two dimensional example

We now present the simulation results for the second-order highly nonlinear system

$$\begin{aligned} \dot{x}_1 &= -4x_1 - x_2 - (x_1^2 + x_2^2)^2 x_1 \\ \dot{x}_2 &= x_1 - 4x_2 - (x_1^2 + x_2^2)^2 x_2 \\ y &= [x_1, x_2]^T \end{aligned} \tag{24}$$

In this simulation, the initial condition for both estimators is $\hat{x}_0 = (0.1, 0.1)$ while the real system starts at $x_0 = (2, 2)$, $Q_0 = diag([1\ 1])$ and the time step is $\Delta_t = 0.1$. For the EKF, we used $P_0 = [0.1023\ \ 0.0012; 0.012\ \ 0.0212]$ (generated randomly) and $Q = R = diag([0.1\ 0.1])$. Figures 5-8 display the results obtained, which agree with the one-dimensional case.

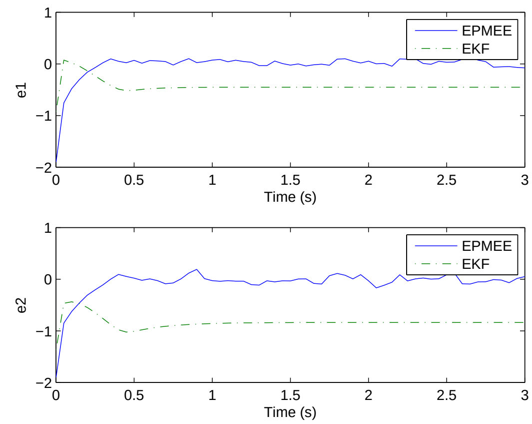

Fig. 5. 2D example - First Order Approximation Error

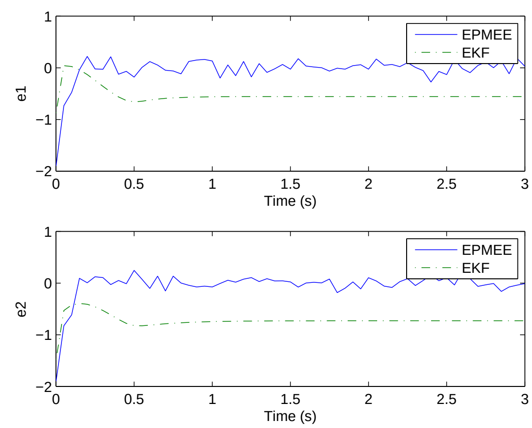

Fig. 6. 2D example - Second Order Approximation Error

The MEE is a particular case of the H^∞ observer, see for instance [18]. To provide an insight about the robustness of the EPMEE, Fig. 9 shows the case when there exist uniformly distributed measurement errors in the interval $[-0.5, 0.5]$. As one can see, even in the presence of significant noise the EPMEE do not diverge and the estimation errors converges to a small neighborhood of zero.

C. Discussion of Results

The EPMEE derived in this paper has the desired feature of being iterative and filtering-like. It continuously improves the state estimate as more measurements arise. In fact, it resembles the EKF (see expression (7)). However, there is a main diference. In the EKF, it is the Riccati equation

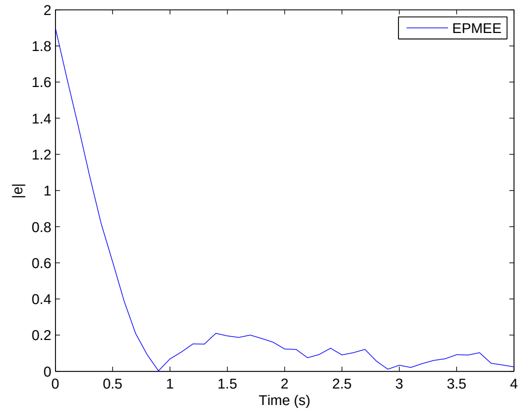

Fig. 7. 2D example - Absolute EPMEE First Approximation Error

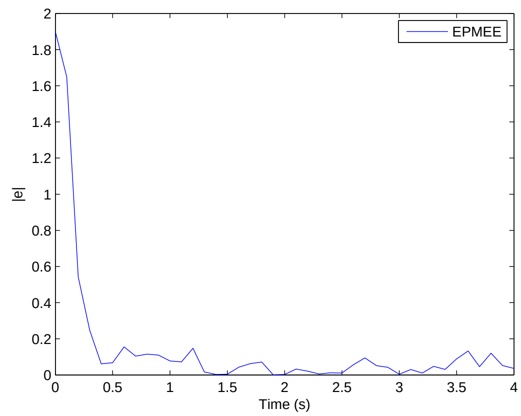

Fig. 8. 2D example - Absolute EPMEE Second Approximation Error

that provides an estimative of the evolution of the value function, which may give poor results if the system is highly nonlinear. In approach proposed in the paper, the fact that the system is nonlinear is taken into consideration by computing an estimate of the evolution of an approximation of the viscosity solution. The simulations show that the EPMEE is less sensitive to the nonlinear nature of the plant compared with the EKF. The drawback is that the convergence speed of the EPMEE in some cases is slightly slower than the EKF (when this one converge).

V. Conclusions and Further Research

We have considered the state estimation problem of nonlinear systems using a minimum energy estimator approach combined with an entropy penalized scheme to approximate the viscosity solution of the Hamilton-Jacobi equation. We derived a computationally efficient procedure to estimate the state and the Hessian of the value function for the first and even second order approximations of the state equation and linearized output equation. The simulation results illustrate and contrast the good performance of our algorithms compared with the EKF. Preliminary convergence results of the EPMEE can be found in [17]. Future work will address

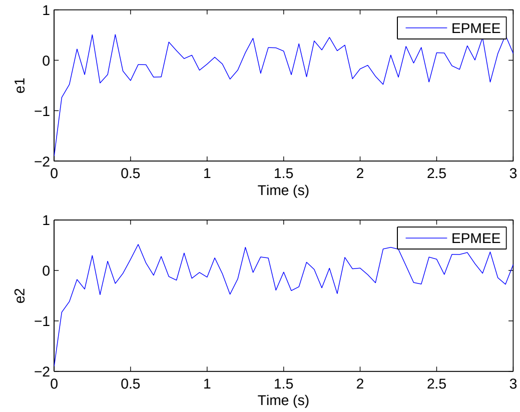

Fig. 9. 2D example - Second Order Approximation Error with Uniform Errors

the development of new techniques to compute the value function using Monte Carlo approximation methods for the entropy penalized method.

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