Minimum Cost Input-Output and Control Configuration Selection: A Structural Systems Approach

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Abstract— **In this paper we provide solutions to two different (but related) design problems involving large-scale linear dynamical systems: 1) the optimal input/output structural design ensuring structural controllability/observability and incurring in the minimal cost under generic assumptions; and 2) the optimal structural control configuration design for decentralized control, i.e., the sparsest information pattern or the minimal communication between outputs and inputs, such that the closed-loop system has no structurally fixed modes and incurring in the minimal cost under the assumption that the communication devices have the same cost. We show that the proposed solution can be implemented efficiently, i.e., using an algorithm with polynomial time complexity in the number of the state variables. We illustrate the obtained results with an example.**

I. INTRODUCTION

In complex large-scale systems, like for example power electric grids, chemical plants or multi-agent networks, to name a few, the first steps of control systems design consist in addressing the following questions [1]:

Q1 Which subset of variables need to be manipulated to ensure structural controllability?

Q2 Which subset of variables need to be measured to ensure structural observability?

Q3 What is the *information structure* required? (i.e., which feedback links should be made between the measured outputs and the control inputs?).

The subsets of variables satisfying the requirement in Q1-Q2 we refer to as *feasible dedicated input configuration* and *feasible dedicated output configuration*, respectively. In other words if a *dedicated input* (i.e., an input that manipulates a single state variable) is assigned to each of the variables in the feasible dedicated input configuration, then the system is structurally controllable[1]. Similar reasonings applies to the dedicated feasible output configuration, using the notion of *dedicated output* and structurally observability. We also address the case where one input (resp. output) actuates (resp. measures) several state variables. Once the structural requirements are ensured, the next step is the control configuration selection problem (question Q3). To this end, it is important to consider that in real physical systems, manipulating and measuring different types of state variables can incur in different costs. In addition, there is also the associated communication cost, which is the cost of collecting the measurements and forward them to the corresponding local controllers/actuators, which can be done, for instance, by a transmitter/receiver device or wired communication. This paper proposes a unified solution to structural control systems design composed by Q1-Q3 and where we take into account the cost of manipulating and measuring different state variables, as well as the communication cost. This problem is solved under the following mild assumptions (that hold in a large class of physical systems):

A1 The manipulating (respectively measuring) cost associated to each state variable is independent of the actuator (respectively sensor) performing the task;

A2 The communication cost between any pair comprising a sensor and an actuator is constant regardless of the pair considered.

More specifically, in this paper we address the following problem:

Problem Statement

Consider a given (possible large-scale) plant (without yet inputs and outputs) model by

$$\dot{x} = Ax$$

where $x \in \mathbb{R}^n$ is the state. Let $\bar{A} \in \{0,1\}^{n\times n}$ be the binary matrix that represents the structural pattern of A. Find efficiently (polynomial time complexity in the number of state variables) the triple $(\bar{B}, \bar{K}, \bar{C})$ that solves the following optimization problem

$$\begin{aligned} \min_{\bar{B},\bar{C},\bar{K}\in\{0,1\}^{n\times n}} \quad & \|\bar{B}\|_{\mathcal{C}^I} + \|\bar{C}\|_{\mathcal{C}^O} + \|\bar{K}\|_{\mathcal{C}^C} \\ \text{s.t.} \quad & (\bar{A}, \bar{B}, \bar{K}, \bar{C}) \text{ has no SFMs} \end{aligned} \qquad (1)$$

where $\|\bar{M}\|_P = \sum_{i,j=1}^{n} \bar{M}_{ij}P_{ij}$ with $\bar{M}_{ij} \in \{0,1\}$, and SFM stands for structurally fixed modes (see Section II for a formal definition). In (1), $\mathcal{C}^I, \mathcal{C}^O, \mathcal{C}^C$ are $n \times n$ matrices that denote the costs of manipulating state variables, measuring state variables and communicating between outputs and inputs, respectively. More precisely, $\mathcal{C}^I_{ij}$ represents the cost to manipulate the state variable i by the actuator j, $\mathcal{C}^O_{ij}$ represents the cost to measure a state variable i by sensor j, and $\mathcal{C}^C_{ij}$ represent the cost of establishing a communication link from a sensor j to an actuator i. Note that assumptions A1-A2 impose the following restrictions: $\mathcal{C}^I$ has its ith row with elements with the same (positive) constant value $\mathcal{C}^I(x_i)$. Similarly, jth column has its elements with the same (positive) constant value $\mathcal{C}^O(x_j)$ and $\mathcal{C}^C$ a matrix with all elements with equal values. Finally, notice that some columns of $\bar{B}, \bar{C}$ can be composed only by zeros, in which case such input/output can be disregarded, leading to the notion of *effective inputs/outputs*, i.e., those that actuate/measure at least one state variable.

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[1]A pair (A, B) is said to be structurally controllable if there exists a pair (A', B') with the same structure as (A, B), i.e., same locations of zeroes and non-zeroes, such that (A', B') is controllable.

Due to the combinatorial nature of the problem in (1), a common strategy (see for instance [2]) is to go through a suboptimal solution that consists in solving independently 1) the input/output selection problem and 2) the control configuration selection problem. Typically, approaches to solve 1) include suboptimal methods such as heuristics, genetic algorithms or relaxations, see for instance [2-6], and references therein. The closest work to the one presented here is the one in [7], where the optimal cost minimal feasible dedicated input/output configuration is found. On the other hand, very few works have considered the control configuration under cost constraints. The work in [8] is the closest to the work presented in this paper, in the sense that it considers the same constraint as in (1). In [8], a suboptimal approach is provided with a cost of at most twice of the optimal one.

The rest of the this paper is organized as follows. Section II reviews some concepts and introduces results in structural systems theory. Subsequently, in Section III we present the main technical results (proofs are omitted due to space constraints), followed by an illustrative example in Section IV. Conclusions and discussion avenues for further research are presented in Section V.

II. PRELIMINARIES AND TERMINOLOGY

The following standard terminology and notions from graph theory can be found, for instance in [9]. In particular, let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ be the digraph representation of $\bar{A}$ in (1), where the vertex set $\mathcal{X}$ represents the set of state variables (also referred to as state vertices) and $\mathcal{E}_{\mathcal{X},\mathcal{X}} = \{(x_i, x_j) : A_{ji} \neq 0\}$ denotes the set of edges. Similarly, we define the following digraphs: $\mathcal{D}(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$ where $\mathcal{U}$ represents the set of input vertices and $\mathcal{E}_{\mathcal{U},\mathcal{X}} = \{(u_i, x_j) : \bar{B}_{ji} \neq 0\}$; $\mathcal{D}(\bar{A}, \bar{C}) = (\mathcal{X} \cup \mathcal{Y}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{X},\mathcal{Y}})$ where $\mathcal{Y}$ represents the set of output vertices and $\mathcal{E}_{\mathcal{X},\mathcal{Y}} = \{(x_i, y_j) : \bar{C}_{ji} \neq 0\}$; and $\mathcal{D}(\bar{A}, \bar{B}, \bar{K}, \bar{C}) = (\mathcal{X} \cup \mathcal{U} \cup \mathcal{Y}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{X},\mathcal{Y}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}} \cup \mathcal{E}_{\mathcal{Y},\mathcal{U}})$ denotes the digraph associated with the closed-loop system and the set of feedback edges/links is given by $\mathcal{E}_{\mathcal{Y},\mathcal{U}} = \{(y_i, u_j) : \bar{K}_{ji} \neq 0\}$.

A digraph $\mathcal{D}_s = (V_s, E_s)$ with $V_s \subset V$ and $E_s \subset E$ is called a *subgraph* of $\mathcal{D}$. If $V_s = V$, $\mathcal{D}_s$ is said to *span* $\mathcal{D}$. A sequence of edges $\{(v_1, v_2), (v_2, v_3), \cdots, (v_{k-1}, v_k)\}$, in which all the vertices are distinct, is called *an elementary path* from v_1 to v_k. When v_k coincides with v_1, the sequence is called a *cycle*.

In addition, we will require the following graph theoretic notions [10]: A digraph $\mathcal{D}$ is said to be strongly connected if there exists a directed path between any two pairs of vertices. A strongly connected component (SCC) is a maximal subgraph $\mathcal{D}_S = (V_S, E_S)$ of $\mathcal{D}$ such that for every $v, w \in V_S$ there exists a path from v to w and from w to v. Note that, an SCC may have several paths between two vertices and the path from v to w may comprise some vertices not in the path from w to v. Visualizing each SCC as a virtual node (or supernode), one may generate a *directed acyclic graph* (DAG), in which each node corresponds to a single SCC and a directed edge exists between two SCCs *iff* there exists a directed edge connecting the corresponding SCCs in the original digraph. The DAG associated with $\mathcal{D} = (V, E)$ may be efficiently generated in $\mathcal{O}(|V| + |E|)$ [10], where $|V|$ and $|E|$ denote the number of vertices in V and the number of edges in E, respectively. In the DAG representation, we refer to an SCC that has no incoming edge from any state in a different SCC as a *non-top linked SCC* and, similarly, we have a *non-bottom linked SCC* if the SCC does not have an edge from its states to the states of another SCC.

For any two vertex sets $S_1, S_2 \subset V$, we define the *bipartite graph* $\mathcal{B}(S_1, S_2, E_{S_1,S_2})$ associated with $D = (V, E)$, to be a directed graph (bipartite), whose vertex set is given by $S_1 \cup S_2$ and the edge set E_{S_1,S_2} by $E_{S_1,S_2} = \{(s_1, s_2) \in E : s_1 \in S_1, s_2 \in S_2\}$.

Given $\mathcal{B}(S_1, S_2, E_{S_1,S_2})$, a matching M corresponds to a subset of edges in E_{S_1,S_2} that do not share vertices, i.e., given edges $e = (s_1, s_2)$ and $e' = (s_1', s_2')$ with $s_1, s_1' \in S_1$ and $s_2, s_2' \in S_2$, $e, e' \in M$ only if $s_1 \neq s_1'$ and $s_2 \neq s_2'$. A maximum matching M^* may then be defined as a matching M that has the largest number of edges among all possible matchings. The maximum matching problem may be solved efficiently in $\mathcal{O}(\sqrt{|S_1 \cup S_2|}|E_{S_1,S_2}|)$ [10]. Vertices in S_1 and S_2 are *matched vertices* if they belong to an edge in the maximum matching M^*, otherwise, we designate the vertices as *unmatched vertices*. If there are no unmatched vertices, we say that we have a *perfect match*. It is to be noted that a maximum matching M^* may not be unique.

For ease of referencing, in the sequel, the term *right-unmatched vertices* (w.r.t. $\mathcal{B}(S_1, S_2, E_{S_1,S_2})$ and a maximum matching M^*) will refer to only those vertices in S_2 that do not belong to a matched edge in M^*.

Now, consider the linear time invariant (LTI) system

$$\dot{x} = Ax + Bu, \quad y = Cx. \tag{2}$$

Let $\bar{K}$ denote an *information pattern*, i.e., $\bar{K}_{ij} = 1$ if the sensor j is available to actuator i, and zero otherwise and denote by $[\bar{M}] = \{M : M_{ij} = 0 \text{ iff } \bar{M}_{ij} = 0\}$ an equivalence class of matrices of appropriate dimensions. The set of fixed modes of the closed-loop system (2) w.r.t. an information pattern $\bar{K}$ is given by $\sigma_{\bar{K}} = \bigcap_{K \in [\bar{K}]} \sigma(A + BKC)$ see [11]), where $\sigma(M)$ denotes the set of eigenvalues of the matrix M. It is known that (see [11]) if, for a non-empty symmetric open set $\mathcal{W} \subset \mathbb{C}$, $\sigma_{\bar{K}} \subset \mathcal{W}$, then there exists a gain $K \in [\bar{K}]$ such that all the eigenvalues (also known as the poles) of the closed-loop system $A + BKC$ are in $\mathcal{W}$.

The structural version of fixed modes was introduced in [12] which, essentially, are the fixed modes attributed to the structural pattern, i.e., location of zeroes and non-zeroes, of a system, as opposed to fixed modes that originate from a *perfect matching* of the numerical parameters. Specifically, a structural linear system $(\bar{A}, \bar{B}, \bar{C})$ is said to have structurally fixed modes (SFMs) with respect to (w.r.t.) an information pattern $\bar{K}$ if, for all $A \in [\bar{A}]$, $B \in [\bar{B}]$, $C \in [\bar{C}]$, we have $\bigcap_{K \in [\bar{K}]} \sigma(A + BKC) \neq \emptyset$.

Conversely, a structural system $(\bar{A}, \bar{B}, \bar{C})$ has no structurally fixed modes w.r.t. $\mathcal{K}$ if there exists at least one instantiation $A \in [\bar{A}]$, $B \in [\bar{B}]$, $C \in [\bar{C}]$ which has no fixed modes, i.e., $\cap_{K \in [\bar{K}]} \sigma(A + BKC) = \emptyset$. In this latter case, it may be shown (see [13]) that almost all systems in the sparsity class $(\bar{A}, \bar{B}, \bar{C})$ have no fixed modes, and, hence, allow pole-placement arbitrarily close to any pre-specified set of eigenvalues. This also justifies our constraint of designing systems with no SFMs in problem (1).

The following graphical condition for a structured LTI system to be SFM free is provided in [14], where we assume that the system is both structurally controllable and observable.

Theorem 1 ([14]): The structural system $(\bar{A}, \bar{B}, \bar{C})$ associated with (2) has no structurally fixed modes with respect to an information pattern $\bar{K}$, if and only if both of the following two conditions hold:

a) each state vertex $x \in \mathcal{X}$ is contained in a strongly connected component of $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ which includes an edge of $\mathcal{E}_{\mathcal{Y},\mathcal{U}}$;

b) there exists a finite disjoint union of cycles $\mathcal{C}_k = (\mathcal{V}_k, \mathcal{E}_k)$ (subgraph of $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$) with $k \in \mathbb{N}$ such that $\mathcal{X} \subset \bigcup_k \mathcal{V}_k$. □

We will also require the following general results on structural control design from [9] (see also [15]).

Theorem 2: Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ denote the system digraph and $\mathcal{B} \equiv \mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ its bipartite representation. Let $\mathcal{S}_u \subset \mathcal{X}$, then the following statements are equivalent:

1) The set $\mathcal{S}_u$ is a feasible dedicated input configuration;

2) There exists a subset $\mathcal{U}_R \subset \mathcal{S}_u$ corresponding to the set of right-unmatched vertices of some maximum matching of $\mathcal{B}$, and a subset $\mathcal{A}_u \subset \mathcal{S}_u$ comprising one state variable from each non-top linked SCC of $\mathcal{D}(\bar{A})$. □

By duality, the result can be stated to *feasible dedicated output configurations* in terms of the left-unmatched vertices of a graph and its non-bottom linked SCCs (see [9] for details).

The following results on general non-dedicated structural input/output and control configuration design were obtained in [9]. To ease the presentation, we denote by m the number of right/left-unmatched vertices in any maximum matching of $\mathcal{B}$, β, the number of non-top linked SCCs in $\mathcal{D}(\bar{A})$ and β', the number of non-bottom linked SCCs in $\mathcal{D}(\bar{A})$.

R1 Construction of feasible $\bar{B}$ and $\bar{C}$: A pair $(\bar{A}, \bar{B})$ is structurally controllable if and only if there exists a maximum matching of $\mathcal{B}$ with a set of right-unmatched vertices $\mathcal{U}_R$, such that, $\bar{B}$ has (at least) m non-zero entries, one in each of the rows corresponds to the different state variables in $\mathcal{U}_R$ and located at different columns, and (at least) β non-zero entries, each of which belongs to a row (state variable) corresponding to a distinct non-top linked SCC and located in arbitrary columns.

We emphasize that the second set of β non-zero entries may share columns with the first set of m non-zero entries. Also, as a direct consequence of R1, we obtain that any $\bar{B}$, such that $(\bar{A}, \bar{B})$ is structurally controllable, must have at least m distinct non-zero columns (or m distinct control inputs). Similarly, a pair $(\bar{A}, \bar{C})$ is structurally observable if and only if there exists a maximum matching of $\mathcal{B}$ with a set of left-unmatched vertices $\mathcal{U}_L$, such that, $\bar{C}$ has (at least) m non-zero entries, one in each of the columns corresponds to the different state variables in $\mathcal{U}_L$ and located at different rows, and (at least) β' non-zero entries, each of which belongs to a column (state variable) corresponding to a distinct non-bottom linked SCC and located in arbitrary rows.

R2 Separation principle of structural input/output selection: For any two distinct maximum matchings M^* and $M^{*\prime}$, there exists a (common) maximum matching M^o such that the set of right-unmatched vertices of M^o coincide with those of M^* and the set of left-unmatched vertices of M^o coincide with those of $M^{*\prime}$.

As will be shown later, R2 allows us to design $\bar{B}$ and $\bar{C}$ *independently* as far as the solution of (1) is concerned. Finally, we list the following two results on the feasibility of information patterns:

R3 Let $\bar{B}$ and $\bar{C}$ be feasible input and output matrices, and by R1-R2, let M^* be a (common) maximum matching, such that $\bar{B}$ and $\bar{C}$ have a correspondence (in the sense of R1) with the set $\mathcal{U}_R$ of right and the set $\mathcal{U}_L$ of left-unmatched vertices of M^* respectively. Then, for any information pattern $\bar{K}$, which consists of non-zero entries corresponding to a disjoint (but arbitrary) pairing of the state variables in $\mathcal{U}_R$ and $\mathcal{U}_L$, the digraph $\mathcal{D}(\bar{A}, \bar{B}, \bar{C}, \bar{K})$ satisfies condition b) of Theorem 1. Additionally, a subset of those pairings also ensures condition a) of Theorem 1. A pairing satisfying both conditions of Theorem 1 is referred to as *mix-pairing*.

Remark 1: Notice that a mix-pairing can be efficiently obtained, and incurs on $|\mathcal{U}_R|$ (or equivalently $|\mathcal{U}_L|$) feedback edges. ⋄

III. MAIN RESULTS

In this section we present the main results of the paper, and provide a solution to problem (1). The key idea consists of using the separation principle R2, which allows us to *independently* design optimal structurally controllable and observable input and output configurations respectively (achieved in Lemma 1-2 and Theorems 3). Subsequently, the resulting optimal $\bar{B}$ and $\bar{C}$ are used to identify the sparsest information pattern $\bar{K}$ and the solution of problem (1) (achieved in Theorem 5). Through a series of intermediate results, we will show that the solution of (1) can be obtained by performing the following steps:

S1 From the bipartite graph representation of system structure, determine, in the class of all possible maximum matchings, a maximum matching (and the corresponding set of right-unmatched vertices $\mathcal{U}_R^*$) that incurs in the minimum sum-cost of actuating each of the state variables in $\mathcal{U}_R^*$;

S2 From the class of all subsets $\mathcal{A}_u$'s of state variables each of which consists of at least one state variable in each non-top linked SCC, determine a subset $\mathcal{A}_u^*$ that incurs in the minimum sum-cost of actuating the corresponding variables.

Note that since the actuating costs are positive, the set $\mathcal{A}_u^*$ obtained in S2 will consist only of a single state variable from each non-top linked SCCs. Further, note that by Theorem 2, the set $\mathcal{U}_R^* \cup \mathcal{A}_u^*$ is a feasible dedicated input configuration. Both S1-S2 may be implemented using Algorithm 1. Next,

S3 Verify if there exists a maximum matching of the system bipartite graph $\mathcal{B}$ (with associated set of right-unmatched vertices $\mathcal{U}_R^{\text{opt}}$) and a set $\mathcal{A}_u^{\text{opt}}$ of state variables, such that $\mathcal{S}_R^{\text{opt}} = \mathcal{U}_R^{\text{opt}} \cup \mathcal{A}_u^{\text{opt}}$ is a feasible dedicated input configuration, and the sum-cost of actuating each of the state variables in $\mathcal{S}_R^{\text{opt}}$ is strictly smaller than the sum-cost of actuating each of the state variables in the feasible dedicated input configuration $\mathcal{S}_R^* = \mathcal{U}_R^* \cup \mathcal{A}_u^*$.

Step S3 may be implemented by Algorithm 2.

S4 From the sets $\mathcal{U}_R^{\text{opt}}$ and $\mathcal{A}_u^{\text{opt}}$ obtained in S3, we may design (see Theorem 3) a feasible $\bar{B}$ such that $\bar{B}$ is minimal (in the sense of the cost $\|\bar{B}\|_{C^I}$) in the class of all input matrices $\bar{B}$ such that $(\bar{A}, \bar{B})$ is structurally controllable. Notice that, the $\bar{B}$ obtained above has minimum number of non-zero columns (i.e., effective inputs) in the class of all minimal $\bar{B}$'s (in the sense of the cost $\|\bar{B}\|_{C^I}$) that ensure structural controllability of the pair $(\bar{A}, \bar{B})$. By duality, the same reasoning applies to find the output matrix structure $\bar{C}$ (by the dual of Theorem 3 applied to the output design) such that $\|\bar{C}\|_{C^O}$ is minimum and has the minimum number of non-zero rows (i.e., effective outputs) among all possible feasible output matrix structures.

Finally, the optimal control configuration may be obtained by pairing the inputs and outputs obtained in Step S4 (formalized in Theorem 4), and briefly described as follows.

S5 Let $\bar{B}$ and $\bar{C}$ be obtained as in Step S4. Then, using R1+R2 it may be shown that the number of (non-zero) columns in $\bar{B}$ (i.e., the minimum number of inputs required to attain structural controllability) and the rows in $\bar{C}$ (i.e., the minimum number of outputs required to attain structural observability) are equal. Finally, using R3 and R4, we show that the triple $(\bar{B}, \bar{C}, \bar{K})$ is a solution to (1), where $\bar{K}$ is the information pattern that consists of a mix-pairing of the effective inputs and effective outputs in $\bar{B}$ and $\bar{C}$.

ALGORITHM 1: Determine the sets $\mathcal{U}_R^*$ and $\mathcal{A}_u^*$ in Steps S1 - S2.

Input: $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ and $\mathcal{C}^I$
Output: Minimal sum-cost set of right-unmatched vertices $\mathcal{U}_R^*$ (in the class of all possible maximum matchings of $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$), and $\mathcal{A}_u^*$, the minimum sum-cost set of state variables which consists of at least one variable in each of the non-top linked SCCs.

% consider one dedicated input for each state variable
$\mathcal{U} = \{u_1, \cdots, u_n\}$;
$\mathcal{E}_{\mathcal{U},\mathcal{X}} = \{(u_i, x_i) : i = 1, \cdots, n\}$;

(1) Compute the weighted bipartite graph $\mathcal{W} \equiv \mathcal{B}(\mathcal{X} \cup \mathcal{U}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$ where the weight w of each edge is given by $w(e_x) = 0$ for each $e_x \in \mathcal{E}_{\mathcal{X},\mathcal{X}}$ and $w(e_{u_i} \equiv (u_i, x_i)) = \mathcal{C}^I(x_i)$ for each $e_{u_i} \in \mathcal{E}_{\mathcal{U},\mathcal{X}}$;

Compute a minimum-weight maximum matching $M_{\mathcal{W}}$ of $\mathcal{W}$ and the set $\mathcal{U}_R^* = \{x_i \in \mathcal{X} \ : \ (u_i, x_i) \in M_{\mathcal{W}}\}$;

(2) Perform the DAG representation of $\mathcal{D}(\bar{A})$ and let $\mathcal{N}^j$ with $j = 1, \cdots, \beta$ denote its non-top linked SCCs;

For each non-top linked SCC $\mathcal{N}^j$ with $j = 1, \cdots, \beta$ find a state variable $x \in \mathcal{N}^j$ that incurs in the minimum $\mathcal{C}^I(x)$ and, let $\mathcal{A}_u^*$ be the collection of such state variables;

Due to the non-uniqueness of minimum-weight maximum matchings (see Algorithm 1-Step (1)), the set $\mathcal{U}_R^*$ may not be unique. Similarly, the minimization in Algorithm 1-Step (2) may have non-unique solutions $\mathcal{A}_u^*$.

We now show the correctness and complexity of Algorithm 1, i.e., the sets $\mathcal{U}_R^*$ and $\mathcal{A}_u^*$ obtained indeed satisfy the specifications in Steps S1 and S2, respectively.

Lemma 1: Let $\mathcal{B} \equiv \mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ be the bipartite graph representation of $\mathcal{D}(\bar{A})$ and consider the sets $\mathcal{U}_R^*,\ \mathcal{A}_u^* \subset \mathcal{X}$ given by Algorithm 1. Then:

(i) There exists a maximum matching M^* of $\mathcal{B}$, such that $\mathcal{U}_R^*$ corresponds to the set of its right-unmatched vertices.

(ii) The maximum matching M^* and the corresponding set $\mathcal{U}_R^*$ of right-unmatched vertices obtained above is minimal (in the sense of Step S1), i.e., for any other maximum matching M' of $\mathcal{B}$, the sum-cost of actuating its right-unmatched vertices cannot be smaller than the sum-cost of actuating the state variables in $\mathcal{U}_R^*$.

(iii) The set $\mathcal{A}_u^*$ is minimal as far as the requirement in Step 2 is concerned.

(iv) The complexity of implementing Algorithm 1 is $\mathcal{O}(|\mathcal{X}|^3)$. □

Note that, by Theorem 2 and Lemma 1, $\mathcal{S}^* = \mathcal{U}_R^* \cup \mathcal{A}_u^*$ is a feasible dedicated input configuration. A feasible dedicated input configuration that has the minimum sum-cost of actuating its state variables is referred to as a *minimum-cost feasible dedicated input configuration*. We present an intuitive justification that why $\mathcal{S}^*$ is not, in general, a minimum-cost feasible dedicated input configuration and the necessity of refining it through Algorithm 2. Note that, by Theorem 1, a minimum-cost feasible dedicated input configuration $\mathcal{S}^{\text{opt}}$ necessarily satisfies $\mathcal{S}^{\text{opt}} = \mathcal{U}_R^{\text{opt}} \cup \mathcal{A}_u^{\text{opt}}$, where $\mathcal{U}_R^{\text{opt}}$ corresponds to the set of right-unmatched vertices of some maximum matching of $\mathcal{B}$ and $\mathcal{A}^{\text{opt}}$ is a set of state variables which contains at least one state variable in each of the non-top linked SCCs of $\mathcal{D}(\bar{A})$. Then, by Lemma 1 it readily follows that $c(\mathcal{U}^{\text{opt}}) \geq c(\mathcal{U}_R^*)$ and $c(\mathcal{A}^{\text{opt}}) \geq c(\mathcal{A}^*)$, where $c(\cdot)$ is the element-wise sum actuation cost. However, the above is not sufficient to guarantee (as desired) that $c(\mathcal{S}^{\text{opt}}) \geq c(\mathcal{S}^*)$, because the sets $\mathcal{U}^{\text{opt}}$ and $\mathcal{A}^{\text{opt}}$ are not required to be disjoint. In a sense, separately optimizing over maximum matchings (the corresponding set of right-unmatched vertices) and state variables in non-top linked SCCs (as is done for $\mathcal{S}^*$) may not ensure optimality; in fact, more cost savings may be obtained by determining maximum matchings whose right-unmatched vertices are spread across the non-top linked SCCs. Algorithm 2 achieves this by essentially iteratively refining the sets $\mathcal{U}_R^*$ and $\mathcal{A}_u^*$, where at each iteration we check the possibility of replacing a pair of state variables in the (current) sets $\mathcal{U}_R^*$ and $\mathcal{A}_u^*$ by a single state variable to maximize the overlap. Specifically, at each iteration of Algorithm 2 (the outer loop indexed by k), we consider all possible variable pairs $(u, a) \in \mathcal{U}_R^* \times \mathcal{A}_u^*$ (the inner loops indexed by i and j in Algorithm 2) and, for each such pair (u, a), we determine if there exists a single variable v in the same non-top linked SCC such that $\mathcal{U}_R^* \cup \{v\} - \{u\}$ continues to be the set of right-unmatched vertices of a maximum matching of $\mathcal{B}$ and the cost of actuating $\{v\}$ is smaller than the joint cost of actuating the pair (u, a); after scanning through all such pairs, the replacement with the best cost savings is used to refine the (current) sets $\mathcal{U}_R^*$ and $\mathcal{A}_u^*$. An efficient way of obtaining the best refinement over all such pairs is achieved by setting up a weighted maximum matching problem between the current right-unmatched vertices and the state variables in the non-top linked SCCs. This is executed once per iteration of the main cycle, where the edge weights take into account the cost savings for replacing any pair with a single state variable. The sets $\mathcal{U}^{\text{opt}}$ and $\mathcal{A}_u^{\text{opt}}$ obtained after m iterations/refinements of the sets $\mathcal{U}_R^*$ and $\mathcal{A}_u^*$ lead to a minimum-cost feasible dedicated input configuration $\mathcal{S}^{\text{opt}} = \mathcal{U}_R^{\text{opt}} \cup \mathcal{A}_u^{\text{opt}}$ as shown in the following.

Lemma 2: Let $\mathcal{U}_R^{\text{opt}}$ and $\mathcal{A}_u^{\text{opt}}$ be given by Algorithm 2. Then $\mathcal{U}_R^{\text{opt}}$ is the set of right-unmatched vertices of some maximum matching of $\mathcal{B}$, the set $\mathcal{U}_R^{\text{opt}} \cup \mathcal{A}_R^{\text{opt}}$ contains at least one state variable in each non-top linked SCC, and $\mathcal{S}^{\text{opt}} = \mathcal{U}_R^{\text{opt}} \cup \mathcal{A}_u^{\text{opt}}$ is a feasible dedicated input configuration with minimum sum-cost. Furthermore, Algorithm 2 may be implemented with complexity in $\mathcal{O}(|\mathcal{X}|^4)$. □

Using output of Algorithm 2 and result R1, we perform step S4, i.e., construct feasible matrices $\bar{B}$ and $\bar{C}$ that incur in the minimal cost $\|\bar{B}\|_{\mathcal{C}^I}$ and $\|\bar{C}\|_{\mathcal{C}^O}$, respectively.

To this end, recall that $\mathcal{U}_R^{\text{opt}}$ is the set of m right-unmatched vertices associated with a maximum matching M^{opt} of $\mathcal{B}$. Further, note that, by the minimality of $\mathcal{S}^{\text{opt}}$ (see Lemma 2), the set $\mathcal{A}^{\text{opt}}$ consists of exactly k state variables belonging to k distinct non-top linked SCCs of $\mathcal{D}(A)$, where k is the number of non-top linked SCCs that do not contain a right unmatched vertex from $\mathcal{U}_R^{\text{opt}}$.

Theorem 3: Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ be the system digraph and, let $\mathcal{U}_R^{\text{opt}}$ and $\mathcal{A}_u^{\text{opt}}$ be the sets determined by Algorithm 2. Let $\sigma : \{1, \cdots, n\} \mapsto \{1, \cdots, n\}$ be a permutation

ALGORITHM 2: Determine $\mathcal{U}_R^{\text{opt}}$ and $\mathcal{A}_u^{\text{opt}}$

Input: $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$, $\mathcal{U}_R^* = \{x_{\sigma(1)}, \cdots, x_{\sigma(m)}\}$ and $\mathcal{A}_u^* = \{x_{\sigma(m+1)}, \cdots, x_{\sigma(m+\beta)}\}$ (Algorithm 1) and $\mathcal{C}^I$ (i.e., the cost of each actuator to manipulate a state variable)
Output: Sets $\mathcal{U}_R^{\text{opt}}$ and $\mathcal{A}_u^{\text{opt}}$
$\mathcal{U}_R^{\text{opt}} = \mathcal{U}_R^*$;
$\mathcal{A}_u^{\text{opt}} = \mathcal{A}_u^*$;
Compute the non-top linked SCC $\mathcal{N}^j$ with $j = 1, \cdots, \beta$ where $x_{\sigma(m+j)} \in \mathcal{N}^j$.
$\mathcal{N} = \{1, \cdots, \beta\}$;
$\mathcal{U} = \{1, \cdots, m\}$;
$M^* = \emptyset$; % Let M^* contains the edges of the optimal matching between right-unmatched vertices and non-top linked SCCs. Let $M^*[1]$ denote the set of indices of the left end of the edges and $M^*[2]$ denote the set of indices of the right end of the same edges.
for $k = 1, \cdots, m$ **do**
  $\mathcal{E}_{\mathcal{N},\mathcal{U}} = \emptyset$;
  % Record of which state variable is simultaneously used in a non-top linked SCC and a right-unmatched vertex
  $\mathcal{R} = \emptyset$; % each element is composed of a pair where the first entry is an edge in $\mathcal{N} \times \mathcal{U}$ and the second a state variable $x \in \mathcal{X}$
  for $i \in \mathcal{U}$ **do**
    for $j \in \mathcal{N}$ **do**
      Compute the set of
      $\Xi^{i,j} = \{x \in \mathcal{N}^j : \mathcal{U}_R^i(u) \equiv \{x\} \cup \mathcal{U}_R^{\text{opt}} - \{x_{\sigma(i)}\}$ is the set of right-unmatched vertices for some max. matching of $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})\}$;
      if $\Xi^{i,j} \neq \emptyset$ **then**
        Find $x^* = \arg\min_{x \in \Xi^{i,j}} \mathcal{C}^I(x)$;
        $\mathcal{E}_{\mathcal{N},\mathcal{U}} = \mathcal{E}_{\mathcal{N},\mathcal{U}} \cup \{(i,j)\}$;
        % Set the weight as
        $w((i,j)) = \mathcal{C}^I(x_{\sigma(i)}) + \mathcal{C}^I(x_{\sigma(m+j)}) - \mathcal{C}^I(x^*)$;
        $\mathcal{R} = \mathcal{R} \cup \{((i,j), x^*)\}$;
  Compute a maximum-weight maximum matching M associated with $\mathcal{B}(\mathcal{N} - M^*[1], \mathcal{U} - M^*[2], \mathcal{E}_{\mathcal{N},\mathcal{U}} - \{(m_1, .) : m_1 \in M^*[1]\} - \{(., m_2) : m_2 \in M^*[2]\})$.
  if $M = \emptyset$ **then**
    STOP and return $\mathcal{U}_R^{\text{opt}}, \mathcal{A}_u^{\text{opt}}$
  Find $e^* = (j^*, i^*)$ where $e^* = \arg\max_{e \in M} w(e)$;
  if $w(e^*) < 0$ **then**
    STOP and return $\mathcal{U}_R^{\text{opt}}, \mathcal{A}_u^{\text{opt}}$
  $\mathcal{A}_u^{\text{opt}} = \mathcal{A}_u^{\text{opt}} - \{x_{\sigma(m+j^*)}\}$;
  % Denote by $\mathcal{R}_{(j^*,i^*)}$ the second entry of the pair in $\mathcal{R}$, the first entry being (j^*, i^*);
  $\mathcal{U}_R^{\text{opt}} = \{\mathcal{R}_{(j^*,i^*)}\} \cup \mathcal{U}_R^{\text{opt}} - \{x_{\sigma(i^*)}\}$;

of the indices such that $\mathcal{U}_R^{\text{opt}} = \{x_{\sigma(1)}, \cdots, x_{\sigma(m)}\}$ and $\mathcal{A}_u^{\text{opt}} = \{x_{\sigma(m+1)}, \cdots, x_{\sigma(m+k)}\}$. Then, if $\bar{B} \in \{0,1\}^{n \times n}$ is constructed as

a) exactly one entry in each row of $\bar{B}$ indexed by $\sigma(1), \cdots, \sigma(m)$ must be 1 and placed in different columns;

b) exactly one entry in each row of $\bar{B}$ indexed by $\sigma(m+1), \cdots, \sigma(m+k)$ must be 1 and placed in the columns that were filled in a),

then $\bar{B}$ is a minimum (in the sense of the cost $\|\bar{B}\|_{C^I}$) feasible (i.e., the pair $(\bar{A}, \bar{B})$ is structurally controllable) structural input matrix with the minimum number of non-zero columns (i.e., corresponding to distinct control inputs).

Conversely, if $\bar{B}$ is a minimum feasible structural input matrix with the minimum number of non-zero columns, then there exist subsets $\mathcal{U}_R$ and $\mathcal{A}_u$ (not necessarily disjoint) of state variables, such that, $\mathcal{U}_R$ corresponds to the set of right unmatched vertices of a maximum matching of $\mathcal{B}$, $\mathcal{A}_u$ consists of exactly one state variable from each of the non-top linked SCCs of $\mathcal{D}(\bar{A})$ that do not contain vertices in $\mathcal{U}_R$ and $\mathcal{S}_u = \mathcal{U}_R \cup \mathcal{A}_u$ is a minimum cost feasible dedicated input configuration from which $\bar{B}$ may be recovered using steps a) and b). □

By duality we can obtain $\bar{C}$ incurring in the minimum-cost, which we simply referred to as the dual of Theorem 3 applied to the output design.

Remark that by Lemma 1 and Lemma 2, the feasible input/output constructions in Theorem 3 and its dual applied to the output design can be efficiently implemented with polynomial complexity. The same reasoning applies to our final and main result, which uses Theorem 3 and its dual applied to the output design to construct the solution to (1).

Theorem 4: Let the sets $\mathcal{S}_u^{\text{opt}}$, $\mathcal{S}_y^{\text{opt}}$, $\mathcal{U}_R^{\text{opt}}$, $\mathcal{U}_L^{\text{opt}}$, $\mathcal{A}_u^{\text{opt}}$, $\mathcal{A}_y^{\text{opt}}$ be as in the hypotheses of Theorem 3 and its dual applied to the output design. In addition, let $\bar{B}$ and $\bar{C}$ be designed as stated in Theorem 3 and its dual applied to the output design, respectively, and $\bar{K}$ be constructed to be a mix-pairing between effective outputs and effective inputs. Then, $(\bar{B}, \bar{C}, \bar{K})$ is a solution to (1). □

IV. AN ILLUSTRATIVE EXAMPLE

Consider the example in Figure 1, in which the manipulating costs for each state variable is given by: $\mathcal{C}^I(x_1) = 3$, $\mathcal{C}^I(x_2) = 10$, $\mathcal{C}^I(x_3) = \infty$, $\mathcal{C}^I(x_4) = 1$, $\mathcal{C}^I(x_5) = 10$, $\mathcal{C}^I(x_6) = 5$, $\mathcal{C}^I(x_7) = 10$, $\mathcal{C}^I(x_8) = \infty$, $\mathcal{C}^I(x_9) = 5$, $\mathcal{C}^I(x_{10}) = 6$, $\mathcal{C}^I(x_{11}) = 10$, $\mathcal{C}^I(x_{12}) = 5$, $\mathcal{C}^I(x_{13}) = 20$ and $\mathcal{C}^I(x_{14}) = 2$.

The output costs is given by: $\mathcal{C}^O(x_1) = 3$, $\mathcal{C}^O(x_2) = 10$, $\mathcal{C}^O(x_3) = \infty$, $\mathcal{C}^O(x_4) = 10$, $\mathcal{C}^O(x_5) = 1$, $\mathcal{C}^O(x_6) = 5$, $\mathcal{C}^O(x_7) = 10$, $\mathcal{C}^O(x_8) = \infty$, $\mathcal{C}^O(x_9) = 5$, $\mathcal{C}^O(x_{10}) = 8$, $\mathcal{C}^O(x_{11}) = 10$, $\mathcal{C}^O(x_{12}) = 5$, $\mathcal{C}^O(x_{13}) = 20$ and $\mathcal{C}^O(x_{14}) = 2$.

In addition, we assume that the communication cost $\mathcal{C}^C$ is constant and with value 0.1.

To solve (1), first, we compute an optimal set of right-unmatched vertices as stated in Algorithm 1, which is given by $\mathcal{U}_R^* = \{x_6, x_9, x_{10}\}$ (depicted in Figure 1-a) by red circled vertices). In Figure 1-a), the gray vertices, $\mathcal{A}_R^* = \{x_1, x_4\}$ correspond to the cheapest variables in the non-top linked SCCs. Now, let us consider the execution of Algorithm 2, in particular the first iteration of the main cycle. It can be seen that $\Xi^{1,1}, \Xi^{2,2}, \Xi^{3,1} = \emptyset$ whereas,

- $\Xi^{1,2} = \{x_4\}$, i.e., $x_4 = \arg\min_{x \in \Xi^{1,2}} \mathcal{C}^I(x)$, and the edge $e = (2,1)$ should be considered in the bipartite representation with weight $w(e) = \mathcal{C}^I(x_6) + \mathcal{C}^I(x_1) - \mathcal{C}^I(x_4) = 5 + 3 - 1 = 7$. In addition, $\mathcal{R} = \{((2,1), x_4)\}$.
- $\Xi^{2,1} = \{x_2\}$, i.e., $x_2 = \arg\min_{x \in \Xi^{2,1}} \mathcal{C}^I(x)$, and the edge $e = (1,2)$ should be considered in the bipartite representation with weight $w(e) = \mathcal{C}^I(x_9) + \mathcal{C}^I(x_1) - \mathcal{C}^I(x_2) = 5+3-10 = -2$. In addition, $\mathcal{R} = \{((2,1), x_4), ((1,2), x_2)\}$.
- $\Xi^{3,1} = \{x_2\}$, i.e., $x_2 = \arg\min_{x \in \Xi^{3,1}} \mathcal{C}^I(x)$, and the edge $e = (1,3)$ should be considered in the bipartite representation with weight $w(e) = \mathcal{C}^I(x_{10}) + \mathcal{C}^I(x_1) - \mathcal{C}^I(x_2) = 6 + 3 - 10 = -1$. In addition, $\mathcal{R} = \{((2,1), x_4), ((1,2), x_2), ((1,3), x_2)\}$.

By computing the maximum-weight matching computed using the Hungarian algorithm, the edge $(2,1)$ (i.e., $(\mathcal{N}^2, x_6)$) has maximum weight equal to 7, and because its weight is positive we have to proceed with the algorithm by

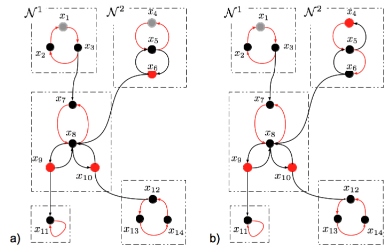

Fig. 1. An illustrative example of a digraph $\mathcal{D}(\bar{A})$, where the red edges correspond to the edges in a possible maximum matching M of the bipartite representation of $\mathcal{D}(\bar{A})$. The red vertices represent the right-unmatched vertices with respect to the maximum matching M whereas the gray vertices represent the cheapest variables in the non-top linked SCCs. In a) we depict the set of right-unmatched vertices $\mathcal{U}_R^*$, the vertices $\mathcal{A}_u^*$ in the non-top linked SCCs and the feasible dedicate input configuration $\mathcal{S}_u^* = \mathcal{U}_R^* \cup \mathcal{A}_u^*$ obtained after the execution of Algorithm 1; and b) reports the set of right-unmatched vertices $\mathcal{U}_R^{\text{opt}}$ and vertices $\mathcal{A}_u^{\text{opt}}$ in the non-top linked SCC that originates a feasible dedicated input configuration $\mathcal{S}_u^{\text{opt}} = \mathcal{U}_R^{\text{opt}} \cup \mathcal{A}_u^{\text{opt}}$ obtained after the execution of Algorithm 2.

updating $\mathcal{U}_R^{\text{opt}} = \{x_4\} \cup \{x_6, x_9, x_{10}\} - \{x_6\} = \{x_4, x_9, x_{10}\}$, where $x_4 = \mathcal{R}_{(2,1)}$ and $\mathcal{A}_u^{\text{opt}} = \{x_1, x_4\} - \{x_4\} = \{x_1\}$. This concludes one iteration of the main cycle in Algorithm 2. The second and third iterations (note that $m = 3$) are similar because the bipartite graph corresponding to the assignment of the right-unmatched vertices in $\mathcal{U}_R^*$ with the non-top linked SCCs has no positive weighted edge belonging to the maximum-weight maximum matching, which is a terminal condition for Algorithm 2. Therefore, by Theorem 3 we can construct a feasible $\bar{B}$ (i.e., such that $(\bar{A}, \bar{B})$ is structurally controllable) that incurs in the minimal cost $\|\bar{B}\|_{\mathcal{C}^I}$. For instance, take $\bar{B}_{4,1} = \bar{B}_{9,2} = \bar{B}_{10,3} = 1$ (by the dual of Theorem 3 applied to the output design) and, for instance, $\bar{B}_{1,1} = 1$ (by b) in the dual of Theorem 3) with all other entries equal to zero.

Now, the next step is to find the variables that are required to be measured such that $\bar{C}$ is feasible and incurs in the minimal cost $\|\bar{C}\|_{\mathcal{C}^O}$. This may be done by executing Algorithm 1 and Algorithm 2 with the digraph $\mathcal{D}(\bar{A}^T)$ and the output cost metric $\mathcal{C}^O$. Consequently, a possible $\bar{C}$ may be constructed by setting $\bar{C}_{1,6} = \bar{C}_{2,9} = \bar{C}_{3,14} = 1$ and zero elsewhere.

Finally, we just need to pair the different outputs and inputs as suggested by Theorem 4 to solve (1), i.e., a mix-pairing between the effective outputs and effective inputs (see Remark 1). One example of such $\bar{K}$ is, for instance, $\bar{K}_{1,3} = \bar{K}_{3,1} = \bar{K}_{2,2} = 1$ and all other entries equal to zero. This scenario is depicted in Figure 2.

V. CONCLUSIONS AND FURTHER RESEARCH

In this paper we have provided a systematic method with polynomial complexity (in the number of the state variables) to jointly solve the input-output and control configuration selection problem that incurs in a overall minimal cost, under the mild assumptions that the manipulating (respectively measuring) costs for state variables are arbitrary but independent of the actuator (respectively sensor) performing the task. We have also considered that the communication cost is uniform. We suspect that, to the best of our knowledge, relaxing any of these constraints would lead to a strictly combinatorial problem for which no polynomial algorithms are expected to exist. A natural direction for future research concerns the development of approximation algorithms under relaxed cost assumptions, especially, for scenarios involving non-uniform communication costs between actuator and sensor pairs.

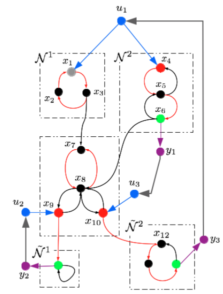

Fig. 2. Depiction of the closed-loop system corresponding to the solution of (1) where the inputs are represented by blue vertices and assigned to either right-unmatched vertices or vertices in the non-top linked SCCs that are not right-unmatched vertices (as is the case for x_1). Similarly, in magenta we have the outputs that are linked from the left-unmatched vertices. Blue edges indicate the variables that are actuated by a specific input, whereas the magenta edges correspond to the variables measured by the outputs. The bold gray edges represent the feedback edges used to close the loop. It may be readily seen that the conditions of Theorem 1 are verified: 1) we can go from any vertex to any other, since, the closed-loop structure consists of a single SCC which contains at least one feedback edge (in fact three); 2) there are cycles constituted by either red edges only, or by a mix or red/blue/magenta that span the entire state space.

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