Minimum Number of Information Gatherers to Ensure Full Observability of a Dynamic Social Network: A Structural Systems Approach

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***Abstract*—This paper studies the problem of identifying the minimum number of entities (agents), referred to as *information gatherers*, that are able to infer all the states in a dynamical social network. The information gatherers can be, for instance, service providers and the remaining agents the clients, each comprising several dynamic states associated with the services and personal information. The problem of identifying the minimum number of information gatherers can constitute a way to create coalitions to oversee the entire state of the system, and consequently the behavior of the agents in the social network. The dynamical social network is assumed to be modelled as a linear time-invariant system, and we will make use of the structural systems concept, i.e., by considering only the sparsity pattern (location of zeroes/non-zeroes) of the system coupling matrix. As a consequence, the design guarantees derived hold for almost all numerical parametric realizations of the system. In this paper, we show that this problem is NP-hard: in addition, we provide a reduction of the coalition problem to a minimum set covering problem that, in practice, leads to efficient (polynomial complexity) approximation schemes for solving the coalition problem with guaranteed optimality gaps. Finally, an example is provided which illustrates the analytical findings.**

***Index Terms*—Structural Systems, Observability, Coalition, Privacy**

I. Introduction

In today's world, there exist several tasks that demand the cooperation between individuals or entities (agents) to exert a joint action towards a specific goal, as for instance a multi-agent network to achieve formation. Yet, there are other forms of *coalition*: the cooperation between agents to acquire information about all the agents in the network, in order to obtain leverage in economic markets or advertisement strategies, just to name a few. Hereafter, we focus in networks of agents, where the interaction between the agents consists of linear updates of their own state consisting of a quantity of interest. This interaction occurs over time, leading to a dynamic evolution of their states, which we briefly refer to as a *dynamical social network*. Additionally, each agent may have access to information about the network, either the updating rules of the agents in the network as well as partial information of a collection of agents in the network, yet each agent only has partial measurements of the network state, i.e., linear combinations of the agents' states: consequently, each agent gathers information not only about itself, but also about other agents. These agents play the role of *information gatherers* that can compile information and try to infer general characteristics of the network, for instance, the states of all other agents at a particular instance of time. In this context, we address the problem of determining the minimum number of information gatherers, whose collective information is sufficient to retrieve all agents' states. We refer to this problem as the *coalition* problem. Formally, consider the social network dynamics, represented by the linear time-invariant system

$$x_{k+1} = Ax_k,\ x_0 \in \mathbb{R}^n,\ k \in \mathbb{N}, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the collection of agents' states, i.e., its ith entry corresponds to the scalar state of agent i. Further, each agent i is able to collect a set of measurements from the agents in the network, given by

$$y_k^i = C_i x_k, \quad i = 1, \ldots, n, \tag{2}$$

where $y_k^i \in \mathbb{R}^{m_i}$, with $m_i \in \mathbb{N}$, denotes the output vector measured by agent i. Notice that each agent may (and generally will) observe different quantities from the network, i.e., C_i can be different across agents in the network. In particular, if the matrix A is a consensus-like matrix [5], [2], where each agent i only measures the incoming state from its neighbors $\mathcal{N}_i$ and itself, then $C_i = \mathbb{I}_n^{\mathcal{N}_i \cup \{i\}}$, where $\mathbb{I}_n^{\mathcal{N}_i \cup \{i\}}$ denotes the collection of rows of the identity matrix with indices in $\mathcal{N}_i \cup \{i\}$. Now, let $\bar{A} \in \{0,1\}^{n \times n}$ denote the zero/non-zero or *structural pattern* of the system matrix A, $\bar{C}_i \in \{0,1\}^{m_i \times n}$ the structural pattern of the output matrix of agent i, given in (2). These structural matrices are heavily used in the study of *structural systems theory* [3], where the pair $(\bar{A}, \bar{C}_i)$ is said to be *structurally observable* if there exists an observable numerical realization (A, C_i) in (1)-(2) with the same structure as $(\bar{A}, \bar{C}_i)$ (by density arguments, this entails that almost all pairs with the structure of $(\bar{A}, \bar{C}_i)$ are observable, see for example [3]).

Hereafter, we restrict our attention to the theoretical properties of structural systems, and we consider the structural counterpart to the coalition problem described above. More specifically, given the structural dynamics matrix and the collection of output configurations of each agent, the coalition problem consists of identifying the smallest subset of agents $\mathcal{J} \subset \{1, \ldots, n\}$, such that the union of their outputs ensure structural observability of the system, and may be formally posed as follows:

$\mathcal{P}_1$: Given a network composed by n agents, where the structural dynamic matrix is given by $\bar{A} \in \{0,1\}^{n\times n}$ and $\bar{C}_i \in \{0,1\}^{m_i\times n}$ is the structure of the measured outputs by agent i, for $i = 1, ..., n$, determine the minimum collection of agents, indexed by $\mathcal{J}^*$, obtained as follows:

$$\mathcal{J}^* = \arg \min_{\mathcal{J}\subset\{1,\ldots,n\}} \quad |\mathcal{J}| \qquad (3)$$
$$\text{s.t.} \quad (\bar{A}, \bar{C}_{\mathcal{J}}) \text{ is structurally observable,}$$

where $\mathcal{J}$ is a subset of indices associated with the agents' outputs, and $\bar{C}_{\mathcal{J}}$ corresponds to the collection of the measurements of agents indexed by $\mathcal{J}$, i.e.,

$$\bar{C}_{\mathcal{J}} = [\bar{C}_{j_1}^T \ \bar{C}_{j_2}^T \ \cdots \ \bar{C}_{j_p}^T]^T,$$

where $\{j_1, \ldots, j_p\} \subset \{1, \ldots, n\}$, and $j_1 \neq \ldots \neq j_p$. ◇

Problem $\mathcal{P}_1$ is closely related to the *minimum constrained output selection* (minCOS) problem addressed in [7] (see also the references therein), in which the problem of determining the minimum number of measurements that a single agent needs to obtain to ensure structural observability of the system was studied. In fact, the minCOS problem was shown to be NP-hard in [7], and it will play a key role in showing the coalition problem is at least as difficult to solve, i.e., it is an NP-hard problem.

The main contributions of the present paper consist in showing that the coalition problem is an NP-hard problem, and providing the reduction of the coalition problem to a minimum set covering problem. In practice, such a reduction may lead to efficient (polynomial complexity) approximation schemes for solving the coalition problem with guaranteed optimality gaps.

The rest of this paper is organized as follows: Section II introduces some preliminaries on computational complexity theory. Additionally, it reviews some concepts and results in structural systems theory to be used in the sequel. Section III presents the main results of this paper, i.e., that the coalition problem is NP-hard, and a polynomial reduction from the coalition problem to a minimum set covering problem. The proofs are relegated to the Appendix. Finally, an illustrative example is provided in Section IV.

II. Preliminaries and Terminology

In this section, we review the set covering problem [1], and some necessary and sufficient conditions that ensure system's structural observability.

A (computational) problem is said to be *reducible in polynomial time* to another if there exists a procedure to transform the former into the latter using a number of operations, which is bounded by a polynomial on the size of the input. Such reduction is useful in determining the qualitative complexity class [4] a particular problem belongs to. For instance, recall that a problem $\mathcal{P}$ in NP (i.e., the class of problems for which the feasibility of a solution can be verified in polynomial time) is said to be NP-complete if all other NP problems can be polynomially reduced to $\mathcal{P}$ [4]. These problems are commonly referred to decision problems, since they aim to assess the feasibility of a solution with specified properties. Instead, one may be willing to determine solutions that are optimal with respect to some objective. Those *optimization* problems which decision versions are NP-complete are referred to as NP-hard, see [4].

Lemma 1 ([4]): If a problem $\mathcal{P}_A$ is NP-hard and $\mathcal{P}_A$ is reducible in polynomial time to $\mathcal{P}_B$, then $\mathcal{P}_B$ is NP-hard. ◇

In this paper, we will often use the following well known NP-hard problem as a means of obtaining the desired reductions.

Definition 1 ([1]): (Minimum Set Covering Problem) Given a set of m elements $\mathcal{U} = \{1, 2, \ldots, m\}$ and a set of n sets $\mathcal{S} = \{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$ such that $\mathcal{S}_i \subset \mathcal{U}$, with $i \in \{1, \cdots, n\}$, and $\bigcup_{i=1}^{n} \mathcal{S}_i = \mathcal{U}$, the set covering problem consists of finding a set of indices $\mathcal{I}^* \subseteq \{1, 2, \ldots, n\}$ corresponding to the minimum number of sets covering $\mathcal{U}$, i.e.,

$$\mathcal{I}^* = \arg\min_{\mathcal{I}\subseteq\{1,2,\ldots,n\}} \quad |\mathcal{I}|$$
$$\text{subject to:} \quad \mathcal{U} = \bigcup_{i\in\mathcal{I}} \mathcal{S}_i \ .$$

◇

Now, we revise some concepts from structural systems theory. Structural systems provide an efficient representation of a linear time invariant system as a directed graph (digraph). A digraph consists of a set of *vertices* $\mathcal{V}$ and a set of *directed edges* $\mathcal{E}_{\mathcal{V},\mathcal{V}}$ of the form $(v^i, v^j) \in \mathcal{V} \times \mathcal{V}$. If a vertex v belongs to the endpoints of an edge $e \in \mathcal{E}_{\mathcal{V},\mathcal{V}}$, we say that the edge e is incident to v. We represent the *state digraph* by $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$, i.e., the digraph that comprises only the state variables as vertices denoted by $\mathcal{X} = \{x^1, \cdots, x^n\}$ and a set of directed edges between the state vertices denoted by $\mathcal{E}_{\mathcal{X},\mathcal{X}} = \{(x^i, x^j) \in \mathcal{X} \times \mathcal{X} : \ \bar{A}_{j,i} \neq 0\}$. In addition, we represent the ith *agent digraph* by $\mathcal{D}(\bar{A}, \bar{C}_i) = (\mathcal{X} \cup \mathcal{Y}_i, \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup \mathcal{E}_{\mathcal{X},\mathcal{Y}_i})$, where $\mathcal{Y}_i = \{y^{i,1}, \cdots, y^{i,m_i}\}$ corresponds to the output vertices and $\mathcal{E}_{\mathcal{X},\mathcal{Y}_i} = \{(x^j, y^{i,k}) \in \mathcal{X} \times \mathcal{Y}_i : \ [\bar{C}_i]_{k,j} \neq 0\}$ the edges identifying which state variables are measured by which outputs of the ith agent. Similarly, we define the *system digraph* given by $\mathcal{D}(\bar{A}, [\bar{C}_1 \ \cdots \ C_n) = (\mathcal{X} \cup (\bigcup_{i=1}^n \mathcal{Y}_i), \mathcal{E}_{\mathcal{X},\mathcal{X}} \cup (\bigcup_{i=1}^n \mathcal{E}_{\mathcal{X},\mathcal{Y}_i}))$.

We also require the following graph theoretic notions [1]: A digraph $\mathcal{D}$ is strongly connected if there exists a directed path between any two vertices. A *strongly connected component* (SCC) is a maximal subgraph $\mathcal{D}_S = (\mathcal{V}_S, \mathcal{E}_S)$ of $\mathcal{D}$ such that for every $u, v \in \mathcal{V}_S$ there exists a direct path from u to v.

By visualizing each SCC as a virtual node, we can build a *directed acyclic graph* (DAG) representation, in which a directed edge exists between two SCCs *if and only if* there exists a directed edge connecting two vertices in the respective

SCCs in the original digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$. The construction of the DAG associated with $\mathcal{D}(\bar{A})$ can be performed efficiently in $\mathcal{O}(|\mathcal{V}|+|\mathcal{E}|)$ [1]. The SCCs in the DAG may be categorized as follows.

Definition 2 ([6]): An SCC is said to be *non-bottom linked* if it has no outgoing edges from its vertices to the vertices of another SCC. ⋄

Given $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, we can construct a *bipartite graph* $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$ [6], where $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{V}$ and the edge set $\mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2} = \{(s_1, s_2) \in \mathcal{E} \ : \ s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2 \ \}$.

Given $\mathcal{B}(\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2})$, a matching M corresponds to a subset $M \subseteq \mathcal{E}_{\mathcal{S}_1,\mathcal{S}_2}$ so that no two edges in M have a vertex in common, i.e., given edges $e = (s_1, s_2)$ and $e' = (s'_1, s'_2)$ with $s_1, s'_1 \in \mathcal{S}_1$ and $s_2, s'_2 \in \mathcal{S}_2$, $e, e' \in M$ only if $s_1 \neq s'_1$ and $s_2 \neq s'_2$. A maximum matching M^* is a matching M with the largest number of edges among all possible matchings.

Given a matching M, an edge is said to be matched with respect to (w.r.t.) M, if it belongs to M. In addition, we say that a vertex $v \in \mathcal{S}_1 \cup \mathcal{S}_2$ is *matched* if it is incident to some matched edge in M, otherwise we say that the vertex is *free* w.r.t. M. A matching is said to be *perfect* if there are no free vertices.

Corollary 1 ([6]): Given $\mathcal{D}(\bar{A}) = (\mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ and its DAG representation, constituted by k SCCs, denoted by $\{\mathcal{N}^i\}_{i=1,\ldots,k}$, where $\mathcal{N}^i = (\mathcal{X}_i, \mathcal{E}_{\mathcal{X}_i,\mathcal{X}_i})$, let $\mathcal{N}^{i_1}, \ldots \mathcal{N}^{i_m}$ be the non-bottom linked SCCs in the DAG representation with $\{i_1, \ldots, i_m\} \subset \{1, \ldots, k\}$ and $\mathcal{B}(\bar{A}) = \mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ the state bipartite graph. If $\mathcal{B}(\bar{A})$ has a perfect matching, then $(\bar{A}, \bar{C})$ is structurally observable if and only if for each non-bottom linked SCC there exists an output (corresponding to a row in $\bar{C}$) measuring at least one of its state variables. ⋄

III. Main Results

In this section, we present the main results of this paper: first, we show that the coalition problem is NP-hard, and, then, we provide a reduction to a set covering problem, which may be used to determine a solution to the coalition problem. Additionally, the reduction may lead to efficient (polynomial complexity) approximation schemes for solving the coalition problem with guaranteed optimality gaps.

We start by showing that the coalition problem presented in $\mathcal{P}_1$ is NP-hard.

Theorem 1: The coalition problem presented in $\mathcal{P}_1$ is NP-complete. ⋄

Now, we assume that the structure of the dynamic matrix, i.e., $\bar{A}$ in the coalition problem, satisfies the following condition:

Assumption 1 The structural dynamic matrix $\bar{A}$ is such that the state bipartite graph $\mathcal{B}(\bar{A}) = \mathcal{B}(\mathcal{X}, \mathcal{X}, \mathcal{E}_{\mathcal{X},\mathcal{X}})$ associated with $\bar{A}$ has a perfect matching. ⋄

Notice that Assumption 1 holds in general for social dynamical networks in which each agent uses its own past state information in addition to its neighbors' states in the update protocol (for instance, consensus or gossip type protocols, see [2]): more precisely, since each agent uses its own information to update its state, the matrix $\bar{A}$ has non-zero diagonal entries, thus leading to a perfect matching of $\mathcal{B}(\bar{A})$ comprised by the edges (self-loops in $\mathcal{D}(\bar{A})$) associated with those diagonal entries.

The polynomial reduction of the coalition problem to a minimum set covering problem is presented next.

Theorem 2: Consider the coalition problem $\mathcal{P}_1$ with system matrix instance $\bar{A} \in \{0,1\}^{n\times n}$, where $\bar{A}$ satisfies Assumption 1, and output matrices $\bar{C}_i \in \{0,1\}^{m_i\times n}$, with $i = 1, \ldots, n$ and $m_i \in \mathbb{N}$. Denote by $\mathcal{N}^i$, $i = 1, \ldots, k$, the k non-bottom linked SCCs of $\mathcal{D}(\bar{A})$. Then, the coalition problem can be polynomially reduced to the set covering problem with universe $\mathcal{U} = \{1, \ldots, k\}$ and sets $\{\mathcal{S}^*_j\}_{j=1,\ldots,p}$, where $\mathcal{S}^*_j = \{i \in \mathcal{U} : \ \exists_{r\in\{1,\ldots,n\}} \exists_{k\in\{1,\ldots,m_i\}} [\bar{C}_j]_{r,k} = 1 \text{ and } x_r \in \mathcal{N}^i\}$. ⋄

In the next section, we provide an illustrative example in which Theorem 2 is used to determine the solution to a coalition problem, with the social dynamic network satisfying Assumption 1.

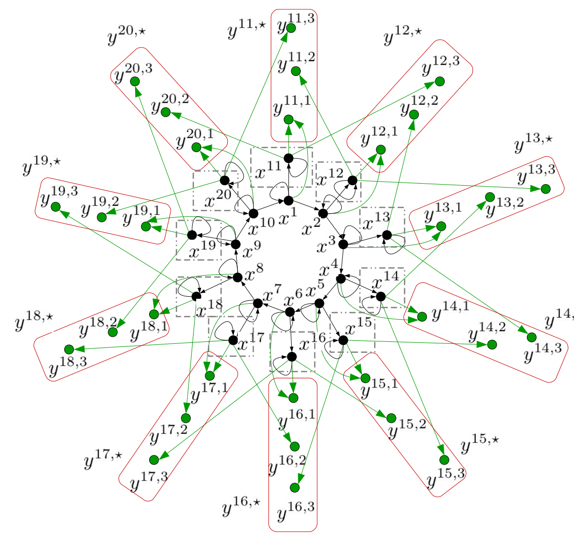

Fig. 1. This figure represents a digraph $\mathcal{D}(\bar{A}, [\bar{C}_1^T \ \ldots \ \bar{C}_n^T]^T)$: the agent states are depicted by black vertices and the inter-agent dynamical coupling by the black directed edges. The non-bottom linked SCCs are depicted by gray dashed boxes, and the red boxes represent the collection of outputs (depicted by green vertices) that each agent has: the green edges ending in those outputs represent the state variables measured, i.e., the information collected by the agents. In particular, if an output has more than one incoming edge, then it measures a linear combination of the agents' states (those agents from which the edges depart from).

IV. Illustrative Example

Hereafter, we illustrate how the reduction proposed in Theorem 2 can be used to determine the minimum number of information gatherers. Consider the system digraph $\mathcal{D}(\bar{A}, [\bar{C}_1^T \ \cdots \ \bar{C}_n^T]^T)$ depicted in Figure 1 comprising 20 agents. The agents' states are depicted by black vertices and the inter-agent dynamical coupling by the black directed edges. Further, agents' output vertices are depicted by green vertices. Consequently, the agent digraph $\mathcal{D}(\bar{A}, \bar{C}_i)$ comprises the state digraph $\mathcal{D}(\bar{A})$ and the collection of output vertices (and edges ending on them) associated with agent i delineated with the rectangle labeled as $y^{i,\star}$. The peripheral agents ($x_{11} - x_{20}$) have three outputs each, whereas the agents $x_1 - x_{10}$ only have one output each (omitted in the figure) corresponding to the linear combination of its own measurement and the measurement of the agent with an incoming edge on it (similar to those obtained by $y^{j,1}$ for $j = 11, \ldots, 20$). Figure 1 depicts 10 non-bottom linked SCCs, contained in gray dashed boxes, and given by $\mathcal{N}^1 = \{x_{11}\}, \mathcal{N}^2 = \{x_{12}\}, \ldots, \mathcal{N}^{10} = \{x_{20}\}$. Consequently, the sets S_j^* in Theorem 2 are given as follows: for $j = 1, \ldots, 10$, $\mathcal{S}_j^* = \emptyset$, whereas for $j = 11, \ldots, 20$, $\mathcal{S}_j^*$ contains the indices of the non-bottom linked SCCs comprising the state variables with outgoing edges into one of the output vertices in the bundle $y^{j,*}$, for instance, we have $\mathcal{S}_{11}^* = \{1, 2, 10\}$ when $j = 11$. An optimal solution to the corresponding set covering problem is $\{\mathcal{S}_j^*\}$ with $j \in \mathcal{J}^* = \{11, 14, 17, 19\}$; hence, by Theorem 2, $\bar{C}_{\mathcal{J}^*}$ is a solution to the coalition problem. In fact, it is possible to graphically verify that it is solution: more precisely, recall Corollary 1, and notice that each bundle of outputs $y^{k,\star}$ measures the states in $\{k-1, k, k+1\}$ for $k = 12, \ldots, 19$, whereas $y^{11,\star}$ measures the states in $\{11, 12, 20\}$ and $y^{20,\star}$ measures the states in $\{11, 19, 20\}$. In summary, to measure the states in the 10 non-bottom linked SCCs, is the same as measuring the states themselves, which implies that we need at least the collection of measurements from 4 agents.

V. Conclusions and Further Research

In this paper, we have showed that the coalition problem is NP-hard, which implies that (in general) efficient (polynomial complexity) solution procedures to solve it are unlikely to exist. Yet, some approximation algorithms can be used to determine suboptimal solutions with proven optimality gap guarantees, as is the case with the minimum set covering problem proposed to solve the coalition problem. Further research may consist of determining special network structures for which the coalition problem can be solved using efficient algorithms.

Appendix

The minimum constrained output selection (minCOS) problem is the problem of, given $\bar{A}$ and $\bar{C}$, determining the minimum number of outputs that ensure structural observability. Theorem 3 shows that the minCOS in general is a hard problem.

Theorem 3 ([7]): Let $\bar{A} \in \{0,1\}^{n \times n}$ and $\bar{C} \in \{0,1\}^{m \times n}$. The minCOS problem given by:

$$\begin{aligned} \mathcal{J}^* = \arg \min_{\mathcal{J} \subset \{1, \cdots, n\}} & \quad |\mathcal{J}| \\ \text{s.t.} & \quad (\bar{A}, \bar{C}(\mathcal{J})) \text{ is structurally observable,} \end{aligned} \tag{4}$$

where $\bar{C}(\mathcal{J})$ consists of the submatrix of $\bar{C}$ comprising the rows with indices in $\mathcal{J}$, is NP-hard. ◇

Using Theorem 3, we now obtain one of the main results of the present paper.

Proof of Theorem 1: The reduction of the minCOS to the coalition problem depends on the relation between m and n. More precisely, the polynomial reductions are presented for the following cases: (i) $m = n$, (ii) $m < n$, and (iii) $m > n$. In (i), we consider as inputs to the coalition problem the matrix $\bar{A}$ and $\bar{C}$ of the minCOS problem, and set $\bar{C}_i$ in the coalition problem to be $\bar{C}_i = \bar{C}(\{i\})$, with $i = 1, \ldots, n$. In case (ii), we proceed as in (i), where $\bar{C}_i = \bar{C}(\{i\})$ for $i = 1, \ldots, m$, and $\bar{C}_i = \mathbf{0}_{1 \times n}$ for $i = m+1, \ldots, n$, where $\mathbf{0}_{1 \times n}$ is the zero vector. Finally, in (iii) consider as input to the coalition problem the structural $m \times m$ dynamic matrix

$$\bar{A}^\star = \begin{bmatrix} \bar{A}_{n \times n} & \mathbf{1} \\ 0 & \mathbf{I} \end{bmatrix},$$

where $\mathbf{1}$ and I are the matrices comprising only ones and the identity matrix with appropriate dimensions, respectively; whereas $\bar{C}_i = \bar{C}(\{i\})$, with $i = 1, \ldots, m$.

To see that the reduction is correct, we need to verify if a solution to the corresponding coalition problem also provides a solution to the minCOS problem. To see this, notice that (i) follows by noticing that it is just a re-writing of the minCOS problem. In (ii) it is immediate to see that agents i, with $i = m+1, \ldots, n$ will never be considered to form the coalition, since they do not collect any measurements, and, hence by considering only agents $i = 1, \ldots, m$ we can do the same reasoning as we did for (i). Finally, in (iii), the bipartite graph $\mathcal{B}(\bar{A})$ has a perfect matching M^* comprising all the edges of the form (x_j, x_j), with $j = 1, \ldots, m$. In addition, the digraph representation of $\mathcal{D}(\bar{A}^\star)$ has the same non-bottom linked SCCs as $\mathcal{D}(\bar{A})$; hence, recalling Corollary 1, it readily follows that a solution to the coalition problem given above yields a solution to the minCOS problem.

At last, it is easy to see that all the inputs created to the coalition problem in the proposed reductions, require a linear (in $\max\{n, m\}$) computational complexity; hence the reduction is polynomial. Consequently, by Lemma 1, if follows that the coalition problem is NP-hard. ■

Proof of Theorem 2: The present proof follows the same steps as in the proof of Theorem 2 in [7]. ■

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