Analysis and Design of Electric Power Grids with p-Robustness Guarantees using a Structural Hybrid System Approach

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Abstract— **In this paper, we address the analysis of resilience properties related to electric power grids modeled as a (large) dynamical system. To this end, we introduce the notion of p-robustness as the capability of ensuring the proper functioning of the electric power grids, in the sense of guaranteeing generic controllability of the associated dynamical system, under arbitrary p transmission line failures. Then, we provide conditions under which the electric power grid is p-robust, and an algorithm that determines the minimum number of transmission lines in the electric power grid that is need to add in order to transform a non-robust (0-robust) electric power grid into a 1-robust electric power grid. Further, we discuss how the methodology can be extended to ensure p-robustness with a relatively small number of additional transmission lines. We present an illustrative example of the proposed analysis and methodology using the IEEE 39-bus system, whose dynamical model is described by 127 state variables.**

I. INTRODUCTION

Electric power grids, as well as many other physical systems, can be modeled as hybrid dynamical systems [1], as long as they present both continuous and discrete behaviors; typical examples include the air conditioning, or circumstances where a control module has to switch [2], such as electric power grid when failure of transmission lines occur, for instance, due to fatigue or/and overheat. In fact, the latter serves as testbed for the present paper.

The *modes* of an hybrid system represent the continuous-time behavior, among which the system switches (or *jump*) – the discrete nature of these systems. If the dynamics described by the modes are linear time-invariant, we have *linear time-invariant switching systems*, which we refer to as *switching systems* and enclose many of the existing hybrid systems [2], [3]. Motivated by the uncertainties in the models' parameters and/or floating point approximations, we resort to structural systems [4], where only the location of the zero/nonzero entries of the system plant matrices is considered. In the context of switching systems, we obtain the so-called *structural switching systems*, a subclass of *structural hybrid systems* introduced in [5]. One of the distinct characteristics of structural systems is the capability of inferring controllability properties that hold almost always. Controllability plays a major role because, once ensured, a control law can be designed to steer the system towards a specified goal. In fact, when considering critical infrastructures, such as electric power grids, certain standards need to be fulfilled over time, for instance, in US it is required that the frequency lies within the 60 ± 5Hz; thus, by ensuring controllability of the system, those standards can be enforced. Nevertheless, before such control law can be specified, we need to ensure that the inputs placed in our system are capable of ensuring systems' controllability. Given the aforementioned discussion, in this paper, we aim to analyze and design a dynamical model of an electric power grid guarantying *p-robustness*, i.e., the capability of ensuring the proper functioning of a system under p transmission line failures, and an initial placement of inputs ensuring generic controllability of the system (also referred to as *structural controllability*, formally introduced in Section II).

Our approach differs from others that aim to explore structural vulnerability and resilience of the network, where properties of the network topology are assessed through the nodes' degrees, edge failures, among other metrics comparable to random graphs [6], [7]. Although, these do not explicitly consider the dynamics of the electric power grid, we refer the reader to [8] for an overview of the area. In addition, several graph theoretical measures have been proposed to assess the power systems' vulnerabilities, see for example [9]. See also [10] and [11], where some conclusions are drawn in terms of graph connectivity, where the dynamics is not considered.

The structural treatment of dynamical properties of switching systems and, in particular, necessary and sufficient conditions to ensure *weak structural controllability*, i.e., structural controllable for some non-empty window of time, have been explored in [12]. However, if at some time the system is structurally controllable for a given instance of time, then it will be weak structurally controllable for any interval of time containing it. Therefore, this approach does not cope with the critical scenario in electric power grids: suppose that a switching system is working under normal operating conditions at a certain instance of time and a transmission line failure occurs, leading the system to a mode where it is not structurally controllable, then it will be weak structurally controllable. Hence, the system may fail to satisfy certain operating standards, since these cannot be enforced. In this paper, the proposed approach and design scenarios ensure that the system is always structurally controllable and, therefore, capable of maintaining the operation standards. Additional references about structural hybrid systems can be found in [5], and applications of structural switching systems can be found, for instance, in [13], [14].

The main contributions of this paper are fourfold: (i) we provide necessary and sufficient conditions to obtain p-robustness of an electric power grid modeled as a dynamical

This work was partially supported by grant SFRH/BD/33779/2009, from Fundação para a Ciência e a Tecnologia (FCT) and the CMU-Portugal (ICTI) program.

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system [3]; (ii) we provide an algorithm to determine a minimum number of transmission lines in the electric power grid that need to be added, to transform a non-robust (0-robust) electric power grid into a 1-robust electric power grid; (iii) we discuss how the presented methodology can be extended to achieve p-robustness by introducing a relatively small number of transmission lines; and (iv) an illustrative example of the proposed analysis and methodology is presented using the IEEE 39-bus system, whose dynamical model is described by 127 state variables.

The rest of the paper is organized as follows. Section II introduces the problem statement addressed in this paper. Section III reviews and introduces some concepts in structural systems theory. Section IV presents the main contributions of this paper. Section V provides an illustrative example using the IEEE 39-bus system. Finally, Section VI concludes the paper and discusses avenues for further research.

II. Problem Statement

Consider the electric power grid as modeled in [3] that consists in a linearized model under normal operating conditions. It can be written in terms of interconnected dynamical subsystems consisting in generators and loads, denoted by G and L. The network topology of the electric power grid is given by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the vertices in $\mathcal{V}$ identify the buses and $\mathcal{E}$ are the edges representing the transmission lines between buses. Further, the electric power grid can be represented by the triple $\mathcal{N} \equiv (\mathcal{G}, \{G_i\}_{i\in\mathcal{I}}, \{L_j\}_{j\in\mathcal{J}})$, where $\mathcal{I}$ and $\mathcal{J}$ are the indices of the buses that generators and loads are connected to, respectively. The dynamics of the electric power grid and its components, modeled as in [3], is described by the following state variables: $P_{T_{G_i}}$ represents the mechanical power of the turbine of the generator G_i, ω_{G_i} the generator G_i frequency and a_{G_i} its valve opening. In addition, l_{L_j} is the real energy consumed by the load L_j and ω_{L_j} the frequency measured at load L_j location. The different components are connected through the injected/received power to/from the network at the connection site, which dynamics depend on the frequency of the components on the neighboring buses; the injected and received power variables for generator i and load j are P_{G_i} and P_{L_j}, respectively.

The electric power grid dynamics modeled as in [3], is denoted by $A(\mathcal{N}(\mathcal{G}(\sigma(t)))) \in \mathbb{R}^{n\times n}$, with $\mathcal{N}(\mathcal{G}(\sigma(t))) = (\mathcal{G}(\sigma(t)), \{G_i\}_{i\in\mathcal{I}}, \{L_j\}_{j\in\mathcal{J}})$, i.e., only the network topology changes over time, and where $\sigma : \mathbb{R}_0^+ \longrightarrow \{1, \ldots, M\}$ is a piecewise constant and deterministic switching signal that may only switch, at most, once in a given dwell-time $[t, t + \varepsilon]$, $\varepsilon > 0$.

Because the exact values of these parameters are not (in general) known, and considered as nominal value with certain error bounds. Thus, rather than considering the exact values, we seek an approach that considers the structure of the system dynamics which consists only on the location of the zeros/nonzeros of the pair (A, B) which we denote by $(\bar{A}, \bar{B})$. A pair $(\bar{A}, \bar{B})$, with $\bar{A} \in \{0, \star\}^{n\times n}$ and $\bar{B} \in \{0, \star\}^{n\times p}$, where $\star$ stands for a nonzero entry, is said to be *structurally controllable* if there exists a numerical realization of the system's plant matrices (A, B) with the same structure (i.e., location of zero/nonzero entries) as $(\bar{A}, \bar{B})$ that is controllable. In fact, a stronger characterization holds and it can be shown that the set of non-controllable numerical realizations (A, B) of a structurally controllable pair $(\bar{A}, \bar{B})$ has zero Lebesgue measure in the product space $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times p}$, in other words, *almost all* numerical realizations of a structurally controllable pair are controllable [4].

Subsequently, in this paper, we aim to address the following two problems.

Problem Statement

$\mathcal{P}_1$ Given structural matrices $\bar{A}(\sigma(t)) \in \{0, \star\}^{n\times n}$, and $\sigma : \mathbb{R}_0^+ \longrightarrow \{1, \ldots, M\}$, a piecewise constant and deterministic switching signal satisfying the dwell-time property, and $\bar{A}(\sigma(0), \bar{B})$ structurally controllable, we aim to provide conditions on $\mathcal{G}(\sigma(t))$ under which $(\bar{A}(\sigma(t)), \bar{B})$ is *structurally controllable* for an arbitrary interval $[t_1, t_2]$, with $0 < t_1 < t_2$. $\diamond$

Then, provided that a power electric grid is non-robust, i.e., it is 0-robust, we address the following problem.

$\mathcal{P}_2$ Given a 0-robust power electric grid $(\bar{A}(\mathcal{N}(\mathcal{G}(t))), B)$, where $\mathcal{G}(\sigma(0)) = (\mathcal{V}, \mathcal{E})$ and $(\bar{A}(\mathcal{N}(\mathcal{G}(0))), \bar{B})$ is structurally controllable, we want to find a minimum set of extra transmission lines $\widetilde{\mathcal{E}} \subset (\mathcal{V} \times \mathcal{V})$ such that power electric grid $(\bar{A}(\mathcal{N}(\tilde{\mathcal{G}}(t))), B)$ is 1-robust, where $\widetilde{\mathcal{G}}(\sigma(0)) = (\mathcal{V}, \mathcal{E} \cup \widetilde{\mathcal{E}})$.$\diamond$

In addition, we also discuss how the provided solution can be used to obtain an arbitrary p-robust of the electric power grid by adding a relatively small number of transmission lines.

III. Preliminaries and Terminology

In this section, we review some concepts of graph theory and its constructs related with structural systems [15].

Structural systems provide an efficient representation of the system as a directed graph (digraph). A digraph consists of a set of *vertices* $\mathcal{V}$ and a set of *directed edges* $\mathcal{E}_{\mathcal{V},\mathcal{V}}$ of the form (v_i, v_j) where $v_i, v_j \in \mathcal{V}$ (meaning a directed edge that starts in v_i and ends in v_j). If a vertex v belongs to the endpoints of an edge $e \in \mathcal{E}_{\mathcal{V},\mathcal{V}}$, we say that the edge e is incident in v. We represent the *state digraph* by $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$, i.e., the digraph that comprises only the state variables as vertices denoted by $\mathcal{X} = \{x_1, \cdots, x_n\}$ and the set of directed edges between the state vertices denoted by $\mathcal{E}_{\mathcal{X},\mathcal{X}} = \{(x_i, x_j) : x_i,\, x_j \in \mathcal{X} \text{ and } \bar{A}_{j,i} \neq 0\}$. Similarly, we represent the *system digraph* by $\mathcal{D}(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$, where $\mathcal{U} = \{u_1, \cdots, u_p\}$ corresponds to the input vertices and $\mathcal{E}_{\mathcal{U},\mathcal{X}} = \{(u_i, x_j) : u_i \in \mathcal{U},\, x_j \in \mathcal{X} \text{ and } \bar{B}_{i,j} \neq 0\}$ to the edges, identifying which state variables are actuated by which input.

A *directed path* between the vertices v_1 and v_k is a sequence of edges $\{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\}$. If all the vertices in a directed path are different, then the path is said to be an *elementary path*. A *cycle* is a directed path such that $v_1 = v_k$ and all remaining vertices are distinct.

We also require the following graph theoretic notions [16]: Given a digraph $\mathcal{D}$, we say that a digraph $\mathcal{D}_s = (\mathcal{V}_s, \mathcal{E}_s)$ such that $\mathcal{V}_s \subset \mathcal{V}$ and $\mathcal{E}_s \subset \mathcal{E}$ is a *subgraph* of $\mathcal{D}$. If

$\mathcal{V}_s = \mathcal{V}$, then $\mathcal{D}_s$ is said to *span* $\mathcal{D}$. In addition, a digraph $\mathcal{D}$ is strongly connected if there exists a directed path between any two vertices. A *strongly connected component* (SCC) is a maximal subgraph, i.e., there is no other subgraph with more edges having the same property, $\mathcal{D}_S = (\mathcal{V}_S, \mathcal{E}_S)$ of $\mathcal{D}$ such that for every $u, v \in \mathcal{V}_S$ there exists a path from u to v and from v to u. By visualizing each SCC as a virtual node, we can build a *directed acyclic graph* (DAG) representation, where there is a directed edge between vertices belonging to two SCCs *if and only if* there exists a directed edge connecting the corresponding SCCs in the original digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$. The construction of the DAG associated with $\mathcal{D}(\bar{A})$ can be computed efficiently in $\mathcal{O}(|\mathcal{V}| + |\mathcal{E}|)$, [16]. The SCCs in the DAG can be categorized as follows.

Definition 1 ([15]): An SCC is said to be linked if it has at least one incoming/outgoing edge from another SCC. In particular, an SCC is *non-top linked* if it has no incoming edges to its vertices from the vertices of another SCC. ⋄

The controllable pairs $(\bar{A}, \bar{B})$ can be characterized as follows.

Theorem 1 ([5], [15]): Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ be the state digraph and its DAG representation with k SCCs, denoted by $\{\mathcal{N}_i\}_{i=1}^{k}$, $\mathcal{N}_i = (\mathcal{X}_i, \mathcal{E}_{\mathcal{X}_i,\mathcal{X}_i})$, where $\mathcal{N}_{i_1}, \ldots, \mathcal{N}_{i_m}$ be the non-top linked SCCs in the DAG representation with $\{i_1, \ldots, i_m\} \subset \{1, \ldots, k\}$. If $\mathcal{D}(\bar{A})$ is spanned by cycles, then $(\bar{A}, \bar{B})$ is structurally controllable if and only if there exists an edge from an input in $\mathcal{D}(\bar{A}, \bar{B})$ to a state variable in each non-top linked SCC $\mathcal{N}_j$, $j \in \{i_1, \ldots, i_m\}$. ⋄

Now, we have the following result that characterizes the structural controllability of structural switching systems.

Proposition 1 ([5]): Let the structural switching system be given by $(\bar{A}(\sigma(t)), \bar{B})$, with $\sigma : \mathbb{R}_0^+ \longrightarrow \{1, \ldots, M\}$. A structural switching system is structurally controllable *if and only if* the pair $(\bar{A}_q, \bar{B}_q)$ is structurally controllable for all $q \in \{1, \ldots, M\}$. ⋄

Finally, we present some additional concepts necessary to study the structure of the network topology constraints and its robustness to transmission line failures in Section IV. For both directed and undirected graph, with some abuse of notation, we denote an edge from vertex x_1 to vertex x_2 by (x_1, x_2). We say that vertex v_i is connected to vertex v_j if there is a sequence of one or more undirected edges of the form (v_i, v_1), (v_1, v_2), …, (v_l, v_j), i.e., a sequence of edges starting in v_i and ending in v_j, which we refer to as path. A tree $\mathcal{T}(\mathcal{V}, \mathcal{E}) = \mathcal{G}(\mathcal{V}, \mathcal{E})$ is an undirected graph in which any two vertices are connected by exactly one path. Given an undirected connected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, a cut C consists in a partition of the vertices $\mathcal{V}$ of the graph into two disjoint subsets $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{V}$, i.e., $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ and $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$. A minimum cut C^* is a cut C with sets $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{V}$ with the minimum number of edges between vertices in the set $\mathcal{V}_1$ and $\mathcal{V}_2$. The size of a cut C, denoted by $|C|$, is the number of edges between the sets of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$. Given an undirected connected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, the complexity of computing a minimum cut is $\mathcal{O}(|\mathcal{V}|^2|\mathcal{E}|)$, using the Dinic's algorithm [16]. In addition, determining the minimum number of edges to be added to a graph to ensure that its minimum cut size increases by one can be achieved using an algorithm with complexity $\mathcal{O}(|\mathcal{C}||\mathcal{V}|^2|\mathcal{E}|)$ where $\mathcal{C}$ denotes the collection of all minimum cuts [17].

Finally, let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, we say that $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ is a contraction of $\mathcal{G}$ if $\mathcal{G}'$ consists of transforming each SCC in $\mathcal{G}$ into a single vertex. This way there is an edge in $\mathcal{G}'$ between two vertices if in $\mathcal{G}$ there is an edge between the vertices of the corresponding SCCs. We say that the tree $\mathcal{T} = (\mathcal{V}', \mathcal{E}')$ is a reduced graph of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if each SCC is contracted into a vertex, without considering edges that are critical cuts of size 1.

IV. Main Results

In this section, we provide the main results of the paper. First, we characterize the state digraph associated with the dynamics of a power electric grid modeled as in [3]. Then, we show that the p-robustness of a system depends on the cuts of the network topology of the electric power grid, but not necessarily the minimum cut. First, sufficient conditions to ensure p-robustness are presented in Theorem 3 and Corollary 1 in terms of the minimum cuts of the network topology. Subsequently, necessary and sufficient conditions are provided in Theorem 4 and Corollary 2 that resort to the notion of *critical cut* introduced in Definition 2. Next, we provide an algorithm (Algorithm 2) that determines a minimum number of transmission lines in the electric power grid that should be considered, such that a non-robust (0-robust) electric power grid becomes a 1-robust electric power grid. Finally, we discuss how the methodology can be extended to ensure p-robustness with a relatively small number of additional transmission lines, see Section IV-E.

A. Digraph Representation of the Dynamical Model of the Power Electric Grid

The state digraph representation $\mathcal{D}(\bar{A}_{G_i})$ and $\mathcal{D}(\bar{A}_{L_j})$ of a generator i and a load j are depicted in Figure 1, with dynamics structure given by $\bar{A}_{G_i}$ and $\bar{A}_{L_j}$, respectively. If these are annexed to a bus, hence to the network, then the induced dynamics is coupled with the power injected/received to/from the network, as described in Section II. Thus, the digraph representation of the dynamics has bidirectional connection between these new variables and the frequency of the corresponding components (corresponding to the injected/received power to/from the network), as depicted in Figure 1. Further, if the load j and generator i are attached to the same bus, or different buses but there exists a transmission line between those, there exists an edge (i, j) in the network topology $\mathcal{G}$; consequently, the frequency of the components at bus i affects the dynamics of the power of the components at bus j, which implies that the digraph of the interconnected dynamics has outgoing edges from the frequencies of the components into the power of the components in the neighboring buses, for example, in Figure 1 is depicted a state digraph, where a generator is connected to bus i, a load is connected to bus j and both buses are connected through a transmission line.

From the above description, we obtain the following result.

Theorem 2: Consider a power electric grid with network topology $\mathcal{N} \equiv (\mathcal{G}, \{G_i\}_{i\in\mathcal{I}}, \{L_j\}_{j\in\mathcal{J}})$, where $\{G_i\}_{i\in\mathcal{I}}$ and

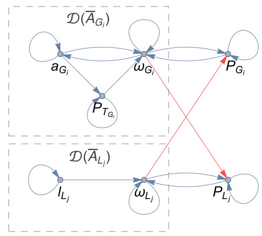

Fig. 1. Coupling between two neighboring components, a load and a generator; $\mathcal{D}(\bar{A}_{G_i})$ and $\mathcal{D}(\bar{A}_{L_j})$ are the generator and load state digraphs, and the state variables are described in Section IV-A.

$\{L_j\}_{j\in\mathcal{J}}$ denote the set of $|\mathcal{J}|$ loads and $|\mathcal{I}|$ generators, respectively. If the network topology $\mathcal{N}$ is such that $\mathcal{G}$ is connected, then $\mathcal{D}(\bar{A}(\mathcal{N}))$, corresponding to the digraph representation of the dynamics of the electric power grid, is composed of several SCCs, where $\max\{1, |\mathcal{J}|\}$ are non-top linked SCCs, and $\mathcal{D}(\bar{A}(\mathcal{N}))$ is spanned by cycles. $\diamond$

Proof: From the description above, if there exists a load L_j, then there is a non-top linked SCC given by l_{L_j}, see Figure 1. Further, a generator G_i with state variable P_{G_i} form an SCC and connects either to a similar (in structure) SCC, when the neighboring component is another generator, or to an SCC originated by a load containing ω_{L_j} and P_{L_j}. Hence, each load contributes with a non-top linked SCC, and if there are no loads and $\mathcal{G}$ is connected, we obtain a single SCC (composed by the state variables associated with generators and the corresponding power state variables), also a non-top linked SCC. Finally, since every state variables have self-loops, the digraph is spanned by cycles. ■

B. Sufficient Conditions to Ensure p-Robustness

Upon the description we provided in Section IV-A, we now explore some sufficient conditions that ensure p-robustness of the electric power grid. We start with the following result.

Theorem 3: Consider an electric power grid with network topology $\mathcal{N} \equiv (\mathcal{G} = (\mathcal{V}, \mathcal{E}), \{G_i\}_{i\in\mathcal{I}}, \{L_j\}_{j\in\mathcal{J}})$, where $\{G_i\}_{i\in\mathcal{I}}$ and $\{L_j\}_{j\in\mathcal{J}}$ denote the set of $|\mathcal{J}|$ loads and $|\mathcal{I}|$ generators, respectively. Let $(\bar{A}(\mathcal{N}), \bar{B})$ be a structurally controllable mode of the structural switching system, and $(\bar{A}(\mathcal{N}'(\mathcal{L})), \bar{B})$ a mode for which the structural switching system transits to after transmission line failures, where $\mathcal{N}'(\mathcal{L}) \equiv (\mathcal{G}' = (\mathcal{V}, \mathcal{E} \setminus \mathcal{L}), \{G_i\}_{i\in\mathcal{I}}, \{L_j\}_{j\in\mathcal{J}})$ and $\mathcal{L}$ is the collection of undirected edges representing the transmission lines that failed. If $\mathcal{L}^* \subset \mathcal{E}$ is a minimum cut of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, then $(\bar{A}(\mathcal{N}'(\mathcal{L})), \bar{B})$ is structurally controllable for any $\mathcal{L} \subset \mathcal{E}$ such that $|\mathcal{L}| < |\mathcal{L}^*|$. $\diamond$

Proof: The proof follows by first noticing that by Theorem 1 and Theorem 2 there exists an input edge from an input to a state variable in each non-top linked SCC of $\mathcal{D}(\bar{A}(\mathcal{N}), \bar{B})$, since it is structurally controllable. Now, $\mathcal{D}(\bar{A}(\mathcal{N}'(\mathcal{L}), \bar{B})$ contains the same the same SCCs as $\mathcal{D}(\bar{A}(\mathcal{N}), \bar{B})$ if $|\mathcal{L}| < |\mathcal{L}^*|$; hence, there exists an input edge from an input to a state variable in each non-top linked SCC, which implies that, by Theorem 1, $\mathcal{D}(\bar{A}(\mathcal{N}'(\mathcal{L}), \bar{B})$ is structurally controllable. ■

Invoking Proposition 1, we have the following corollary.

Corollary 1: Under the same assumptions as in Theorem 3. If $\mathcal{G}(\sigma(t))$ is connected and $(\bar{A}(\sigma(0)), \bar{B})$ is structurally controllable, then $(\bar{A}(\sigma(t)), B)$ is also structurally controllable. $\diamond$

Although, suppose a set of transmission lines that are lost corresponds to one minimum cut on the network topology, in the next subsection we provide conditions to verify if the mode the system transits to after the transmission line failure is structurally controllable.

C. Necessary and Sufficient Conditions to Ensure p-Robustness

Previously, we only provided sufficient conditions to ensure robustness to transmission line failure. Hereafter we provide also necessary conditions addressing, this way, addressing problem $\mathcal{P}_1$. We start by analyzing the scenario painted at the end of the previous section. If we loose a set of transmission lines that correspond to one minimum cut (or cuts containing a minimum cut), the mode of the structural switching system that we transition to may be structurally controllable. Therefore, we propose the following classification for the cuts.

Definition 2: Consider an electric power grid with network topology $\mathcal{N} \equiv (\mathcal{G} = (\mathcal{V}, \mathcal{E}), \{G_i\}_{i\in\mathcal{I}}, \{L_j\}_{j\in\mathcal{J}})$, where $\{G_i\}_{i\in\mathcal{I}}$ and $\{L_j\}_{j\in\mathcal{J}}$ denote the set of $|\mathcal{J}| = l$ loads and $|\mathcal{I}| = m$ generators, respectively. Let $\mathcal{D}(\bar{A}(\mathcal{N}), \bar{B})$ be a structurally controllable mode of the structural switching system, and $(\bar{A}(\mathcal{N}'(\mathcal{L})), \bar{B})$ a mode that the structural switching system transitions to after transmission lines failure, where $\mathcal{N}'(\mathcal{L}) \equiv (\mathcal{G}' = (\mathcal{V}, \mathcal{E} \setminus \mathcal{L}), \{G_i\}_{i\in\mathcal{I}}, \{L_j\}_{j\in\mathcal{J}})$ and $\mathcal{L}$ the collection of the edges representing the transmission lines that failed. If $(\bar{A}(\mathcal{N}'(\mathcal{L})), \bar{B})$ is not structurally controllable, then $\mathcal{L}$ is said to be a *critical cut*. $\diamond$

Intuitively, the previous definition precludes the mischaracterization of islands formed after transmission line failures which are structurally controllable. In the sequence, using Definition 2, we can adapt Theorem 3 to obtain necessary and sufficient conditions to ensure p-robustness of the electric power grid. This is the scope of the next result.

Theorem 4: Consider an electric power grid with network topology $\mathcal{N} \equiv (\mathcal{G} = (\mathcal{V}, \mathcal{E}), \{G_i\}_{i\in\mathcal{I}}, \{L_j\}_{j\in\mathcal{J}})$, where $\{G_i\}_{i\in\mathcal{I}}$ and $\{L_j\}_{j\in\mathcal{J}}$ denote the set of $|\mathcal{J}| = l$ loads and $|\mathcal{I}| = m$ generators, respectively. Let $(\bar{A}(\mathcal{N}), \bar{B})$ be a structurally controllable mode of the structural switching system, and $(\bar{A}(\mathcal{N}'(\mathcal{L})), \bar{B})$ a mode that the structural switching system transits to after transmission lines failure, where $\mathcal{N}'(\mathcal{L}) \equiv (\mathcal{G}' = (\mathcal{V}, \mathcal{E} \setminus \mathcal{L}), \{G_i\}_{i\in\mathcal{I}}, \{L_j\}_{j\in\mathcal{J}})$ and $\mathcal{L}$ the collection of the edges representing the transmission lines that failed. If $\mathcal{L}$ is a non-critical cut of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, then $(\bar{A}(\mathcal{N}'(\mathcal{L})))$ is structurally controllable. $\diamond$

Proof: The proof follows similar steps to that in Theorem 3, where additionally one has to account for the classification of cuts provided in Definition 2. ■

Invoking Proposition 1, we obtain the following corollary.

Corollary 2: Under the same assumptions as in Theorem 4. The structural switching system $(\bar{A}(\sigma(t)), B)$ is structurally controllable (in the sense of Proposition 1) if and only if no jump is due to the failure of a collection of transmission lines containing a critical cut of $\mathcal{G}(\sigma(t))$ occurs, and $(\bar{A}(\sigma(0)), B)$ is structurally controllable. $\diamond$

Observe that Corollary 2 assumes that the set of inputs is fixed. Nevertheless, to preclude the system to lose structural controllability, two strategies are possible: (i) reallocate resources such as inputs to ensure that the dynamical system associated with islands formed after transmission line failures is structurally controllable, see [5] for details; or (ii) redesign the network topology, by adding transmission lines, to ensure that the structural switching system representing the dynamics of the electric power grid has a pre-specified level of robustness. In the next subsection, we explore the latter scenario in further detail.

D. From a Non-robust to a 1*-robust Network Topology*

Now, we address the design problem $\mathcal{P}_2$: first, Algorithm 1 builds a tree containing the edges of the original critical cuts of size 1, used as subroutine of Algorithm 2 that determines how to add a minimum number of edges such that a network topology no longer has critical cuts of size 1.

ALGORITHM 1: Subroutine that produces a tree for which eliminating the edges corresponds to eliminating the critical cuts of size one of the original graph

Input: $\mathcal{N} \equiv (\mathcal{G}, \{G_i\}_{i\in\mathcal{I}}, \{L_j\}_{j\in\mathcal{J}})$, where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has at least one critical minimum cut of size 1
Output: $\mathcal{T} = (\mathcal{V}', \mathcal{E}')$ as the reduced graph obtained from $\mathcal{G}$
1: **remove** the set of edges $\mathcal{E}'$ from $\mathcal{G}$ corresponding to critical cuts of size one, obtaining the graph $\mathcal{G}'$
2: **find** contract each of the SCCs of $\mathcal{G}'$ into a single vertex, renaming the corresponding vertices in $\mathcal{E}'$
3: **return** the tree $\mathcal{T} = (\mathcal{V}', \mathcal{E}')$, where $\mathcal{V}'$ is the set of vertices in which the edges in $\mathcal{E}'$ are incident

Proposition 2: Let $\mathcal{N} \equiv (\mathcal{G}, \{G_i\}_{i\in\mathcal{I}}, \{L_j\}_{j\in\mathcal{J}})$ be the network topology of the electric power grid, where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has at least one critical minimum cut of size 1, and $\mathcal{C}$ the collection of size 1 critical cuts of $\mathcal{G}$, then Algorithm 1 has complexity $\mathcal{O}(|\mathcal{C}||\mathcal{V}|^2|\mathcal{E}|)$, where $\mathcal{O}(|\mathcal{V}|^2|\mathcal{E}|)$ is the complexity of finding a minimum critical cut. ⋄

Proof: Step 1 has complexity $\mathcal{O}(|\mathcal{C}||\mathcal{V}|^2|\mathcal{E}|)$ since we need to compute every critical cut of size 1, where there are $|\mathcal{C}|$ critical cuts of size 1 and $\mathcal{O}(|\mathcal{V}|^2|\mathcal{E}|)$ is the complexity of finding a critical minimum cut; more precisely, it corresponds to determine a minimum cut, partitioning $\mathcal{V}$ in $\mathcal{V}_1$ and $\mathcal{V}_2$, and verify if for both $\mathcal{N}_1 \equiv (\mathcal{G}_1, \{G_i\}_{i\in\mathcal{I}}, \{L_j\}_{j\in\mathcal{J}})$ and $\mathcal{N}_2 \equiv (\mathcal{G}_2, \{G_i\}_{i\in\mathcal{I}}, \{L_j\}_{j\in\mathcal{J}})$, the conditions in Theorem 1 hold. $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}|_{\mathcal{V}_1})$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}|_{\mathcal{V}_2})$ are the subgraphs of $\mathcal{G}$ with sets of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$, and the original edges incident to $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively ($\mathcal{E}|_{\mathcal{V}_1}$ and $\mathcal{E}|_{\mathcal{V}_2}$). Step 2 has complexity bounded by $\mathcal{O}(|\mathcal{E}|)$, corresponding to the complexity of computing the SCCs of $\mathcal{G}'$, the final complexity of the Algorithm 1 is $\mathcal{O}(|\mathcal{C}||\mathcal{V}|^2|\mathcal{E}|)$. ■

Proposition 3: Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph with at least one critical minimum cut of size 1, let $\mathcal{C}$ denote the set of size 1 critical cuts of $\mathcal{G}$, then Algorithm 2 has complexity $\mathcal{O}(|\mathcal{C}||\mathcal{V}|^2|\mathcal{E}|)$, where $\mathcal{O}(|\mathcal{V}|^2|\mathcal{E}|)$ is the complexity of finding a critical minimum critical cut. ⋄

Proof: Step 1 has complexity $\mathcal{O}(|\mathcal{C}||\mathcal{V}|^2|\mathcal{E}|)$, by Proposition 2. The remaining steps have complexity bounded by $\mathcal{O}(|\mathcal{E}|)$, since we visit all the edges in the tree $\mathcal{T}$. ■

Proposition 4: Algorithm 2 is correct, i.e., given a non-robust electric power grid and, in particular, its network topology, Algorithm 2 computes a minimum set of extra transmission lines we need to consider in order to achieve a network topology of a 1-robust electric power grid.

ALGORITHM 2: Determine the minimum number of edges to eliminate critical cuts of size 1

Input: $\mathcal{N} \equiv (\mathcal{G}, \{G_i\}_{i\in\mathcal{I}}, \{L_j\}_{j\in\mathcal{J}})$, where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has at least one critical minimum cut of size 1
Output: $\mathcal{G}' = (\mathcal{V}, \mathcal{E}')$ resulting from adding the minimum possible number of edges to $\mathcal{G}$ such that $\mathcal{G}'$ does not have critical cuts of size 1
1: **build** $\mathcal{T}$, using Algorithm 1 and $\mathcal{G}$
2: **set** $\mathcal{G}' = \mathcal{G}$
3: **find** the set of leaves of $\mathcal{T}$, denoted by $\mathcal{L}$
4: **choose** two vertices $u, v \in \mathcal{L}$
5: **add** the edge (u, v) to $\mathcal{G}'$
6: **add** the edge (u, v) to $\mathcal{T}$ and contract cycles into a single vertex
7: **goto** 3: until $\mathcal{T} = \emptyset$
8: **return** $\mathcal{G}'$

Proof: The proof of the correctness of Algorithm 2 is a known result, see Chapter 4 of [17]. ■

E. Redesigning the Network Topology to Ensure p*-Robustness*

The previous algorithms (Algorithm 1 and Algorithm 2) add the minimum number of transmission lines such that a 0-robust electric power grid becomes 1-robust. Next, we extend these algorithms to achieve p-robustness of the electric power grid.

ALGORITHM 3: Subroutine that produces a tree for which eliminating the edges corresponds to eliminating the critical cuts of size $k-1$ of the original graph

Input: $\mathcal{N} \equiv (\mathcal{G}, \{G_i\}_{i\in\mathcal{I}}, \{L_j\}_{j\in\mathcal{J}})$, where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ has at least one critical minimum cut of size $k-1$
Output: $\mathcal{T} = (\mathcal{V}', \mathcal{E}')$ the tree of the reduced graph obtained from $\mathcal{G}$
1: **remove** the set of edges $\mathcal{E}'$ from $\mathcal{G}$ corresponding to critical cuts of size $k-1$, obtaining the graph $\mathcal{G}'$
2: **find** contract each of the SCCs of $\mathcal{G}'$ into a single vertex, renaming the corresponding vertices in $\mathcal{E}'$
3: **return** the tree $\mathcal{T} = (\mathcal{V}', \mathcal{E}')$, where $\mathcal{V}'$ is the set of vertices in which the edges in $\mathcal{E}'$ are incident

The extension to turn a network topology p-robust by adding a relatively small number of transmission lines can be obtained iteratively as follows: given a $(k-1)$-robust network, to achieve k-robustness we only need to invoke Algorithm 2, where its first instruction uses Algorithm 3 instead of Algorithm 1.

In the next section, we provide an illustrative example that applies the main results presented in this paper.

V. Illustrative Example

IEEE 39-bus power system

The IEEE 39-bus electric power system, with the network topology $\mathcal{N} \equiv (\mathcal{G}, \{G_i\}_{i\in\mathcal{I}\}}, \{L_j\}_{j\in\mathcal{J}})$ depicted in Figure 2, is used as benchmark model in power systems. It corresponds to an electric power grid composed by 39 buses, interconnected through transmission lines (depicted by solid lines between buses in Figure 2). Here, we consider $|\mathcal{I}| = 10$ generators and $|\mathcal{J}| = 29$ loads, coupled through the network topology, depicted in Figure 2. Let $\mathcal{D}(\bar{A}(\mathcal{N}(\sigma(0))))$ denote the state digraph, where $\bar{A}(\mathcal{N}(\sigma(0)))$ is the dynamic of the electric power grid with network topology given by $\mathcal{N}(\sigma(0))$ as depicted in Figure 2.

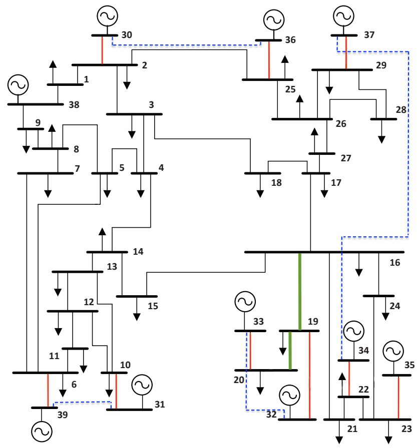

Fig. 2. Network topology $\mathcal{N}(\sigma(0)) \equiv (\mathcal{G}(\sigma(0)), \{G_i\}_{i\in\mathcal{I}}, \{L_j\}_{j\in\mathcal{J}})$ of a 39-bus system consisting of 10 synchronous generators connected to 29 loads. Each red edge corresponds to a critical minimum cut (of size 1), whereas each green edges correspond to the non-critical cuts of size 1. The dashed blue edges represent a possible minimum set of additional transmission lines that, if considered, represent a network topology without critical cuts of size one, and the network becomes robust to one transmission line failure, i.e., 1-robust

Moreover, let the initial mode of the structural switching system be given by the digraph $\mathcal{D}(\bar{A}(\mathcal{N}), \bar{B})$, where $\bar{B} = [e_{L_j}]_{j\in\mathcal{J}}$ denotes a matrix comprehending canonical vectors in which the non-zero entries correspond to the state variables l_{L_j} (i.e., the real load consumption), with $j \in \mathcal{J}$. Therefore, recalling Theorem 2, it follows that $\mathcal{D}(\bar{A}(\mathcal{N}(\sigma(0))), \bar{B})$ is structurally controllable by invoking Theorem 1. Further, a transmission line in $\mathbb{T} = \{(2,30), (25,36), (29,37), (23,35), (19,32), (22,34), (19,32), (20,33), (10,31), (6,39)\}$ (red lines between buses depicted in Figure 2) corresponds to a critical minimum cut (of size 1) of the network topology; contrarily, the green lines between buses depicted in Figure 2 correspond to the non-critical minimum cuts (of size 1). Consequently, the network topology is 0-robust to transmission line failures, recall Theorem 3 and Theorem 4.

In fact, for each possible transmission line failure, except the ones in the set $\mathbb{T}$, the network topology is still connected, hence, by Corollary 1 (see also Corollary 2) we obtain a transition that steers the structural switching system to a structurally controllable mode.

To obtain a 1-robust electric power grid, the model of the network should comprise more transmission lines, i.e., we should redesign it in order to eliminate the critical cuts of size 1. In that case, we use Algorithm 2, as described in Section IV-D, to obtain the network topology with the additional transmission lines depicted in blue dashed edges in Figure 2 and the electric power grid becomes 1-robust.

VI. Conclusions And Further Research

In this paper, we used structural switching systems to model a linearized model of the electric power grid under transmission line failures. We explored conditions of the network topology in the electric power grid that ensure this to be structurally controllable after at most p transmission line failures, i.e., p-robust. Further, we provided a systematic procedure to transform a 0-robust network into a 1-robust considering the lowest subset of transmission lines that need to be added. In addition, we discussed the general procedure to obtain an arbitrary p-robust electric power grid with a relatively small number of additional transmission lines. In this paper, the transmission lines were considered without accounting for geographical or costs constraints, which constitutes part of future research. In addition, an interesting open question consists in integrating fault detection and isolation schemes for the jumps in the structural hybrid system.

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