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FROM LOCAL TO GLOBAL STABILIZATION: A HYBRID SYSTEM APPROACH ⋆

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Abstract: This paper addresses the problem of nonlinear system stabilization using hybrid control. Its main contribution is the derivation of a new methodology to solve the problem of global stabilization of a specific class of nonlinear systems. Using classical Lyapunov techniques and some recent results on switching design of hybrid controllers, it is shown how a locally asymptotically stabilizing controller can be suitably modified to yield global asymptotic stability. The resulting control laws avoid chattering and capture the interplay between the original locally stabilizing control loop and an outer control loop that comes into effect when the system trajectories deviate too much from the origin. The methodology proposed is used to derive a simple control design method for global stabilization of a particular class of single input single output (SISO) nonlinear systems coupled by a given term. Simulation results are presented and discussed.

Keywords: Nonlinear Control, Hybrid Control, Lyapunov Stability, SISO Systems, Stability Analysis.

1. INTRODUCTION

The last few years have witnessed increasing interest in the subject of hybrid control. Much of this interest has been motivated by applications in such diverse fields as car automation and aeronautics, real time software, communication protocols, transportation, traffic control, power distribution, robotics, and consumer electronics. At the same time, in many areas of industrial control, traditional methods of structuring large control systems are being gradually replaced by hybrid system methodologies that explicitly address the interplay between time and event driven phenomena. Modeling, analysis, control and synthesis of such systems pose a considerable number of challenging problems. In fact, while classical control theory and discrete-event system theory provide efficient tools for analysis and synthesis of time-driven and event-driven systems, much work remains to be done before powerful tools for hybrid systems become available. See for example (Antsaklis and Nerode, 1998), (Morse *et al.*, 1999), (Lemmon *et al.*, 1999), and (Schaft and Schumacher, 2000) and the references therein for a comprehensive survey of the field.

In the initial phase of hybrid control systems theory, most of the discussions focused on modeling. In the early work of (Brockett, 1993), a class of models is introduced as an attempt to model a range of phenomena that cut across the usual boundary between control engineering and computer engineering. Branicky, Borkar, and Mitter (Branicky *et al.*, 1994) proposed a very general framework for hybrid control problems that encompasses several types of such hybrid phenomena. Some classical results from dynamical systems have been extended to hybrid systems. In (Ye *et al.*, 1998) a model for hybrid dynamical systems was presented that encompasses a very large class of systems and is suitable for qualitative analysis. The work in (Ye *et al.*, 1998) introduces also several types of Lyapunov-like stability concepts for an invariant set and establishes sufficient and necessary conditions (converse theorems) for these types of stability. See also (Branicky, 1998) where several tools for the analysis and synthesis of hybrid systems were developed. In (Schaft and Schumacher, 1998) and (Lygeros *et al.*, 1999), questions on the existence and uniqueness of solutions to hybrid systems are addressed.

Hybrid controllers that combine continuous with discrete event features have been developed by a number of authors. In (Kolmanovsky and McClamroch, 1996), a hybrid controller for so called cascade systems was proposed. The methodology can be also used for stabilizing a class of nonholonomic systems, as well as for solving tracking problems. In (Lygeros *et al.*, 1996), a game-theoretic framework for designing hybrid con-

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trollers was proposed and latter applied to intelligent highway systems (Lygeros *et al.*, 1998) and air traffic control systems (Tomlin *et al.*, 1998). In another approach to hybrid control system synthesis, a family of continuous-time subsystems are switched on the basis of some supervisory control logic. A general overview of this area can be found in (Morse, 1995). This partly classical topic has regained new interest, in part due to its potential to overcome the basic limitation introduced by Brockett's celebrated result in the area of nonholonomic system control (Brockett, 1983). This is clearly shown in (Hespanha, 1996) where the problem of stabilization of the nonholonomic integrator is solved using a simple logic-based hybrid controller.

Hybrid control is a good candidate to deal with complex systems. Some methods for simulation verification and analysis have been developed in the last years, but the derivation of mathematical tools of sufficient rigor for controller synthesis is an open research issue.

Motivated by the above considerations, this paper addresses the problem of nonlinear system stabilization using hybrid control. *For a specific class of nonlinear plants, it is shown how a locally asymptotically stabilizing controller can be suitably modified to yield global asymptotic stability.* The new methodology proposed for nonlinear system stabilization builds on classical Lyapunov techniques and borrows from recent results on switching design of hybrid controllers. *The resulting control laws avoid chattering and embody in themselves the interplay between the original locally stabilizing control loop and an outer control loop that comes into effect when the system trajectories deviate too much from the origin.* An application which illustrates the potential of the control scheme presented and proposes a simple design method to globally stabilize a particular class of single input single output nonlinear systems coupled by a suitable term is described.

2. CONTROL PROBLEM FORMULATION

Before proceeding with the control problem formulation, consider the following definitions which will be used in the sequel.

Given $x \in \mathbb{R}^n$, let $z \in \mathbb{R}^l$; $l \leq n$ be a *subvector of* x, that is, a vector that is obtained from x by retaining l of its components. Further, let $z^c \in \mathbb{R}^{n-l}$ denote the *complementary vector of* z, that is, the vector that consists of the elements of x that are not in z. A scalar continuous function $V(x) : D \subset \mathbb{R}^n \to \mathbb{R}$ is said to be *locally positive definite with respect to* z if $V(0) = 0$, $V(x) \geq 0$ for $x \neq 0$, and $V(x) > 0$ in the set $\{x \in D : z \neq 0\}$. If $V(x)$ satisfies the weaker condition $V(x) \geq 0$ for $z \neq 0$, then it is said to be *positive semidefinite with respect to* z. A function $V(x)$ is said to be *negative definite* (respectively *negative semidefinite with respect to* z) if $-V(x)$ is positive definite (respectively positive semidefinite with respect to z). If $\dim(z) = n$, then the above definitions recover the usual concepts of positive/negative definite/semidefinite scalar continuous functions.

The class of nonlinear systems considered in this paper is described by

$$\dot{x} = f(x, u) \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz in its arguments, $x(t) \in \mathbb{R}^n$ is the state, and $u(t) \in \mathbb{R}^m$ is the control input. It is assumed that (1) satisfies the following key assumption:

Assumption 1. Given (1), there exists a feedback control law

$$u = \alpha(x) \tag{2}$$

such that the origin $x = 0$ of the closed loop system

$$\dot{x} = f(x, \alpha(x)) \tag{3}$$

is an equilibrium point and system (3) satisfies the following properties:

i) There exists a subvector $z(t) \in \mathbb{R}^l$ of $x(t)$ with $l \leq n$ and a continuously differentiable, positive semidefinite function $V(x) : D \subset \mathbb{R}^n \to \mathbb{R}$ defined in an open set D that contains the origin such that
   a) $V(x)$ is positive definite with respect to z,
   b) $\dot{V}(x)$ is negative definite with respect to z, and
   c) $\|z\| \to 0 \Rightarrow V(x) \to 0$.

ii) For any initial condition $x(t_0) \in D$ such that the resulting trajectory $x(t) \in D$ and $z(t)$ is bounded, there exists a positive constant ζ such that the solution $z^c(t)$ satisfies

$$\|z^c(t)\| \leq \zeta \|z^c(t_0)\| + \sup_{t_0 \leq \tau \leq t} \|z(\tau)\|. \tag{4}$$

   Furthermore, the set $D_0 = \{x \in D : z = 0\}$ is positively invariant and in this set the solution $z^c(t)$ converges to zero as $t \to \infty$.

Assumption 1 implies that the origin $x = 0$ is a locally asymptotically stable equlibrium point of closed loop system (3); the proof is a straightforward application of classical results on system stability using positive semidefinite Lyapunov functions. See for example (Khalil, 1996) and (Sepulchre *et al.*, 1997).

The main problem considered in this paper can now be briefly stated as follows:

Modify the control law (2) *in order to globally asymptotically stabilize* (1).

A solution to this problem will be obtained by resorting to hybrid control techniques, as shown in the next sections.

3. HYBRID CONTROLLER DESIGN

This section proposes a simple piecewise smooth controller to globally asymptotically stabilize system (1) that borrows from hybrid system theory. Hybrid systems are specially suited to deal with the combination of continuous dynamics and discrete events. The literature on hybrid systems is extensive and discusses different modeling techniques. In this paper, a continuous-time autonomous hybrid system Σ is defined as (Hespanha, 1996)

$$\Sigma := \begin{cases} \dot{x}(t) = f_{\sigma(t)}\big(x(t)\big), & t \geq t_0 \quad (5a) \\ \sigma(t) = \phi\big(x(t), \sigma(t^-)\big) & \quad (5b) \end{cases}$$

where $\sigma(t) \in \mathcal{I} \triangleq \{1, \ldots, N\}$ and $x(t) \in \mathcal{X} \triangleq \cup_{\sigma=1}^{N} \mathcal{X}_\sigma \subset \mathbb{R}^n$. In this setup the differential equation

(5a) models the continuous dynamics. Each of the vector fields $f_\sigma : \mathcal{X}_\sigma \to \mathcal{X}$ is locally Lipschitz continuous maps from $\mathcal{X}_\sigma$ to $\mathcal{X}$. The algebraic equation (5b), where $\phi : \mathcal{X} \times \mathcal{I} \to \mathcal{I}$, models the state of the decision-making logic. The discrete state $\sigma(t)$ is piecewise constant. The notation t^- indicates that the discrete state is piecewise continuous from the right. The dynamics of the system Σ can now be described as follows: starting at (x_0, i) with $x_0 \in \mathcal{R}_i \subset \mathcal{X}_i$, the continuous state trajectory $x(t)$ evolves according to $\dot{x} = f_i(x,t)$. When $\phi\big(x(\cdot), i\big)$ becomes equal to $j \neq i$, (and this could only happen when x and σ satisfies a switching rule that is incorporated in the switching logic algorithm), the continuous dynamics switches to $\dot{x} = f_j(x,t)$, from which the process continues. The "logical dynamics" are determined recursively by equation (5b) with $\sigma^-(t_0) = \sigma_0 \in \mathcal{I}$, where $\sigma^-(t)$ denotes the limit of $\sigma(\tau)$ from below as $\tau \to t$. The transition function ϕ is defined with the help of the diagram of Figure 2.

Consider now the control problem described in Section 2 and define the compact set

$$\tilde{D}(\epsilon) = \big\{x \in \mathbb{R}^n : \|z\| \leq \epsilon,\, |z_i^c| \leq \delta_i\big\},$$

for some $\epsilon > 0$ and $\delta_i > 0$, $i = 1, 2, \cdots, n-l$.

Assumption 2. Given system $\dot{x} = f(x,u)$ in (1) and the set D in Assumption 1, there is a control law of the form

$$u = \beta(x),$$

and arbitrarily large positive constants δ_i; $i = 1, 2, \cdots, n-l$ such that

i) There exists $\epsilon > 0$ such that the set $\tilde{D}(\epsilon)$ is strictly inside D, and
ii) For all $x(t_0)$ and for any arbitrarily small positive constant ϵ, a finite time $T \geq t_0$ exists such that $x(T) \in \tilde{D}(\epsilon)$.

Notice that the above conditions do not rule out the possibility that $z_i^c(t)$ grow unbounded. However these conditions imply that $z_i^c(t)$ will not blow up in finite time.

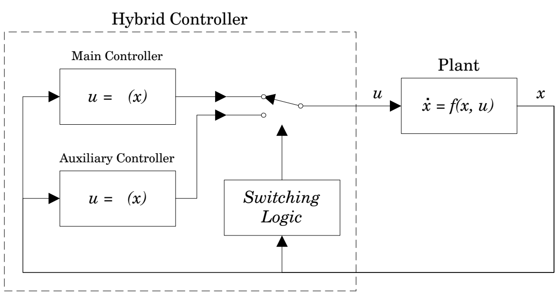

Fig. 1. Block diagram of the hybrid system Σ.

It will be shown that the hybrid control law

$$u = g_\sigma(x), \tag{6}$$

where the vector fields $g_\sigma : \mathbb{R}^n \to \mathbb{R}^m$, $\sigma \in \mathcal{I} = \{1,2\}$ are given by

$$g_1(x) = \alpha(x), \quad g_2(x) = \beta(x), \tag{7}$$

globally asymptotically stabilizes (1). See the block diagram in Figure 1, where V denotes the positive semidefinite function of Assumption 1. The variable σ is a piecewise constant switching signal taking values in $\mathcal{I} = \{1,2\}$ and determines, at each instant of time, which of the feedback laws in (7) should be used. Its value is determined according to the switching logic S_l described by the computer diagram in Figure 2, where S denotes an open set that contains the origin and is strictly inside D, and $S \supseteq \tilde{D}(\epsilon)$ for some $\epsilon > 0$. The block *Initialize* sets $\gamma = +\infty$ and selects A or B as the entry point to the lower blocks of Figure 2 as follows: if $x_0 \in D$, $\sigma(t_0^+) = 1$ and the entry point is A; otherwise, $\sigma(t_0^+) = 2$ and the entry point is B.

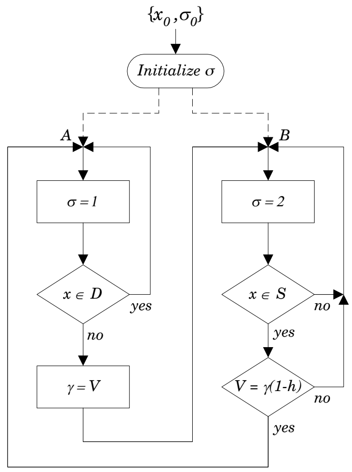

Fig. 2. Switching logic S_l.

The rationale for the control law above can be briefly explained as follows. Assume for example that A is the starting point. If $x(t)$ remains in D for all $t \geq t_0$, then it will be later shown that $x(t)$ will, under the action of control law $\alpha(x)$, tend to 0 as $t \to \infty$. Suppose now that $x(t)$ leaves the region D at time $t = t_s$ and therefore that σ switches to 2. Let $\gamma = \lim_{t \to t_s} V(x(t))$. According to the switching logic proposed, the control law $\beta(x)$ will come into play and force the variable z to approach 0. As a consequence, x will enter $S \subset D$ and since $\|z\| \to 0 \Rightarrow V(x) \to 0$ it follows that V will reach the value $\gamma_h = \gamma(1-h)$, where $h \in (0,1)$ is a constant variable. At that moment, σ will switch back to 1. Since $\dot{V}(x) < 0$ on the set $D_z = \{x \in D : z \neq 0\}$ it follows that the successive values of γ_h will be decreasing and z will tend to 0. Furthermore, the bound (4) on $z_c(t)$ guarantees that $z_c(t)$ and therefore $x(t)$ will tend to zero. The switching logic described above generates a piecewise constant signal σ that is continuous from the right everywhere. Notice how h and the construction of $S \subset D$ introduce a hysteresis in the switching logic that avoids chattering.

Consider the following definition of Lyapunov stability for hybrid systems that borrows from the concepts introduced in (Ye *et al.*, 1998).

Definition 1. The equilibrium point $x = 0$ of the hybrid system Σ of (5) is said to be Lyapunov stable if for every $\epsilon > 0$ and any $t_0 \in \mathbb{R}^+$ there exists $\delta = \delta(\epsilon) > 0$ such that for every initial condition $\{x_0, \sigma_0\} \in \mathcal{X} \times \mathcal{I}$ with $\|x_0\| < \delta$, the solution $\{x(t), \sigma(t)\}$ satisfies $\|x(t)\| < \epsilon$ for all $t \geq t_0$. If in addition the equilibrium point $x = 0$ is attractive

i.e., there exists $\eta(t_0) > 0$ and, for each $\epsilon > 0$, there exists $T(\epsilon) > 0$ such that $\|x_0\| < \eta \Rightarrow \|x(t)\| < \epsilon$, for all $t \geq t_0 + T$, then the origin is said to be asymptotically stable. If these properties hold for any initial conditions, then the equilibrium point $x = 0$ is called globally asymptotically stable.
The next theorem shows existence and uniqueness of solutions, as well as convergence to the origin and stability of the closed loop hybrid dynamical system.

Theorem 2. Consider the hybrid system Σ described by (1), (6), (7), and the switching logic S_l defined by the computer diagram shown in Figure 2. Suppose that (1) satisfies Assumptions 1-2. Let $\{x(t), \sigma(t)\} = \{x : [t_0, \infty) \to \mathbb{R}^n, \sigma : [t_0, \infty) \to \mathcal{I}\}$ be a solution to Σ. Then, the following properties hold.

i) $\{x(t), \sigma(t)\}$ exists, is unique and defined for all $t \geq t_0$ and all $\{x(t_0), \sigma^-(t_0)\} = \{x_0, \sigma_0\} \in \mathbb{R}^n \times \mathcal{I}$.
ii) For any set of initial conditions $\{x_0, \sigma_0\} \in \mathbb{R}^n \times \mathcal{I}$, there exists a finite time $T \geq t_0$ such that $\sigma(t) = 1$ for all $t \geq T$, *i.e.*, the switching stops in finite time. Furthermore, the continuous state $x(t) \in \mathbb{R}^n$ converges to zero as $t \to \infty$.
iii) The origin $x(t) = 0$ is a Lyapunov globally asymptotically stable equilibrium point of Σ.

Proof. See (Aguiar, 2002). □

4. AN ILLUSTRATIVE EXAMPLE

This section illustrates an application of the hybrid control law developed in Section 3. See (Isidori, 1989) for standard notation. Consider the two single-input single output (SISO) systems

$$\Sigma_i := \begin{cases} \dot{x}_i = f_i(x_i) + g_i(x_i)u_i, \\ y_i = h_i(x_i), \end{cases} \qquad i = 1, 2$$

with $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}$, $y_i \in \mathbb{R}$, f_i, and g_i are smooth vector fields on $\mathbb{R}^{n_i}$, h_i a smooth nonlinear function, and $f_i(0) = 0$, $h_i(0) = 0$ for all $i = 1, 2$. Consider also that both systems Σ_1, Σ_2 have relative degree n_1, n_2 at $x_1 = 0$, $x_2 = 0$, respectively. Then, the nonlinear coordinate transformation $\Phi_i : x_i \to \xi_i = \left(h_i, L_{f_i} h_i, \cdots, L_{f_i}^{n_i - 1} h_i\right)'$; $i = 1, 2$ is a local diffeomorphism and transforms Σ_i into the "normal form" (Isidori, 1989)

$$\begin{aligned}
\dot{\xi}_{i_1} &= \xi_{i_2}, \\
\dot{\xi}_{i_2} &= \xi_{i_3}, \\
&\vdots \\
\dot{\xi}_{i_{n_i}} &= b_i(\xi_i) + a_i(\xi_i)u_i, \\
y_i &= \xi_{i_1},
\end{aligned}$$

where $\xi_i = (\xi_{i_1}, \cdots, \xi_{i_{n_i}})'$, $b_i = L_{f_i}^{n_i} h_i\left(\Phi_i^{-1}(\xi_i)\right)$, and $a_i(\xi_i) = L_{g_i} L_{f_i}^{n_i - 1} h_i\left(\Phi_i^{-1}(\xi_i)\right)$. Note that at the point $\xi_i = \Phi_i(0) = 0$, we have $a_i(0) \neq 0$, and therefore the coefficient $a_i(\xi_i)$ is nonzero for all ξ_i in a neighborhood of $\xi_i = 0$. Consequently, choosing the state feedback control law

$$u_i = \frac{1}{a_i(\xi_i)}\left[-b_i(\xi_i) + v_i\right],$$

yields a closed loop system that is linear and controllable from the input v_i. Thus, the closed loop system can be made to have its poles placed at the the zeros of a desired polynomial $s^{n_i} + c_{i_{n_i - 1}} s^{n_i - 1} + \cdots + c_{i_0}$ by choosing $v_i = -c_{i_0}\xi_{i_1} - c_{i_1}\xi_{i_2} - \cdots - c_{i_{n_i - 1}}\xi_{n_i}$. Suppose now that there is a coupling term between the two SISO systems as given by

$$\dot{x} = f(x) + g(x)u + \varphi(x, y), \tag{8}$$

where $x = (x_1', x_2')'$, $f = (f_1', f_2')'$, $g = (g_1', g_2')'$, $\varphi(x, y) = (\varphi_1', \varphi_2')'$, and

$$\begin{aligned}
\varphi_1 &= \vartheta_1(x_1)(y_2 - k), \\
\varphi_2 &= \vartheta_2(x_2)y_1,
\end{aligned}$$

with ϑ_i; $i = 1, 2$ are smooth vector field on $\mathbb{R}^{n_i}$, k is an arbitrary non null constant, and ϑ_i satisfies the "matching condition"

$$L_{\vartheta_i} L_{f_i}^k h_i(x_i) = 0; \quad 0 \leq k \leq n_i - 2; \quad i = 1, 2$$

Further assume that the coordinate transformation $\xi_2 = \Phi_2(x_2)$ is a global diffeomorphism on $\mathbb{R}^{n_2}$, $\xi_1 = \Phi_1(x_1)$ is a local diffeomorphism, and that the subsystem

$$\dot{x}_1 = f_1(x_1) + \vartheta_1(x_1)\tilde{u}; \quad \tilde{u} = y_2 - k \tag{9}$$

of (8) that corresponds to the evolution of state x_1 with $u_1 = 0$ can be rewritten as

$$\begin{aligned}
\dot{\xi}_{3_1} &= \xi_{3_2}, \\
\dot{\xi}_{3_2} &= \xi_{3_3}, \\
&\vdots \\
\dot{\xi}_{3_r} &= b_3(\xi_3) + a_3(\xi_3)(y_2 - k), \\
\dot{\xi}_{3_{r+1}} &= q_{r+1}(\xi_3), \\
&\vdots \\
\dot{\xi}_{n_1} &= q_{n_1}(\xi_3),
\end{aligned}$$

where $\xi_3 = \Phi_3(x_1)$, $\Phi_3(x_1) = \left(\phi_{3_1}, \cdots, \phi_{3_{n_1}}\right)'$, $\phi_{3_1}(x_1) = h_1(x_1)$, $\phi_{3_2}(x_1) = L_{f_1} h_1(x_1)$, $\cdots$, $\phi_{3_r} = L_{f_1}^{r-1} h_1(x_1)$, and $\phi_{3_{r+1}}(x_1), \cdots, \phi_n(x_1)$ are chosen in such a way as to make $L_{\vartheta_1}\phi_{3_i} = 0$ for all $r + 1 \leq i \leq n_1$ and all $x_1 \in \mathbb{R}^{n_1}$. The functions $b_3(\xi_3)$, $a_3(\xi_3)$ are given by $L_{f_1}^r h_1(x_1)$, $L_{\vartheta_1} L_{f_1}^{r-1} h_1(x_1)$, and $q_i(\xi_3) = L_{f_1}\phi_{3_i}(x_1)$ for all $r + 1 \leq i \leq n_1$. Consider that the zero dynamics $\dot{\eta} = q(\bar{\xi}_3, \eta)$, with $\bar{\xi}_3 = \left(\xi_{3_1}, \cdots, \xi_{3_r}\right)'$, $\eta = \left(\xi_{r+1}, \cdots, \xi_{n_1}\right)'$, are globally asymptotically stable, and that $a_3(\xi_3) \neq 0$ for all $\xi_3 \neq 0$. However, for $\xi_3 = 0$, $a_3(0)$ can be zero.

The goal is to globally asymptotically stabilize system (8) at the equilibrium point $(x_1, x_2) = 0$ applying the methodology proposed in section 3.

4.1 *Main controller $\alpha(x)$*

A simple local controller that exponentially asymptotically stabilizes the origin and satisfies Assumption 1 is given by $u = \alpha(x) = \left(\alpha_1(x), \alpha_2(x)\right)'$ where

$$\begin{aligned}
\alpha_1(x) = \frac{1}{L_{g_1} L_{f_1}^{n_1 - 1} h_1(x_1)} &\left[-L_{f_1}^{n_1} h_1(x_1) - c_{1_0} h_1(x_1)\right. \\
&- c_{1_1} L_{f_1} h_1(x_1) - \cdots - c_{1_{n_1 - 1}} L_{f_1}^{n_1 - 1} h_1(x_1) \\
&\left.- L_{\vartheta_1} L_{f_1}^{n_1 - 1} h_1(x_1)(y_2 - k)\right], \\
\alpha_2(x) = \frac{1}{L_{g_2} L_{f_2}^{n_2 - 1} h_2(x_2)} &\left[-L_{f_2}^{n_2} h_2(x_2) - c_{2_0} h_2(x_2)\right. \\
&- c_{2_1} L_{f_2} h_2(x_2) - \cdots - c_{2_{n_2 - 1}} L_{f_2}^{n_2 - 1} h_2(x_2) \\
&\left.- L_{\vartheta_2} L_{f_2}^{n_2 - 1} h_2(x_2)y_1\right],
\end{aligned}$$

and the constants $c_{i_0}, c_{i_1}, \cdots, c_{i_{n_i-1}}$; $i = 1,2$ are chosen so as to make the matrix

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_{i_0} & -c_{i_1} & -c_{i_2} & \cdots & -c_{i_{r-1}} \end{bmatrix}; \quad i = 1,2 \qquad (10)$$

Hurwitz.

Let $z = x_1$ and $z^c = x_2$ be the states defined in Section 2. A candidate Lyapunov function that satisfies the conditions of Assumption 1.i) can be found by solving the equation

$$PA_1 + A_1' P = -Q \qquad (11)$$

for P, where Q is a real symmetric positive definite matrix. Let $D = \{x \in \mathbb{R}^{n_1+n_2} : \Phi_1(x_1)$ is invertible and $\Phi_1(x_1)$ and $\Phi_1^{-1}(\xi_1)$ are both smooth mappings$\}$. The quadratic function $V(x) : D \subset \mathbb{R}^{n_1+n_2} \to \mathbb{R}$,

$$V(x) = \Phi_1(z)' P \Phi_1(z) \qquad (12)$$

is positive definite with respect to z, its derivative $\dot{V}(x) = -\Phi_1(z)' Q \Phi_1(z)$ is negative definite with respect to z, and $\|z\| \to 0 \Rightarrow V(x) \to 0$. Assumption 1.ii) is easily seen to be satisfied by noting that the dynamics of z^c are governed by the linear differential equation $\dot{\xi}_2 = A_2 \xi_2$, with $z^c = \Phi_2^{-1}(\xi_2)$. Since A_2 is a Hurwitz matrix and Φ_2 is a global diffeomorphism on $\mathbb{R}^{n_2}$, it follows that z_c is bounded and converges to zero.

4.2 *Auxiliary controller $\beta(x)$*

In this section, a feedback control law is derived such that for any initial condition $x(t_0)$ the state $z(t)$ reaches any small neighborhood of the origin $z = 0$ in finite time. Notice that convergence to the origin is not required. Notice also that when compared to the local controller one now has an extra "degree of freedom" given by the state z^c. This vector need not converge to zero and can be viewed as another control input. In fact, by imposing $u_1 = 0$ and observing equation (9), one can see y_2 as a virtual control input and

$$\alpha_{3_1} = k + \frac{1}{a_3}\left[-b_3 - c_{3_0}\xi_{3_1} - c_{3_1}\xi_{3_2} - \cdots - c_{3_{r-1}}\xi_{3_r}\right]$$

as a virtual control law, where the constants $c_{3_i}; i = 0, 1, \cdots, r$ are chosen in such way that the matrix A_3 (see (10) for $i = 3$) is Hurwitz. Notice that if $a_3(0) = 0$, the virtual control law is not valid. However, if $x_1 = 0$ then σ must be 1 and therefore it is the local controller that is active (see Section 3). Introducing the error variable $z_1 = y_2 - \alpha_{3_1}$, and following the backstepping approach (see the book of (Krstić *et al.*, 1995)), let V_1 be a partial Lyapunov function given by

$$V_1 = \xi_3' P \xi_3 + \frac{1}{2} z_1^2,$$

where the symmetric positive definite matrix P is the solution of the Lyapunov equation $PA_3 + A_3'P = -Q$, and Q is a real symmetric positive definite matrix. Computing the time derivative of V_1, yields

$$\dot{V}_1 = -\xi_3' Q \xi_3 + z_1\left[2\xi_3' Pb + \dot{y}_2 - \dot{\alpha}_{3_1}\right],$$

where $b = (0, \cdots, 0, 1)'$. Resorting to the coordinate transformation $\xi_2 = \Phi_2(x_2)$, it follows that $\dot{y}_2 = \dot{\xi}_{2_1} = \xi_{2_2}$. Defining $z_2 = \xi_{2_2} - \alpha_{3_2}$, where

$$\alpha_{3_2} = -2\xi_3' Pb + \dot{\alpha}_{3_1} - k_1 z_1$$

one obtains $\dot{V}_1 = -\xi_3' Q \xi_3 - k_1 z_1^2 + z_1 z_2$.

Let now V_2 be the augmented Lyapunov function $V_2 = V_1 + \frac{1}{2} z_2^2$. Its derivative is given by

$$\dot{V}_2 = -\xi_3' Q \xi_3 - k_1 z_1^2 + z_2\left[z_1 + \xi_{2_3} - \dot{\alpha}_{3_2}\right].$$

Proceeding recursively, at the i-th step define $z_{i+1} = \xi_{2_{i+1}} - \alpha_{3_{i+1}}$, where $\alpha_{3_{i+1}} = -z_{i-1} + \dot{\alpha}_{3_i} - k_i z_i$ to obtain

$$\dot{V}_i = -\xi_3' Q \xi_3 - k_1 z_1^2 - \cdots - k_i z_i^2 - z_i z_{i+1}.$$

At the last step, one has

$$\begin{aligned} \dot{V}_{n_2} = & -\xi_3' Q \xi_3 - k_1 z_1^2 - \cdots - k_{n_2-1} z_{n_2-1}^2 \\ & + z_{n_2}\left[z_{n_2-1} + b_2(\xi_2) + a_1(\xi_2) u_2 - \dot{\alpha}_{3_{n_2}} \right. \\ & \left. - L_{\vartheta_2} L_{f_2}^{n_2-1} h_2(x_2) y_1\right]. \end{aligned}$$

By defining

$$u_2 = \frac{1}{a_2(\xi_2)}\left[-b_2(\xi_2) - z_{n_2-1} + \dot{\alpha}_{3_{n_2}} + L_{\vartheta_2} L_{f_2}^{n_2-1} h_2(x_2) y_1\right], \qquad (13)$$

the time derivative of the composite Lyapunov function

$$V_{n_2} = \xi_3' P \xi_3 + \frac{1}{2}\left(z_1^2 + \cdots + z_{n_2}^2\right)$$

given by $\dot{V}_{n_2} = -\xi_3' Q \xi_3 - k_1 z_1^2 - \cdots - k_{n_2} z_{n_2}^2$ is negative definite. Thus, the closed loop system described by (8), together with $u = \beta(x) = (0, u_2)'$, where u_2 is given by (13) satisfies Assumption (2).

4.3 *Simulation results*

This section illustrates the performance of the proposed control scheme using computer simulations with the nonlinear system

$$\begin{aligned} \dot{x}_{1_1} &= x_{1_1}^2 + x_{1_2} \\ \dot{x}_{1_2} &= \cos(x_{1_1}) u_1 + p(x_1)(x_2 + 1) \\ \dot{x}_2 &= u_2 \end{aligned}$$

where $p(x_1) = x_{1_1}^2 + x_{1_2}^2$, $x = (x_{1_1}, x_{1_2}, x_2)'$ and $u = (u_1, u_2)'$. The objective is to globally asymptotically stabilize the origin $x = 0$. Following the approach developed in the previous sections one has

$$\alpha(x) = \begin{bmatrix} \frac{1}{\cos(\xi_{1_1})}(-2\xi_{1_1}\xi_{1_2} - p(x_1)(x_2 + 1) + v_1) \\ -c_{2_0} x_2 \end{bmatrix},$$

$$\beta(x) = \begin{bmatrix} 0 \\ -k_1 z_1 - 2P_{12}\xi_{3_1} - 2P_{22}\xi_{3_2} + \dot{\alpha}_{3_1} \end{bmatrix},$$

where $\xi_{1_1} = \xi_{3_1} = x_{1_1}$, $\xi_{1_2} = \xi_{3_2} = x_{1_1}^2 + x_{1_2}$, $v_1 = -c_{1_0}\xi_{1_1} - c_{1_1}\xi_{1_2}$, $z_1 = x_2 - \alpha_{3_1}$, $\alpha_{3_1} = -1 + \frac{1}{p(x_1)}(-2\xi_{3_1}\xi_{3_2} - c_{3_0}\xi_{3_1} - c_{3_1}\xi_{3_2})$, and P_{ij} denotes the ij-element of the solution of the Lyapunov matrix $PA_3' + A_3'P = -I$, with $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $A_3 = \left(\begin{smallmatrix} 0 & 1 \\ -c_{3_0} & -c_{3_1} \end{smallmatrix}\right)$. Consider the sets

$$\begin{aligned} D &= \left\{x \in \mathbb{R}^3 : |x_{1_1}| \leq \bar{x}_{1_1}\right\}, \\ S &= \left\{x \in \mathbb{R}^3 : |x_{1_1}| \leq \bar{x}_{1_1}(1 - h_{x_1})\right\}, \end{aligned}$$

with $\bar{x}_{1_1} < \frac{\pi}{2}$. Notice that with S defined as above, S is strictly contained in D.

The control parameters were selected as following: $c_{1_0} = c_{3_0} = 1$, $c_{1_1} = c_{3_1} = 2$, $c_{2_0} = 1$, $k_1 = 1$, $h = h_{x_1} = 0.1$, and $\bar{x}_{1_1} = 85\pi/180$.

Figures 3-4 show the simulation results for the initial condition $x(0) = (70\pi/180, 0, 1)'$. The time evolution of the states $x_{1_1}(t)$, $x_{1_2}(t)$, and $x_2(t)$ is depicted in Figure 3. To better understand the performance of the hybrid control law, Figure 4 displays the time evolution of the Lyapunov variable $V(x) = \xi_1' P \xi_1$ and the switching signal $\sigma(t)$. Initially, σ takes the value 1. While σ is 1, one can see that V is decreasing. However, after a finite time $t \simeq 0.248\, s$, the variable x_{1_1} takes the value $\bar{x}_{1_1}$ and therefore $x(t)$ leaves the region D. As a consequence, σ switches to 2 and $z = (x_{1_1}, x_{1_2})'$ will be forced to approach 0. At $t \simeq 1.264\, s$, it can be seen that $x(t)$ is already in S and $V(x)$ reaches $\gamma(1-h)$. Therefore, σ switches again to 1, the main controller dictates the subsequent trajectory evolution.

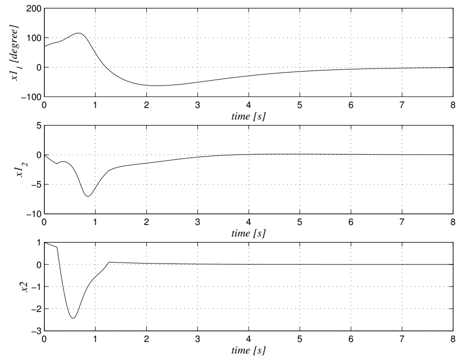

Fig. 3. Time evolution of the state variables $\frac{180}{\pi} x_{1_1}(t)$, $x_{1_2}(t)$, and $x_2(t)$.

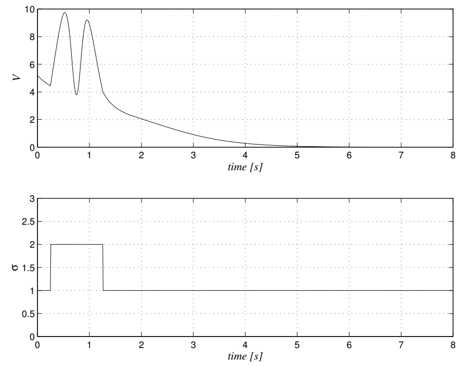

Fig. 4. Time evolution of the Lyapunov function $V(x)$ and the switching signal $\sigma(t)$.

5. CONCLUSIONS

The problem of nonlinear system stabilization using hybrid control was investigated in this paper. For a specific class of nonlinear plants, it was shown how a locally asymptotically stabilizing controller can be suitably modified to yield global asymptotic stability. The new methodology proposed for nonlinear system stabilization builds on classical Lyapunov techniques and borrows from recent results on switching design of hybrid controllers. The resulting control laws avoid chattering and embody the interplay between the original locally stabilizing control loop and an outer control loop that comes into effect when the system trajectories deviate too much from the origin. An application was described that illustrates the potential of the control scheme presented and proposes a simple design method to globally stabilize a particular class of coupled single input single output nonlinear systems Simulation results show that the control objectives were achieved successfully.

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