A Performance Based Model-Set Design Strategy for Multiple Model Adaptive Estimation

Vahid Hassani, A. Pedro Aguiar, António M. Pascoal, and Michael Athans

***Abstract*— This paper addresses the problem of Multiple Model Adaptive Estimator (MMAE) design for linear process models subjected to parameter uncertainty. MMAE algorithms rely on a finite number of representative models chosen from the original set of possibly infinite plant models. One of the standing issues that arise in the process of MMAE design is the selection of the model-set. Typical questions that arise at this phase are the following: i) what is gained by using a MMAE approach compared with a single model approach?, and ii) for a given required level of performance, what is the minimum number of models required and how should they be selected as a function of the parameter uncertainty region? For discrete-time, linear, time-invariant MIMO plants with parameter uncertainty, we propose a performance-based model-set design strategy. To this effect, we first introduce the concept of an Infinite Model Adaptive Estimation Performance (IMAEP) index that defines the best achievable performance of the MAAE, assuming an ideal MMAE with an infinite number of representative models. Then, based on a specified demanded performance relative to the ideal IMAEP (say, 85% of the IMAEP uniformly over the original parameter uncertainty set), we provide an algorithm that guarantees the demanded performance and yields the corresponding finite number of representative models. An example is described that illustrates the proposed strategy and the improvement in performance that is obtained when compared with other previously proposed design methodologies.**

I. INTRODUCTION

In most practical applications of estimation theory, it is virtually impossible to obtain a highly accurate mathematical model of the physical process of interest. For this reason, a model is often given in terms of its basic structure and a vector of parameters in a compact set that capture plant parameter uncertainty. When state estimation for this type of systems is carried out, the variations of the parameters and their identification play a critical role. Also, it is often necessary to estimate both a system's parameter vector and its state. Many approaches have been proposed in attempting to perform state estimation together with parameter identification. In particular, the Multiple Model Adaptive Estimation (MMAE) algorithm has received considerable attention [1]–[4]. It is cost-effective, robust, and has a parallel structure. In this approach, a set of models (estimators) is designed to cover the possible system behavior "patterns" or structures and the overall output is obtained by adequate combination of the outputs based on each individual model. Namely, in the standard version of the MMAE [1]–[3] a separate discrete-time Kalman filter (KF) is developed based on each different assumed value of the uncertain parameters defining a "model". The resulting set of KFs forms a "model-set" where each local KF generates its own state estimate and an output error (residual) as shown in Fig. 1. All the KFs in the model-set run in parallel and at each sample time the residuals are used to compute for the i^{th} KF the conditional probability p_i that it corresponds to the correct parameter value. The overall state estimate is a probabilistically weighted combination of each KF's estimate. The rationale behind this approach is that the highest probability should be assigned to the state estimation of the most accurate KF, and lower probabilities assigned to the other KFs.

In many applications of MMAE, the set of possible values of the uncertain system parameter, known as parameter space, is not finite. Instead, the parameters lie in some compact subset of $\mathbb{R}^n$. Optimal use of more models in the continuous parameter space does improve the performance of the MMAE algorithm for state estimation. In practice, however, only a limited number of models, say N, can be used. In [3], by introducing an information theoretic measure, the authors analyzed the convergence of the conditional probabilities p_i and showed that the one corresponding to the KF designed for the closest to the actual system (in a stochastic norm sense) converges to 1, while the others tend to 0. We will call the measure described the Baram metric.

The present paper was strongly motivated by the unresolved issue of how to select a proper model-set during the MMAE design phase. Typical questions that arise are the following: i) what is gained by using a MMAE approach compared with a single model approach? and ii) for a given required level of performance, what is the minimum number of models required and how should they be selected as a function of the parameter uncertainty region?

For discrete-time, linear, time-invariant MIMO plants with parameter uncertainty, we propose a performance-based model-set design strategy. To this effect, we first introduce the concept of an Infinite Model Adaptive Estimation Performance (IMAEP) index that defines the best achievable performance of the MAAE, assuming an "ideal" MMAE with an infinite number of representative models. In practice, to determine the IMAEP we use a dense grid of parameters in parameter space and plot the associated Baram metric.

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The authors are with the Institute for Systems and Robotics (ISR), Instituto Superior Técnico (IST), Lisbon, Portugal.
e-mail:{vahid,pedro,antonio,athans}@isr.ist.utl.pt
Tel: (+351) 21 841 8054, Fax: (+351) 21 841 8291

Michael Athans is also Professor of EECS (emeritus), M.I.T., USA.

Then, based on a specified demanded performance relative to the ideal IMAEP (say, 85% of the IMAEP uniformly over the original parameter uncertainty set), we give an algorithm that guarantees the demanded performance and yields the corresponding finite number of representative models. Besides offering a rigorous procedure to determine the latter, the results in the paper are also useful to evaluate the performance of MMAE algorithms.

The structure of the paper is as follows. Section II reviews the key concepts of MMAE. Section III summarizes basic convergence results and the Baram metric definition [3]. Section IV introduces the IMAEP index and gives a procedure to decide on the number of and what models to use in the the MMAE procedure. An example is described in Section V that illustrates the proposed strategy and the improvement in performance that is obtained when compared with other previously proposed design methodologies, via computer simulations. Conclusions and suggestions for future research are summarized in Section VI.

II. THE MULTIPLE-MODEL ADAPTIVE ESTIMATOR

One of the earliest uses of multiple-models was motivated by the need to accurately estimate the state of a stochastic dynamic system subjected to significant parameter uncertainty. In many such applications, the estimation accuracy provided by standard KFs was not adequate. For some early references on MMAE see [1]–[3].

Fig. 1 shows the architecture of the MMAE system. It is assumed that a discrete-time linear time-invariant plant G is driven by white process noise and a known deterministic input signal and generates measurements that are corrupted by white measurement noise. If there is no parameter uncertainty in the plant, then the KF is the optimal state-estimation algorithm in a well-defined sense; see, for example, [2], [5]. Moreover, under the usual linear-gaussian assumptions, the KF state-estimate is the true conditional mean of the state, given the past controls and observations. If the plant has an uncertain real-parameter vector, say κ, one can imagine that it is "close" to one of the elements of a finite discrete representative parameter set, $\boldsymbol{\kappa} := \{\kappa_1, \kappa_2, \ldots, \kappa_N\}$. One can then design a bank of standard KFs, where each KF uses one of the discrete parameters κ_i in its implementation, $i \in \{1, \ldots, N\}$. It turns out that, if indeed the true plant parameter is identical to one of its discrete values then the conditional probability density of the state is the sum of gaussian densities. Then, the MMAE of Fig. 1 will generate the true conditional mean of the state and one can calculate the true conditional covariance matrix; see, for example, [2]. The structure of MMAE, in Fig. 1, consists of: i) The Posterior Probability Evaluator (PPE) and ii) a bank of N discrete-time KFs, where each local estimator is designed based on one of the representative parameters. The state estimate is generated by a probabilistically weighted sum of the local state-estimates produced by the bank of KFs. In what follows, we assume the plant model G is subjected to parameter uncertainty $\kappa \in \mathbb{R}^l$, that is, $G = G(\kappa)$. We

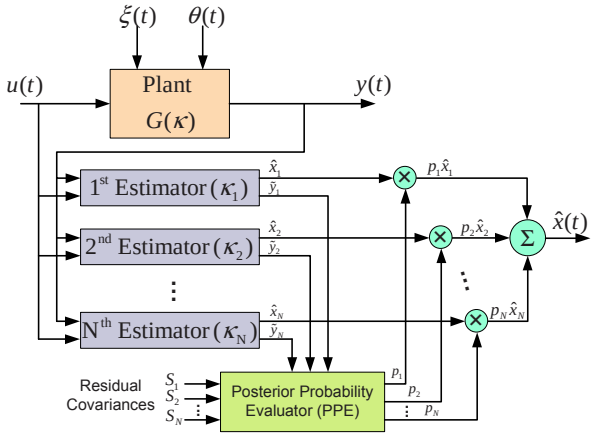

Fig. 1. The MMAE architecture.

consider multiple-input-multiple-output (MIMO) linear time-invariant (LTI) plant models of G the form

$$x(t+1) = A_\kappa x(t) + B_\kappa u(t) + L_\kappa \xi(t), \tag{1a}$$

$$y(t) = C_\kappa x(t) + \theta(t), \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ denotes the state of the system, $u(t) \in \mathbb{R}^m$ its control input, $y(t) \in \mathbb{R}^q$ its measured noisy output, $\xi(t) \in \mathbb{R}^r$ an input plant disturbance that can not be measured, and $\theta(t) \in \mathbb{R}^q$ is the measurement noise. Vectors $\xi(t)$ and $\theta(t)$ are zero-mean white Gaussian sequences, mutually independent with covariances $cov[\xi(t);\xi(\tau)] = \Xi\delta_{t\tau}$ and $cov[\theta(t);\theta(\tau)] = \Theta\delta_{t\tau}$. The initial condition $x(0)$ of (1) is Gaussian random vector with mean and covariance given by $E\{x(0)\} = 0$ and $E\{x(0)x^T(0)\} = \Sigma(0)$. Matrices A_κ, B_κ, L_κ, and C_κ contain *unknown constant parameters* indexed by vector κ.

Consider a finite set of candidate parameter values $\boldsymbol{\kappa} = \{\kappa_1, \kappa_2, \ldots, \kappa_N\}$ indexed by $i \in \{1, \ldots, N\}$. We propose the following MMAE. The state estimate is given by

$$\hat{x}(t|t) := \sum_{i=1}^{N} p_i(t)\hat{x}_{\kappa_i}(t|t), \tag{2}$$

$$\hat{y}(t|t) := \sum_{i=1}^{N} p_i(t)\hat{y}_{\kappa_i}(t|t), \tag{3}$$

where $\hat{x}(t|t)$ and $\hat{y}(t|t)$ are the estimates of the state $x(t)$ and output $y(t)$, respectively (at time t) and $p_i(t)$ is the conditional probability that $\kappa = \kappa_i$, given the measurements record. In (2), each $\hat{x}_{\kappa_i}(t|t)$; $i = 1, \ldots, N$ corresponds to a "local" state estimate generated by the i^{th} steady state Kalman filter [2],

$$\hat{x}_{\kappa_i}(t+1|t) = A_{\kappa_i}\hat{x}_{\kappa_i}(t|t) + B_{\kappa_i}u(t) \tag{4a}$$

$$\hat{x}_{\kappa_i}(t|t) = \hat{x}_{\kappa_i}(t|t-1) + H_{\kappa_i}[y(t) - C_{\kappa_i}\hat{x}_{\kappa_i}(t|t-1)], \tag{4b}$$

$$\begin{aligned} 0 = &-\Sigma_{\kappa_i} + A_{\kappa_i}\Sigma_{\kappa_i}A_{\kappa_i}^T + L_{\kappa_i}\Xi L_{\kappa_i}^T \\ &- A_{\kappa_i}^T\Sigma_{\kappa_i}C_{\kappa_i}^T[C_{\kappa_i}\Sigma_{\kappa_i}C_{\kappa_i}^T + \Theta]^{-1}C_{\kappa_i}\Sigma_{\kappa_i}A_{\kappa_i}, \end{aligned} \tag{4c}$$

$$H_{\kappa_i} = \Sigma_{\kappa_i}C_{\kappa_i}^T[C_{\kappa_i}\Sigma_{\kappa_i}C_{\kappa_i}^T + \Theta]^{-1} \tag{4d}$$

where $[A_{\kappa_i}, L_{\kappa_i}]$ and $[A_{\kappa_i}, C_{\kappa_i}]$ are assumed to be stabilizable and detectable, respectively for $i = 1, \ldots, N$. We next introduce the Posterior Probability Evaluator (PPE) that weighs the local estimations (2).

A. Posterior Probability Evaluator (PPE)

The key to the MMAE algorithm is the so-called *posterior probability evaluator (PPE)* which calculates, in real time, the posterior conditional probability that each model generates the data, i.e. the probability $\kappa = \kappa_i$; $i \in \{1, \ldots, N\}$ (see (5) for more details). Thus, the PPE represents an identification subsystem.

The posterior probabilities can be computed on-line by the PPE using the recursive formula

$$p_i(t+1) = \frac{\beta_i e^{-w_i(t+1)}}{\sum_{j=1}^{N} p_j(t)\beta_j e^{-w_j(t+1)}} p_i(t), \tag{5}$$

where $p_i(0)$ are the prior model probabilities and $w_i(t)$ and β_i are

$$w_i(t) := \frac{1}{2}[y(t) - \hat{y}_{\kappa_i}(t|t-1)]^T \Sigma_{\kappa_i}^{-1} [y(t) - \hat{y}_{\kappa_i}(t|t-1)], \tag{6a}$$

$$\beta_i := \frac{1}{(2\pi)^{\frac{q}{2}} \sqrt{|\Sigma_{\kappa_i}|}}, \tag{6b}$$

where q is dimension of $y(t)$ and Σ_{κ_i} is the covariance matrix of residuals in i^{th} KF given by

$$S_{\kappa_i} = C_{\kappa_i} \Sigma_{\kappa_i} C_{\kappa_i}^T + \Theta. \tag{7}$$

In the sequel we use the following notation. Let $y(t)$ be a sequence of measurements (observations) and let $Y(t) \equiv \{y(0), y(1), \cdots, y(t)\}$. The KF designed based on each κ_i in the parameter set provides a conditional probability density function $f_i\big(y(t)|Y(t-1), \kappa_i\big)$ for each $t \geq 1$, with $f_i\big(y(1)|Y(0), \kappa_i\big) = C_{\kappa_i} E\{x(0)\}$. We call $\kappa_\star$ the true parameter in the plant and the correct conditional probability density function of $y(t)$ is denoted by $f_\star\big(y(t)|Y(t-1), \kappa_\star\big)$. For each KF, we have $f_i\big(Y(t)|\kappa_i\big) = \prod_{k=1}^{t} f_i\big(y(k)|Y(k-1), \kappa_i\big)$.

III. Convergence Properties

As explained in section II, we design N KFs for N nominal values of κ_i, $i = 1, 2, \cdots N$. For two different KFs based on κ_i and κ_j, if

$$f_j\big(Y(t)|\kappa_j\big) > f_i\big(Y(t)|\kappa_i\big) \tag{8}$$

or, equivalently,

$$\log f_j\big(Y(t)|\kappa_j\big) > \log f_i\big(Y(t)|\kappa_i\big)$$

we will say that κ_j is more probable to be the true parameter in comparison with κ_i, based on observation $Y(t)$. We can define the likelihood ratio for the sequence of $Y(t)$ as

$$k_i^j\big(Y(t)\big) = \frac{f_j\big(Y(t)|\kappa_j\big)}{f_i\big(Y(t)|\kappa_i\big)} \tag{9}$$

or, equivalently,

$$\log k_i^j\big(Y(t)\big) = \log f_j\big(Y(t)|\kappa_j\big) - \log f_i\big(Y(t)|\kappa_i\big),$$

where $\log k_i^j\big(Y(t)\big)$ can be regarded as a measure of the information in $Y(t)$ in order to select between j^{th}, i^{th} KFs. Similarly, we can define the conditional likelihood ratio,

$$k_i^j\big(y(t)|Y(t-1)\big) = \frac{f_j\big(y(t)|Y(t-1), \kappa_j\big)}{f_i\big(y(t)|Y(t-1), \kappa_i\big)} \tag{10}$$

or, equivalently,

$$\log k_i^j\big(y(t)|Y(t-1)\big) = \log f_j\big(y(t)|Y(t-1), \kappa_j\big) - \log f_i\big(y(t)|Y(t-1), \kappa_i\big),$$

that can be interpreted as a measure of the information in $y(t)$ for selecting between the j^{th} and i^{th} KFs. We can also define the mean information in $y(t)$ for preferring j^{th} KF over the i^{th} KF as

$$d(j,i) = E\{\log k_i^j\big(y(t)|Y(t-1)\big)\}. \tag{11}$$

When $d(j,i)$ is positive we conclude that the j^{th} KF is more probable to be the true KF when compared with the i^{th} KF. The above measure holds the key to the selection of the "best" KF from the model-set. It is not difficult to see that the true KF is always preferred over other KFs.

Proposition 1. *Let the $\star^{th}$ Kalman filter correspond to the true plant. Then, for all the Kalman filters in the model-set we have that*

$$d_t(\star, i) \geq 0, \qquad \forall i = 1, 2, \cdots N \tag{12}$$

with equality if and only if $f_\star\big(y(t)|Y(t-1), \kappa_\star\big) = f_i\big(y(t)|Y(t-1), \kappa_i\big)$.

See [3] for a proof.

The conditional probability density of $y(t)$ given the past observation $Y(t-1)$ when κ_i is the true parameter is given by [2]

$$f_i\big(y(t)|Y(t-1), \kappa_i\big) = \beta_i e^{-w_i(t)}, \tag{13}$$

where $w_i(t)$ and β_i are defined in (6). Denote by $\Sigma_{\kappa_i}^{\kappa_j}$ the covariance of the innovations when the true parameter in the plant is κ_j but the KF is designed based on κ_i. For each KF we have

$$\begin{aligned} &E\{\log f_i\big(y(t)|Y(t-1), \kappa_i\big)\} \\ &= -\frac{q}{2}\log(2\pi) - \frac{1}{2}\log(|\Sigma_{\kappa_i}|) \\ &- \frac{1}{2} tr(\Sigma_{\kappa_i}^{-1} E\{\tilde{y}_{\kappa_i}^T(t|t-1)\tilde{y}_{\kappa_i}(t|t-1)\}) \\ &= -\frac{q}{2}\log(2\pi) - \frac{1}{2}\log(|\Sigma_{\kappa_i}|) - \frac{1}{2} tr(\Sigma_{\kappa_i}^{-1}\Sigma_{\kappa_i}^{\kappa_\star}). \end{aligned} \tag{14}$$

It is now easy to write $d(j,i)$ as

$$\begin{aligned} d(j,i) &= E\{\log f_j\big(y(t)|Y(t-1),\kappa_j\big)\} \\ &\quad - E\{\log f_i\big(y(t)|Y(t-1),\kappa_i\big)\} \\ &= +\frac{1}{2}\log(|\Sigma_{\kappa_i}|) + \frac{1}{2}tr(\Sigma_{\kappa_i}^{-1}\Sigma_{\kappa_i}^{\kappa_\star}) \\ &\quad - \frac{1}{2}\log(|\Sigma_{\kappa_j}|) - \frac{1}{2}tr(\Sigma_{\kappa_j}^{-1}\Sigma_{\kappa_j}^{\kappa_\star}). \end{aligned} \tag{15}$$

Let

$$\Gamma_{\kappa_i}^{\kappa_\star} \equiv +\frac{1}{2}\log(|\Sigma_{\kappa_i}|) + \frac{1}{2}tr(\Sigma_{\kappa_i}^{-1}\Sigma_{\kappa_i}^{\kappa_\star}), \tag{16}$$

which we call the Baram metric. It follows that

$$d(j,i) = \Gamma_{\kappa_i}^{\kappa_\star} - \Gamma_{\kappa_j}^{\kappa_\star}. \tag{17}$$

It is also useful to mention that

$$d(\star,i) - d(\star,j) = \Gamma_{\kappa_i}^{\kappa_\star} - \Gamma_{\kappa_j}^{\kappa_\star}.$$

It can therefore be stated that

$$d(\star,i) \geq d(\star,j)$$

if and only if

$$\Gamma_{\kappa_i}^{\kappa_\star} \geq \Gamma_{\kappa_j}^{\kappa_\star}.$$

Lemma 1. *For the j^{th} and i^{th} KF in the model-set under the assumption that residuals of KFs are ergodic, we have*

$$\lim_{t\to\infty} k_j^i\big(Y(t)\big) = 0, \tag{18}$$

if and only if

$$\Gamma_{\kappa_i}^{\kappa_\star} \geq \Gamma_{\kappa_j}^{\kappa_\star}. \tag{19}$$

See [3] for a proof (the ergodicity of the KF's residual is proved in [3] for stable plants).

This lemma means that posterior probability $p_i(t)$ of the i^{th} KF whose $\Gamma_{\kappa_i}^{\kappa_\star}$ is less than the others will converge to 1.

IV. Model-Set Design Strategy

This section presents a systematic approach to model selection for the MMAE. To this end, it is first convenient to stress the following property of the MMAE: Suppose that the representative parameter set $\{\kappa_1, \kappa_2 ..., \kappa_N\}$ does not include the true parameter $\kappa_\star$. Then, from Lemma 1, as long as the ergodicity condition holds, it can be concluded that one of the posterior probabilities governed by (5), say p_j, converges to 1 and the rest converge to 0 as $t \to \infty$. Obviously, this does not imply that κ_j is $\kappa_\star$. However, in a well defined sense it can be said that the true value $\kappa_\star$ is closer to κ_j than to any other κ_i in the representative parameter-set. This simple reasoning allows us to conclude that, corresponding to each κ_i, $i = 1 \dots N$ there is a set of plants that are naturally identified as κ_i. Each of these sets is called the set of Equivalently Identified Plants (EIP), denoted S_{EIP}^i [6]. With an obvious abuse of notation, for each κ_i, the corresponding EIP is defined as a subset in the uncertain parameter space with the property that if the uncertain parameter belongs to that subset, then the selected model with parameter κ_i will be identified. For more details and how to compute the set of Equivalently Identified Plants (EIP), see [6].

We now address the following questions: What do we gain by using a MMAE approach compared with the Kalman filter? and, for a given required performance, what is the minimum number of models needed and how should they be distributed in the original parameter space?

To address these questions, we introduce the concept of Infinite Model Adaptive Estimation Performance (IMAEP) index. Consider an "ideal" MMAE that assumes an infinite number of models, $N \to \infty$. Clearly, this is equivalent to knowing exactly the true parameter $\kappa_\star$ and designing the corresponding optimal KF. By computing $\Gamma_{\kappa_\star}^{\kappa_\star}$ in (16) we obtain

$$\Gamma_{\kappa_\star}^{\kappa_\star} = +\frac{1}{2}\log(|\Sigma_{\kappa_\star}|) + \frac{q}{2}. \tag{20}$$

This value can be viewed as the best possible performance if we knew the real parameter exactly and the corresponding model was included in the model-set. In practice, to determine the ideal MMAE one uses a dense grid of points in the original parameter uncertainty set. We denote by $\Gamma_{\kappa_\star}^{\kappa_\star}$ the Infinite Model Adaptive Estimation Performance (IMAEP) index which defines the best possible performance for each parameter value.

At this point we can compute the best performance that can be achieved with MMAE. We now present a systematic approach to decide on the number of models and the design of the model-set, that we refer to as the "*the % IMAEP method*". This method fully exploits the information provided by the IMAEP curve. Furthermore, it has the desired strong property that the number of models required for the MMAE is a direct result of performance design specifications. In the so-called % IMAEP approach, the designer specifies that the performance parameter, $\Gamma_{\kappa_i}^{\kappa_\star}$, should be "equal or greater" than X% of the best possible performance, $\Gamma_{\kappa_\star}^{\kappa_\star}$, as defined by the IMAEP.

For simplicity of presentation we assume that the unknown parameter κ is a scalar parameter that lies in the interval $[\kappa_L\ ,\ \kappa_U]$. The basic idea is illustrated in Fig. 2. Starting from the IMAEP curve and using the designer-specified value of X%, we construct the X% IMAEP curve, shown in Fig. 2, and we proceed as follows. Starting from the lower limit κ_L (or upper limit), of the parameter uncertainty set, we design the first KF, and we increase (or decrease) the value of the representative parameter until the point at which $\Gamma_{\kappa_1}^{\kappa_L}$ intersects the the X% IMAEP curve at κ_L; to design the second model, we start from the first point κ_1^b that $\Gamma_{\kappa_1}^{\kappa_1^b}$ intersects the X% IMAEP curve and exits the area between the IMAEP curve and the X% IMAEP curve, and we repeat the procedure (κ_1^b is the first point in $\big(\kappa_L\ \ \kappa_U\big]$ at which $\Gamma_{\kappa_i}^{\kappa_1^b}$ intersects the X% IMAEP curve).

The % IMAEP method is straightforward for systems involving a single scalar uncertain real parameter. In the case of two, or more, uncertain parameters the procedure has to be modified. In fact, in the latter case the IMAEP and the X% IMAEP become surfaces. The above process yields intersection of surfaces (the equivalent of $\Gamma_{\kappa_1}^{\kappa}$, $\Gamma_{\kappa_2}^{\kappa}$, ... are also surfaces). Unfortunately, the intersection of these surfaces does not occur along rectangular (or parallelepiped)

parameter subsets that lead to having overlap between EIP sets of different models.

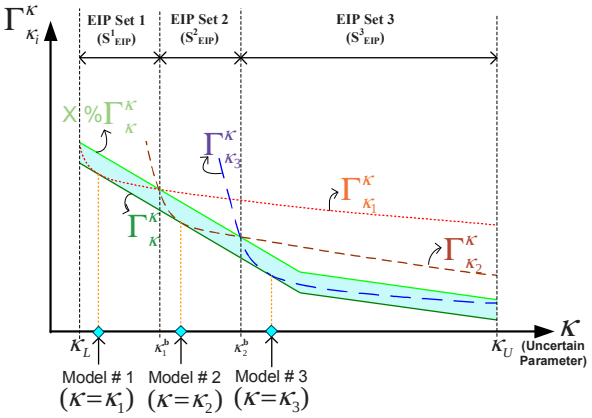

Fig. 2. Model-set Design with the X% IMAEP Strategy.

The following is a procedure by which the number of models and their representative parameters are determined based on the demanded performance.

Algorithm

Input Data: description of the system and the intensity of process and measurement noise and percentage of the demanded performance.

Output Data: number of models and their representative parameters.

Process:

1) For a fine discretization of the parameter space $[\kappa_L \ \ \kappa_U]$, design KF and compute Γ^{κ}_{κ} to obtain the IMAEP.
2) Plot the IMAEP and the X% of IMAEP against the unknown parameter.
3) Let $\kappa_{st} = \kappa_L$, $i = 1$.
4) Design the i^{th} KF at κ_i such that $\Gamma^{\kappa_{st}}_{\kappa_i}$ intersects the X% IMAEP curve and $\Gamma^{\kappa}_{\kappa_i}$ for $\kappa \in [\kappa_{st} \ \ \kappa_i^b]$ is between the curves of the IMAEP and the X% IMAEP. Here, κ_i^b is the first point in $(\kappa_{st} \ \ \kappa_U]$ at which $\Gamma^{\kappa_i^b}_{\kappa_i}$ intersects the X% of IMAEP curve.
5) If $\Gamma^{\kappa}_{\kappa_i}$ for $\kappa \in [\kappa_{st} \ \ \kappa_U]$ is between the curves of the IMAEP and the X% IMAEP, then **stop**; otherwise, let $\kappa_{st} = \kappa_i^b$, $i = i + 1$, and go to 4.

V. ILLUSTRATIVE EXAMPLE

This section illustrates the design methodology described in the previous section. Motivated by [7] we consider a non-minimum phase system described by the transfer function

$$G(s) = \frac{(\frac{1}{0.9}s + 1)(\frac{-1}{Z}s + 1)}{(\frac{1}{2.5}s + 1)(\frac{1}{4.25}s^2 + \frac{1}{4.25}s + 1)}, \qquad (21)$$

which exhibits a non-minimum phase zero at $s = Z$. For this example, we consider that Z is an uncertain zero with a value within the following bounds

$$1 \leq Z \leq 100. \qquad (22)$$

Fig. 3 shows the block diagram of the example adopted where $y(t)$ is the observed output, $u(t)$ is the control input, $d(t)$ is the plant disturbance, and $\theta(t)$ is the sensor noise assumed to be white noise with zero mean and intensity 10^{-3}.

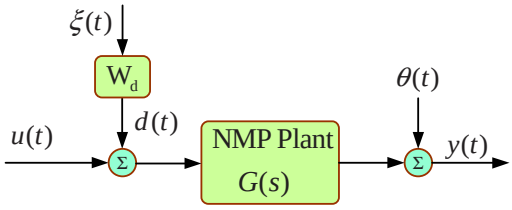

Fig. 3. The non-minimum Phase (NMP) Plant.

The plant is subject to a low-frequency stochastic disturbance input $d(t)$ obtained by filtering white noise $\xi(t)$ with zero mean and unit intensity, as follows:

$$W_d(s) = \frac{d(s)}{\xi(s)} = \frac{0.1}{s + 0.1}. \qquad (23)$$

All simulations for this example were implemented in discrete-time using a zero-order hold with a sampling time of $T_s = 0.01$ secs.

Fig. 5 illustrates the IMAEP curve for the uncertainty zero in the interval (22). As predicted by the analysis in [8], it can be seen that the non-minimum phase zero introduces a performance limitation which gets more accentuated as the zero gets closer to the $j\omega$ axis.

Based on a 99% of IMAEP relaxation, we applied the design strategy proposed in Section IV and obtained three models with the following representative parameter set: $\{1.187, \ 1.88, \ 5.9\}$ (see Fig. 4).

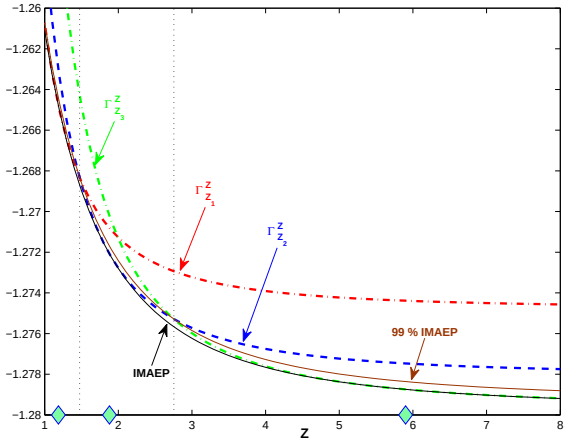

Fig. 4. Model-set Design for 99 % of the IMAEP.

Figures 5-7 illustrate through Monte-Carlo simulations the MMAE using three design strategies: the proposed 99% of IMAEP, a uniform distributed strategy (with representative parameter set $\{17.5, \ 50.5, \ 83.5\}$), and the logarithmically strategy proposed in [9] (with representative parameter set $\{14.68, \ 31.62, \ 68.13\}$); we have also designed , for each

relevant value of κ, the corresponding optimal KF so as to enable comparison with the performance obtained with the MMAE under the different strategies adopted.

It can be seen in Fig. 5 that the RMS of the output estimation error for the three different estimators are consistent with Fig. 5 (see also Figures 6-7 which are zoom-ins on Fig. 5). In particular Fig. 6 shows, in striking contrast with what can be achieved with the other strategies, that the MMAE designed with a 99% IMAEP criterion satisfies the required performance. Fig. 7 shows that only for values of Z greater than 5.3 do the other two MMAE approaches satisfy the 99% performance criteria.

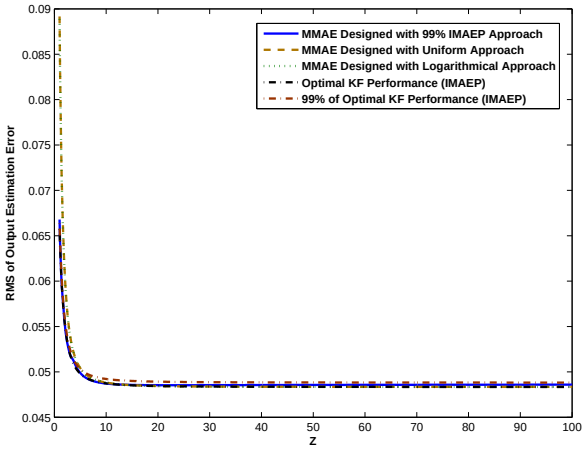

Fig. 5. RMS of the Output Estimation Error for Different Values of Z.

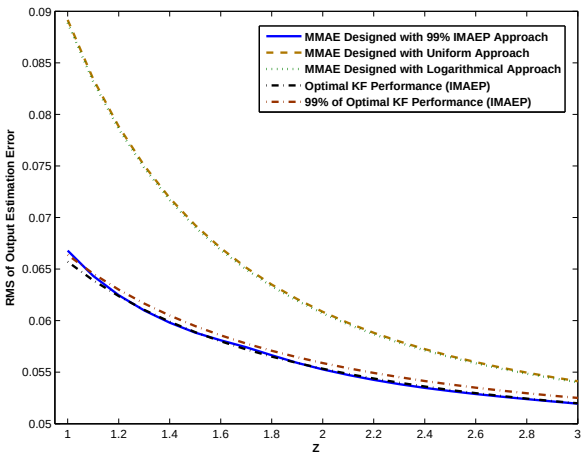

Fig. 6. RMS of the Output Estimation Error for Different Values of Z.

VI. CONCLUSIONS AND FUTURE RESEARCH

For discrete-time, linear, time-invariant MIMO plants with parameter uncertainty, the paper proposed a performance-based model-set design strategy for MMAE design. The methodology addresses in a systematic way the problem of selecting the minimum number of models and how they should be distributed as a function of the parameter uncertainty region (which, in general, does not have to be

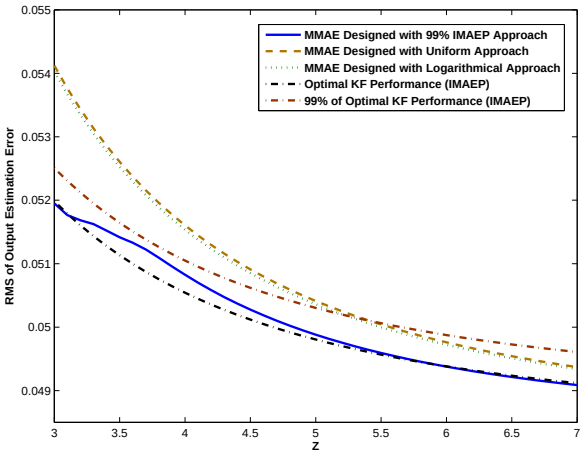

Fig. 7. RMS of the Output Estimation Error for Different Values of Z.

uniform or logarithmic). To this end, the concept of an Infinite Model Adaptive Estimation Performance (IMAEP) index that defines the best achievable performance of the MAAE was introduced. Then, based on a specified demanded performance relative to the ideal IMAEP, an algorithm that guarantees the demanded performance and yields the corresponding finite number of representative models was proposed. The illustrative example showed the improvement in performance that is obtained when compared with other previously proposed design methodologies. Future work will aim at extending the current design methodology to accommodate the case where the unknown parameter is not a scalar, which may imply the study of general geometric properties of the EIP sets. Another topic that warrants further research is the design of adaptive control systems for uncertain plants (with special focus on unstable and non-minimum phase plants) using the multiple model approach.

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