Optimal Cost Actuator/Sensor Placement for Large Scale Linear Time-Invariant Systems: A Structured Systems Approach

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Abstract— **We address the problem of minimal cost actuator/sensor placement for large scale linear time invariant (LTI) systems that ensures structural controllability/observability. In particular, for the *dedicated* actuator placement problem (i.e., each actuator can control only one state variable or dynamic component), we propose a design methodology that provides the optimal placement with minimal cost (with respect to a given placement cost functional), under the requirement that the system be structurally controllable. In addition of obtaining the global solution of the optimization problem, the methodology is shown to be implemented by an algorithm with *polynomial complexity* (in the number of state variables), making it suitable for large scale systems. By duality, the solution readily extends to the structural design of the corresponding sensor placement under cost constraints.**

I. INTRODUCTION

In networked control systems [1], the problem of topology design to meet certain desired specifications is of fundamental importance. Possible specifications include (but are not restricted to) controllability and observability. These specifications ensure the capability of a dynamical system (such as chemical process plants, refineries, power plants, and airplanes, to name a few), to drive its state toward a specified goal or infer its present dynamic state. To achieve these specifications, actuators and sensors must be deployed in the network. More than often, we need to consider the cost per actuator/sensor, that depends on its range of applicability and/or its installation and maintenance cost. The resulting placement cost optimization problem (apparently combinatorial) turns out to be quite non-trivial, and currently applied state-of-art methods typically consider relaxations of the optimization problem, brute force approaches or heuristics, see for instance [2-6].

Given that the precise numerical values of the network parameters are generally not available for the large-scale systems of interest, a natural direction is to consider structured systems [7] based reformulations of the above topology design problems, which we pursue in this paper. Representative work in structured systems theory may be found in [8], [9], [10], [11], see also the survey [12] and references therein. The main idea is to reformulate and study an equivalent class of systems for which system-theoretic properties are investigated based on the location of zeroes/non-zeroes of the state space representation matrices. Properties such as controllability and observability are, in this framework, referred as *structural controllability*[1] and *structural observability*.

Hereafter we focus on the following problem: Given the network topology, where should we place a minimal number of actuators (resp. sensors) such that the dynamical system is structurally controllable (resp. structurally observable) and the assignment cost is minimal?

In this paper we assume that such actuators (resp. sensors) are dedicated, i.e., they manipulate (resp. measure) a single state variable. The mathematical formulation can now be stated is as follows:

<u>*Problem Statement*</u>

Consider a dynamical system

$$\dot{x} = \bar{A}x, \tag{1}$$

where $\bar{A}$ is a $n \times n$ binary matrix that represents the structural pattern of an LTI system dynamics [7].

$\mathcal{P}_1$ Let $C(x) \in \mathbb{R}$, $x \in \mathcal{X} = \{x_1, \cdots, x_n\}$ be the cost of manipulating the state variable x.
Design the input matrix $\bar{B} = \mathbf{I}_{\mathcal{J}^*}$, i.e., find a subset $\mathcal{J}^*$ of columns of the indices of the identity matrix $\mathbf{I}$ such that

$$\begin{aligned} \mathcal{J}^* = \arg \min_{\mathcal{J} \subset \{1,\cdots,n\}} \quad & \mathbf{1}_{\mathcal{J}}^T (\mathcal{C}_{\mathcal{J}} \circ \mathbf{I}_{\mathcal{J}}) \mathbf{1}_{\mathcal{J}} \\ \text{s.t.} \quad & (\bar{A}, \mathbf{I}_{\mathcal{J}}) \text{ is structurally controllable} \\ & |\mathcal{J}| \text{ is minimal} \end{aligned}$$

where $\mathcal{C}_{\mathcal{J}}$ consists of the subset of columns of the cost matrix $\mathcal{C} = \text{diag}(C(x_1), \cdots, C(x_n))$, $\mathbf{1}_{\mathcal{J}}$ is the $1 \times |\mathcal{J}|$ vector of ones and $\circ$ denotes the Hadamard product (i.e., the entry-wise product). By $|\mathcal{J}|$ being minimal, it follows that there exist no other $\mathcal{J}'$, with $|\mathcal{J}'| < |\mathcal{J}|$ such that $(\bar{A}, \mathbf{I}_{\mathcal{J}'})$ is structurally controllable.

Remark 1: Note that the solution procedure for $\mathcal{P}_1$ also addresses the corresponding structural observability output matrix design problem by invoking the duality between estimation and control in LTI systems. $\diamond$

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[1]A pair (A, B) is said to be structurally controllable if there exists a pair (A', B') with the same structure as (A, B), i.e., same locations of zeroes and non-zeroes, such that (A', B') is controllable. By density arguments [10], it may be shown that if a pair (A, B) is structurally controllable, then almost all (with respect to the Lebesgue measure) pairs with the same structure as (A, B) are controllable. In essence, structural controllability is a property of the structure of the pair (A, B) and not the specific numerical values. A similar definition and characterization holds for structural observability (with obvious modifications).

At first view, $\mathcal{P}_1$ seems to be strictly combinatorial since its solution requires to test if a subset $\mathcal{J} \subseteq \{1, \cdots, n\}$ leads to the design of $\bar{B} = \mathbf{I}_{\mathcal{J}}$ such that $(\bar{A}, \mathbf{I}_{\mathcal{J}})$ is structurally controllable and, among these subsets we need to select one that minimizes the cost function that may have several solutions. Note that there exist a total of 2^n subsets which originates an intractable problem from a computational point of view. To the best of authors knowledge, this is the first work that solves the actuator/sensor placement using structural theory with cost constraints. In [13], [14] optimal placement under cost constraints is explored, although under different assumptions and approximations are achieved using distributed Gauss elimination method. On the other hand, optimal sensor and actuator placement with uniform costs (identical, constant across variables) have been studied previously, for instance [2] and references therein; however, these approaches mostly lead to combinatorial implementation complexity in the number of state variables, or are often based on simplified heuristic-based reductions of the optimal design problems. Systematic approaches to structured systems based design were investigated recently in the context of different application scenarios, see, for example, [15-20]; for instance, in network estimation, as in [15], [18], where strategies for output (sensor) placement are provided, ensuring only sufficient (but not necessarily minimal) conditions for structural observability, whereas in [15], [19] applications to power system state estimation are explored. In [21], we provided a characterization of *minimal feasible dedicated input (resp. output) configurations* Θ, in other words, the minimal subset(s) of state variables to which actuators (resp. sensors) need to be assigned to ensure structural controllability (resp. observability). This problem can be understood as a particular case of $\mathcal{P}_1$ where the costs are assumed to be uniform across all state variables, so that, the design goal reduces to obtaining the minimum number of dedicated inputs (outputs) ensuring structural controllability (observability). In [21], a polynomial complexity (in the number of state variables) algorithm was provided to characterize the set of (all) minimal feasible dedicated input configurations Θ and also, a polynomial complexity algorithm to compute such a single minimal feasible dedicated input configuration. However, the techniques developed in [21], although provide useful insights, are not sufficient to address the solution of $\mathcal{P}_1$ with generic cost functionals.

The main contributions of this paper are twofold: first, we show that we can solve $\mathcal{P}_1$ by restricting $\mathcal{S}$ to Θ (the set of all possible minimal feasible dedicated input configurations) and secondly, we provide a polynomial complexity algorithm to solve the above constrained but equivalent version of $\mathcal{P}_1$, thus also recovering the solution of $\mathcal{P}_1$.

The rest of the paper is organized as follows: Section II presents notation and reviews results from optimal actuator-sensor placement in LTI systems with uniform placement cost. Section III presents the main results of the paper, in particular, a procedure to determine the minimal cost placement of actuators in LTI systems as formulated in $\mathcal{P}_1$. In section IV we illustrate the procedure through an example. Finally, Section V concludes the paper.

II. PRELIMINARIES AND TERMINOLOGY

In this section we recall some classical concepts in structural systems [8], to be used in the subsequent development.

Given a dynamical system (1), an efficient approach to the analysis of its structural properties is to associate it with a directed graph (digraph) $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, in which $\mathcal{V}$ denotes a set of *vertices* and $\mathcal{E}$ represents a set of *edges*, such that, an edge (v_j, v_i) is directed from vertex v_j to vertex v_i. Denote by $\mathcal{X} = \{x_1, \cdots, x_n\}$ and $\mathcal{U} = \{u_1, \cdots, u_p\}$ the set of state vertices and input vertices, respectively. Denote by $\mathcal{E}_{\mathcal{X},\mathcal{X}} = \{(x_i, x_j) : [\bar{A}]_{ij} \neq 0\}$ and $\mathcal{E}_{\mathcal{U},\mathcal{X}} = \{(u_j, x_i) : [\bar{B}]_{ij} \neq 0\}$, to define $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ and $\mathcal{D}(\bar{A}, \bar{B}) = (\mathcal{X} \cup \mathcal{U}, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{U},\mathcal{X}})$. A digraph $\mathcal{D}_s = (\mathcal{V}_s, \mathcal{E}_s)$ with $\mathcal{V}_s \subset \mathcal{V}$ and $\mathcal{E}_s \subset \mathcal{E}$ is called a *subgraph* of $\mathcal{D}$. If $\mathcal{V}_s = \mathcal{V}$, $\mathcal{D}_s$ is said to *span* $\mathcal{D}$. A sequence of edges $\{(v_1, v_2), (v_2, v_3), \cdots, (v_{k-1}, v_k)\}$, in which all the vertices are distinct, is called *an elementary path* from v_1 to v_k. When v_k coincides with v_1, the sequence is called a *cycle*.

In addition, we will require the following graph theoretic notions [22]: A digraph $\mathcal{D}$ is said to be strongly connected if there exists a directed path between any two pairs of vertices. A strongly connected component (SCC) is a maximal subgraph $\mathcal{D}_S = (\mathcal{V}_S, \mathcal{E}_S)$ of $\mathcal{D}$ such that for every $v, w \in \mathcal{V}_S$ there exists a path from v to w and from w to v. Note that, an SCC may have several paths between two vertices and the path from v to w may comprise some vertices not in the path from w to v. Visualizing each SCC as a virtual node (or supernode), one may generate a *directed acyclic graph* (DAG), in which each node corresponds to a single SCC and a directed edge exists between two SCCs *iff* there exists a directed edge connecting the corresponding SCCs in the original digraph. The DAG associated with $\mathcal{D} = (\mathcal{V}, \mathcal{E})$ may be efficiently generated in $\mathcal{O}(|\mathcal{V}| + |\mathcal{E}|)$ [22], where $|\mathcal{V}|$ and $|\mathcal{E}|$ denote the number of vertices in $\mathcal{V}$ and the number of edges in $\mathcal{E}$, respectively. The SCCs in a DAG can be characterized as follows:

Definition 1: An SCC is said to be linked if it has at least one incoming/outgoing edge from another SCC. In particular, an SCC is *non-top linked* if it has no incoming edges to its vertices from the vertices of another SCC and *non bottom linked* if it has no outgoing edges to another SCC. $\square$

For any two vertex sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{V}$, we define the *bipartite graph* $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$ associated with $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, to be a directed graph (bipartite), whose vertex set is given by $\mathcal{S}_1 \cup \mathcal{S}_2$ and the edge set $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}$ by $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2} = \{(s_1, s_2) \in \mathcal{E} : s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2 \}$.

Given $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$, a matching M corresponds to a subset of edges in $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}$ that do not share vertices, i.e., given edges $e = (s_1, s_2)$ and $e' = (s_1', s_2')$ with $s_1, s_1' \in \mathcal{S}_1$ and $s_2, s_2' \in \mathcal{S}_2$, $e, e' \in M$ only if $s_1 \neq s_1'$ and $s_2 \neq s_2'$. A maximum matching M^* may then be defined as a matching M that has the largest number of edges among all possible matchings. The maximum matching problem may be solved efficiently in $\mathcal{O}(\sqrt{|\mathcal{S}_1 \cup \mathcal{S}_2|}|E_{\mathcal{S}_1,\mathcal{S}_2}|)$ [22]. Vertices in $\mathcal{S}_1$ and $\mathcal{S}_2$ are *matched vertices* if they belong to an edge in the maximum matching M^*, otherwise, we designate the vertices as *unmatched vertices*. If there are no unmatched vertices, we say that we have a *perfect match*. It is to be noted that a maximum matching M^* may not be unique.

For ease of referencing, in the sequel, the term *right-unmatched vertices* (w.r.t. $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$ and a maximum matching M^*) will refer to only those vertices in $\mathcal{S}_2$ that do not belong to a matched edge in M^*.

In case a graph is composed of multiple SCCs, we define

Definition 2: Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ and M^* be a maximum matching associated with $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$. A non-top linked SCC is said to be a *top assignable SCC* if it contains at least one right-unmatched vertex (with respect to M^*). $\square$

Note that the total number of top assignable SCCs may depend on the particular maximum matching M^* (not unique in general) under consideration; as such we may define:

Definition 3: Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ and M^* a maximum matching associated with $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$. The *maximum assignability index* of $\mathcal{D}(\bar{A})$ is the maximum number of top assignable SCCs that a maximum matching M^* may lead to. $\square$

The following results from [21] which consider the characterization of the minimal feasible dedicated input configurations (or equivalently, the specific case of $\mathcal{P}_1$ with uniform costs, so that, the design goal reduces to obtaining the minimum number of dedicated inputs ensuring structural controllability) will be used in the sequel.

Theorem 1 (Minimum Number of Dedicated Inputs): Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ be the system digraph with β non-top linked SCCs in its DAG representation. Let M^* be a maximum matching associated with the bipartite graph $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ and let $\mathcal{U}_R \subset \mathcal{X}$ be the set of corresponding right-unmatched vertices. Then, the minimum number of dedicated inputs p is given by

$$p = m + \beta - \alpha, \tag{2}$$

where $m = |\mathcal{U}_R|$ and α denotes the maximum top assignability index of $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$. $\diamond$

The characterization of the set of all minimal feasible dedicated input configurations Θ, i.e., the minimal subset(s) of state variables to which actuators need to be assigned to ensure structural controllability, is described as follows:

the minimal subset(s) of state variables to which actuators (resp. sensors) need to be assigned to ensure structural controllability (resp. observability).

Theorem 2 (Naturally Constrained Partitions): Let $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ be a digraph with $|\mathcal{X}| = n$ and $\mathcal{N}^i = (\mathcal{X}^i, \mathcal{E}_{\mathcal{X}^i,\mathcal{X}^i})$, for $i = 1, \cdots, \beta$, be the β non-top linked SCCs of the DAG representation of $\mathcal{D}(\bar{A})$, with $\mathcal{X}^i \subset \mathcal{X}$ and $\mathcal{E}_{\mathcal{X}^i,\mathcal{X}^i} \subset \mathcal{E}_{\mathcal{X},\mathcal{X}}$. In addition, let $\sigma : \{1, \cdots, n\} \to \{1, \cdots, n\}$, a bijective function representing a permutation of the indices and M^* be a maximum matching associated with the bipartite graph $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ with $m = |\mathcal{U}_R|$ right-unmatched vertices, where $\mathcal{U}_R = \{v_1, v_2, \ldots, v_m\} \subset \mathcal{X}$ is the set of right-unmatched vertices with respect to M^* and p denotes the minimum number of dedicated inputs as in Theorem 1.

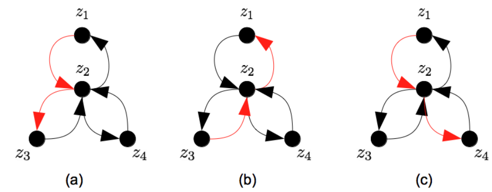

Fig. 1. Some possible maximum matching for a 4-vertices star network.

There exist subsets $\Theta^j \subset \mathcal{X}$, $j = 1, \cdots, p$, given by

- $j = 1, \cdots, m$: $\Theta^j = \{x \in \mathcal{X} : (\mathcal{V} - \{v_j\}) \cup \{x\}$ is the set of right-unmatched vertices for some maximum matching of $\mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})\}$,
- $j = m + 1, \cdots, p$: $\Theta^j = \bigcup\limits_{l \in \{1, \cdots, \beta\}} \mathcal{N}^l$,

such that, the set Θ of minimal feasible dedicated input configurations may be characterized as follows: The subset $\mathcal{S}_u = \{x_{\sigma(1)}, \cdots, x_{\sigma(p)}\} \subset \mathcal{X}$ is a member of Θ if and only if the following *natural constraints* hold:

(i) $x_{\sigma(j)} \in \Theta^j$, for $j = 1, \cdots, p$;
(ii) $x_{\sigma(j)} \in \Theta^j$ and $x_{\sigma(j')} \in \Theta^{j'}$ for $j \neq j'$ with $1 \leq j, j' \leq m$ implies that $x_{\sigma(j)}$ and $x_{\sigma(j')}$ are different two right-unmatched vertices of a maximum matching;
(iii) each non-top linked SCC $\mathcal{N}^l$ has at least one state variable in $\mathcal{S}_u$ that belongs to $\mathcal{N}^l$. $\diamond$

Finally, we have the following result concerning the existence of polynomial complexity algorithms for the instances referred in Theorem 2 and Theorem 3.

Theorem 3 (Complexity): There exist algorithms of polynomial complexity (in the number of state vertices) to implement the following procedures:

1) obtaining the minimum number of dedicated inputs;
2) constructing the natural constrained partitions, Θ^j's;
3) generating a minimal feasible dedicated input configuration iteratively. $\square$

We now illustrate the previous results with the following example: consider the 4-vertices star network $\mathcal{G}$, in other words, one vertex in the centre connected with bidirectional edges to all the others, depicted in Figure 1 together with some possible maximum matchings (depicted by red edges). From Theorem 1 and the fact that $\mathcal{G}$ is an SCC, we have that the minimum number of inputs equals the number of right-unmatched vertices. From Theorem 2, given the minimal feasible dedicated input configuration $\mathcal{S} = \{z_1, z_4\}$ (corresponding to the maximum matching depicted in Fig.1-(a)), we now compute Θ^j as in Theorem 2: $\Theta^1_{\mathcal{S}} = \{z_1, z_3\}$ and $\Theta^2_{\mathcal{S}} = \{z_4, z_3\}$. In other words, by definition of the Θ^j's sets, we can replace z_1 or z_4 by z_3 in $\mathcal{S}$ and we still have a minimal feasible dedicated input configuration. In these sets z_3 belongs to $\Theta^1_{\mathcal{S}}$ (as in Figure 1-b)). On the other hand, z_3 belongs to $\Theta^2_{\mathcal{S}} = \{z_4, z_3\}$ as can be inferred from Figure 1-c). Note that there is no uniqueness in the maximum matching and all relevant information is stated in terms of the right-unmatched vertices. In general, in this way, all possible configurations are described by the naturally constrained partitions.

III. MAIN RESULTS

We first show that solving $\mathcal{P}_1$ reduces to solving

$\mathcal{P}_2$ Find $\mathcal{S} \in \Theta$ (the set of all possible minimal feasible dedicated input configurations) that minimizes

$$\sum_{j=1}^{n} \mathbb{I}_{\mathcal{S}}(x_j)(C(x_j)). \tag{3}$$

It is important to stress that in [21], only a representation of Θ is obtained using a polynomial complexity algorithm and not the Θ itself. In fact, if the set of all possible minimal feasible dedicated input configurations were to be computed with polynomial complexity, $\mathcal{P}_1$ constrained to $\mathcal{S} \in \Theta$ would be straightforward to solve for non uniform costs as in (3), since it will only require to go through the possible minimal feasible dedicated input configurations and select the one that achieves the minimal cost.

The desired reduction is obtained in Theorem 4. Subsequently, without explicitly computing Θ we provide an algorithm (with polynomial complexity in the number of state variables) that solves (3). This procedure is presented in Algorithm 1, and its correctness and complexity are analyzed in Theorem 5 and Theorem 6 respectively.

Theorem 4: Let $C : \mathcal{X} \rightarrow \mathbb{R}$ be the cost function. Then, any solution $\mathcal{S}$ to $\mathcal{P}_1$ satisfies $\mathcal{S} \in \Theta$ and is a solution to $\mathcal{P}_2$. Similarly, any solution $\mathcal{S}'$ to $\mathcal{P}_2$ is a solution to $\mathcal{P}_1$. □

Proof: Follows by definition of feasible dedicated input configuration $\mathcal{S} \in \Theta$, which implies that $(A, \mathbf{I}_{\mathcal{J}})$ is struct. controllable and $|J|$ is minimal. In addition, by developing the cost function in $\mathcal{P}_1$ we obtain the cost in $\mathcal{P}_2$. ■

Remark 2: Note that if an input has the capability to control different variables simultaneously, without increasing the cost, then the problem would be different and much more difficult since one needs to consider in advance the subsets of variables that are controlled by the same input signal.

Algorithm 1 provides a procedure to find the solution to $\mathcal{P}_2$ (and hence, $\mathcal{P}_1$ by Theorem 4). Intuitively, Algorithm 1 is reminiscent of techniques used in the theory of submodular function optimization [23]. Informally, the idea is to sequentially construct a subset of variables $\mathcal{F} \subset \mathcal{X}$ based on the following recursive criteria: for every $x \in \mathcal{X}$ take $\mathcal{F} \cup \{x\}$ and evaluate the cost function on it. Since we would like to minimize the cost function we have to consider the variable that contributes the least in terms of cost increase; add such a variable to $\mathcal{F}$ and proceed in the same way until a specified specification is achieved, for instance a designated number of variables. It is easy to see that in our case we may be selecting variables that do not contribute to ensure structural controllability, increasing the final cost unnecessarily. Nevertheless, using the characterization of Θ as in Theorem 3, our approach rules out this possibility by selecting variables from the Θ^is. This is a key difference from general submodular function optimization procedures, as in our case, instead of selecting variables sequentially from the same set, we select variables from different sets chosen adaptively as the algorithm progresses.

Now consider the following result, concerning the complexity of Algorithm 1.

ALGORITHM 1: Find the minimal feasible dedicated input configuration that incurs in the minimal cost

Input: $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ and cost $C(x)$ for $x \in \mathcal{X}$
Output: Minimal feasible dedicated input configuration $\mathcal{S}^*$ that incurs in the minimal cost
1. Compute the non-top linked SCC of $\mathcal{D}(\bar{A})$ and denote them by $\mathcal{N}^l, l \in \{1, \cdots, k\}$
2. Compute an initial minimal feasible dedicated input configuration $\mathcal{S}$ (see Theorem 2);
Initialize: $\mathcal{S}^* = \mathcal{S}$;
$ThetaSetsUsed = \{\}$;
$nonTopLinkedSCC = \{\}$;
for 1 *to* k **do**
  3. Compute $\Theta^j_{\mathcal{S}^*}$ for $j \in \mathcal{J} = \{1, \cdots, |\mathcal{S}^*|\} - ThetaSetsUsed$;
  $\mathcal{L} = \{1, \cdots, k\} - nonTopLinkedSCC$;
  % non-top linked SCCs to be explored
  4. Let $x^* = \arg \min_{x \in \left(\Theta^j_{\mathcal{S}^*} \cap \bigcup_l \mathcal{N}^l\right),\ j \in \mathcal{J}, l \in \mathcal{L}} C(x)$;
  5. Determine the $\Theta^j_{\mathcal{S}^*}$ for $j \in \mathcal{J}$ and $\mathcal{N}^l$ for $l \in \mathcal{L}$ to which x^* belongs to, say $x^* \in \Theta^{j^*}_{\mathcal{S}^*}$ and $x^* \in \mathcal{N}^{l^*}$, respectively;
  $ThetaSetsUsed = ThetaSetsUsed \cup \{j^*\}$;
  $nonTopLinkedSCC = nonTopLinkedSCC \cup \{l^*\}$;
  $\mathcal{S}^*_{j^*} = x^*$; % entry j^* of $\mathcal{S}^*$ is replaced by x^*
end
for $k+1$ *to* $|\mathcal{C}^*|$ **do**
  3. Compute $\Theta^j_{\mathcal{S}^*}$ for $j \in \mathcal{J} = \{1, \cdots, |\mathcal{S}^*|\} - ThetaSetsUsed$;
  4. Let $x^* = \arg \min_{x \in \Theta^j_{\mathcal{S}^*},\ j \in \mathcal{J}} C(x)$;
  5. Determine the $\Theta^j_{\mathcal{S}^*}$ for $j \in \mathcal{J}$ to which x^* belongs to, say $x^* \in \Theta^{j^*}_{\mathcal{S}^*}$;
  $ThetaSetsUsed = ThetaSetsUsed \cup \{j^*\}$;
  $\mathcal{S}^*_{j^*} = x^*$; % entry j^* of $\mathcal{S}^*$ is replaced by x^*
end

Theorem 5 (Complexity of Algorithm 1): Algorithm 1 has polynomial complexity (in the number of state variables in (1)). □

Proof: Briefly, follows by noticing that in the algorithm we are only required to: 1) compute a DAG of the graph; 2) maximum matchings of some bipartite graph; 3) find the argmin in a finite set; 4) compute a feasible dedicated input configuration and 5) compute the Θ^j's. The computation of 1)-3) can be done in polynomial complexity in the number of vertices of the graph (which are state vertices) and 4)-5) can be done in polynomial complexity in the number of state variables by Theorem 3. Hence the result follows. ■

The correctness of Algorithm 1 is established in the following:

Theorem 6 (Correctness of Algorithm 1): Algorithm 1 is correct. In other words, it provides a minimal feasible dedicated input configuration that incurs in the minimum cost in $\mathcal{P}_2$ (and hence $\mathcal{P}_1$ by Theorem 4). □

Proof: Denote by $\mathcal{S}^t$, the instance of $\mathcal{S}^*$ obtained at the end of iteration t of Algorithm 1, $t = 0, \cdots, p$, with $\mathcal{S}_0 = \mathcal{S}$. We need to show, that $\mathcal{S}_p$ is indeed a solution to $\mathcal{P}_2$.

Note that by construction $|\mathcal{S}^t| = p$ for all t. To show that $\mathcal{S}^p$ is a minimal feasible dedicated input configuration we proceed inductively: at each iteration t for $t = 1, \cdots, p$ we show that $\mathcal{S}^t$ satisfies conditions i) and ii) of Theorem 2, whereas, condition (iii) is shown to be verified for all $\mathcal{S}^t$, $t = k+1, \cdots, p$, i.e., from iteration $k+1$ onwards, after at least one dedicated input has been assigned to at least each of the k non-top linked SCCs.

For the iterations $t = 0, 1, \cdots, p$ we now have that $\mathcal{S}^t$ satisfies condition i) and ii) of Theorem 2. Indeed, $\mathcal{S}^0$ satisfies these conditions (by Theorem 2), being a minimal feasible dedicated input configuration. We now show that $\mathcal{S}^1$ also satisfies these conditions as follows:

i) $\mathcal{S}^0$ is a minimal feasible dedicated input configuration computed in Step 2 of Algorithm 1, so by definition it belongs to Θ and by Theorem 2 it satisfies conditions i) and ii). $\mathcal{S}^1$ results from changing one entry in $\mathcal{S}^0$, say $\mathcal{S}^1_\alpha$ by a variable in Θ^α hence i) holds since $\mathcal{S}^1_\beta = \mathcal{S}^0_\beta \in \Theta^\beta$ for all $\beta \in \{1, \cdots, n\} - \alpha$ (that we verified to satisfy i) in Theorem 2) and $\mathcal{S}^1_\alpha$ is a variable in Θ^α. Hence, $\mathcal{S}^1$ satisfies i).

ii) Note that, $\mathcal{S}^1_\alpha$ is a variable in Θ^α hence, by the definition of Θ^α (in Theorem 2), it corresponds to a right-unmatched vertex or to a vertex in a non-top linked SCC. Moreover, we have, for $\beta \in \{1, \cdots, n\} - \alpha$, $\mathcal{S}^1_\beta = \mathcal{S}^0_\beta$, where each $\mathcal{S}^0_\beta$ satisfies (ii). It then follows that $\mathcal{S}^1$ satisfies (ii) also.

Using the same inductive reasoning, it then follows that conditions i) and ii) hold for $\mathcal{S}^t$ as long as they hold $\mathcal{S}^{t-1}$ for $t = 2, \cdots, p$. As far as condition iii) in Theorem 2 is concerned, note that, it only holds at the end of the first **for**, since this ensures that one state variable is selected from each of the non-top linked SCCs (by restricting the search of variables to the intersection of Θ^j for $j \in \mathcal{J}$ and S^l for $l \in \mathcal{L}$). Indeed, from Step $k+1$ onwards, those variables belonging to the non-top linked SCCs are kept unchanged, and hence, $\mathcal{S}^t$, for $t = k+1, \cdots, p$ has at least one variable from each non-top linked SCC, and thus satisfies condition iii) in Theorem 2. Hence, conditions i)-iii) from Theorem 2 hold, and it follows that $\mathcal{S}^p$, i.e., the final $\mathcal{S}^*$ is a minimal feasible dedicated input configuration.

Now, we show that the minimal feasible dedicated input configuration $\mathcal{S}^* = \{x^*_{i_1}, \cdots, x^*_{i_p}\}$(obtained after the execution of Algorithm 1) incurs in the minimal cost in $\mathcal{P}_2$. The argument follows by contradiction: without loss of generality (since $\mathcal{S}^*$ is order invariant) consider that $C(x^*_{i_1}) \leq \cdots \leq C(x^*_{i_p})$ and suppose there exists another minimal feasible dedicated input configuration $\mathcal{S}' = \{x'_{i_1}, \cdots, x'_{i_p}\}$ that incurs in a cost smaller than $\mathcal{S}^*$. Similarly, let $C(x'_{i_1}) \leq \cdots \leq C(x'_{i_p})$ and because the cost in (3) is additive and strictly positive, it follows that there exists $\mathcal{V} \subset \mathcal{S}'$ comprising of state variables x'_a, for each $a \in \mathcal{A}$ (i.e. the a-th entry of $\mathcal{S}'$), such that $x'_a < x^*_a$. Let $\bar{a} = \arg\min_a \mathcal{A}$, in other words, the smallest entry of $\mathcal{S}'$ that has $x'_a < x^*_a$ for $a \in \mathcal{A}$. Hence, it follows from Algorithm 1 (Step 4) that $\mathcal{S}^{\bar{a}}_{\bar{a}}$ (note that the iteration index coincides with the entry by the assumption that the entries are in increasing order of cost) should be $x'_{\bar{a}}$, which contradicts the fact that $x^*_{\bar{a}}$ is the variable obtained in Step 4 and contained $\mathcal{S}^p$, i.e., $\mathcal{S}^*$. ■

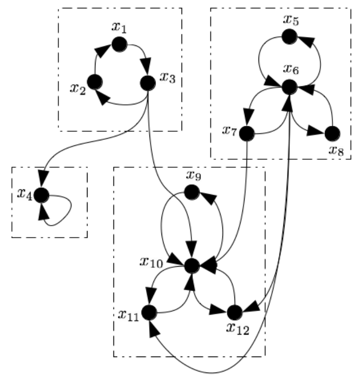

Fig. 2. Graphical representation of $\mathcal{D}(A)$: each rectangle represents an SCC and the two in the top are non-top linked SCCs.

IV. ILLUSTRATIVE EXAMPLE

In this section we present an example demonstration of Algorithm 1. Consider $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ to be as depicted in Figure 2.

Moreover, suppose that assigning a dedicated input to $x \in \mathcal{X}$ incurs the following costs:

$C(x_1)$	$C(x_2)$	$C(x_3)$	$C(x_4)$	$C(x_5)$	$C(x_6)$
10	10	∞	2	20	7
$C(x_7)$	$C(x_8)$	$C(x_9)$	$C(x_{10})$	$C(x_{11})$	$C(x_{12})$
15	13	∞	∞	10	100

Now, consider Algorithm 1 and suppose that Step 2 provides the following minimal feasible dedicated input configuration

$$\mathcal{S} = \{x_3, x_5, x_7, x_9, x_{12}\}.$$

Note that $\mathcal{S}$ is not unique, but by Theorem 4 we can always compute such a minimal feasible dedicated input configuration. Set $\mathcal{S}^* = \mathcal{S}$ and execute the algorithm iterations as follows

(1) Start by computing $\Theta^j_{\mathcal{S}}$ for $j \in \{1, 2, 3, 4, 5\}$. From Figure 1 we have that we need always two dedicated inputs for each of 4-vertices star networks, and the center vertex of each of the 4-vertices star networks is never a right-unmatched vertex. Intuitively, there are two alternatives: first we have two dedicated inputs for each of 4-vertices star networks, or, we have three dedicated inputs for the 4-vertices star network in the top of Figure 2 and one in the 4-vertices star network in the bottom of Figure 2. This last case consists of considering x_6 as a right-unmatched vertex. On the other hand, the cycle that comprises the vertices x_1, x_2, x_3 is structurally controllable by selecting any of its vertices, in accordance with Theorem 2. Because x_4 is a cycle (since it is a self-loop) not in a non-top linked SCC, it will never be a right-unmatched vertex or used in a feasible dedicated input configuration. Finally, we have

$$\Theta^1_{\mathcal{S}^*} = \{x_3, x_2, x_1\},\ \Theta^2_{\mathcal{S}^*} = \{x_5, x_8\},\ \Theta^3_{\mathcal{S}^*} = \{x_7, x_8\},$$
$$\Theta^4_{\mathcal{S}^*} = \{x_9, x_{11}, x_8\},\ \Theta^5_{\mathcal{S}^*} = \{x_{12}, x_{11}, x_8\}.$$

Note that in Figure 1, there exists two non-top linked SCCs. Denote by $\mathcal{N}^1$ the non-top linked SCC corresponding to the cycle comprising x_1, x_2, x_3 and by $\mathcal{N}^2$ the non-top linked SCC that consists of x_5, x_6, x_7, x_8. From the cost table, we have that, the state variables in $\Theta^j_{\mathcal{S}} \cap \bigcup_l \mathcal{N}^l$ for $j \in \{1,2,3,4,5\}, l \in \{1,2\}$ that incur in a smallest cost are x_1, x_2 (cost 10). Since, looking up is done by order, select x_1 that belongs to $\Theta^1_{\mathcal{S}^*}$. Hence, $\mathcal{S}^* = \{x_1, x_5, x_7, x_9, x_{12}\}$ by replacing its first entry by the smallest cost variable found. In addition, set $ThetaSetsUsed = \{1\}$ and $nonTopLinkedSCC = \{1\}$.

(2) Having a new $\mathcal{S}^*$ we need to compute $\Theta^j_{\mathcal{S}^*}$ for $j \in \{2,3,4,5\}$, which are the same as before. Among $\Theta^j_{\mathcal{S}} \cap \bigcup_l \mathcal{N}^l$ for $j \in \{2,3,4,5\}, l \in \{2\}$ the cheapest variable is x_8 (cost 13) in $\Theta^2_{\mathcal{S}^*}$(the first time it appears). Thus, $\mathcal{S}^* = \{x_1, x_8, x_7, x_9, x_{12}\}$, and $ThetaSetsUsed = \{1,2\}$ and $nonTopLinkedSCC = \{1,2\}$ and we are done with looking for the cheapest variables in the non-top linked SCCs.

(3) For the new configuration $\mathcal{S}^*$, we have

$$\Theta^3_{\mathcal{S}^*} = \{x_7, x_5\}, \Theta^4_{\mathcal{S}^*} = \{x_9, x_{11}, x_5\}, \Theta^5_{\mathcal{S}^*} = \{x_{12}, x_9, x_5\}.$$

Now, x_{11} (cost 10) is the cheapest amongst $\Theta^j_{\mathcal{S}^*}$ for $j \in \{3,4,5\}$ and appears for the first time for $j = 4$. Thus, $\mathcal{S}^* = \{x_1, x_8, x_7, x_{11}, x_{12}\}$, and update $ThetaSetsUsed = \{1,2,4\}$.

(4) Repeating the same procedure, we have

$$\Theta^3_{\mathcal{S}^*} = \{x_7, x_5\}, \quad \Theta^5_{\mathcal{S}^*} = \{x_{12}, x_9, x_5\},$$

where x_7 is the cheapest variable with cost 15, leaving $\mathcal{S}^* = \{x_1, x_8, x_7, x_{11}, x_{12}\}$, but setting $ThetaSetsUsed = \{1,2,4,3\}$.

(5) Finally, because the $\mathcal{S}^*$ is the same as in iteration (4), also $\Theta^5_{\mathcal{S}^*}$ remains the same. It follows that x_5 (cost 20) is the cheapest variable in $\Theta^5_{\mathcal{S}}$, hence

$$\mathcal{S}^* = \{x_1, x_8, x_7, x_{11}, x_5\} \implies \sum_{i \in \{1,8,7,11,5\}} C(x_i) = 68.$$

Hence, we conclude that this is the minimal feasible dedicated input configuration that ensures a minimal cost of 68.

V. CONCLUSIONS

In this paper we have provided a systematic method with polynomial implementation complexity (in the number of the state variables) in order to obtain the minimal cost placement with the minimum number of actuators ensuring structural controllability of a given LTI system. We have shown that our method yields the globally optimal dedicated input placement under arbitrary non-homogeneous positive assignment costs. By duality, the results extend to the corresponding structural observability output design under cost constraints. The non-homogeneity of the allocation cost functional makes the framework particularly applicable to actuator (sensor) topology design in large-scale dynamic infrastructures, such as power systems, which consist of a large number of heterogeneous dynamic components with varying overheads for controller (sensor) placement and operation. We believe our proposed framework will lead to cost-efficient controller (sensor) architecture design for these systems, and we intend to pursue such practical design questions in the future.

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