Proc. of MED'02 - 10th Mediterranean Conference on Control and Automation, Lisbon, Portugal, July 2002.

PRACTICAL STABILIZATION OF THE EXTENDED NONHOLONOMIC DOUBLE INTEGRATOR

António Pedro Aguiar **António M. Pascoal**

Institute for Systems and Robotics, Instituto Superior Técnico,
Torre Norte 8, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
E-mail:{antonio.aguiar,antonio}@isr.ist.utl.pt

Keywords: Nonholonomic Systems, Hybrid Control, Practical Stabilization, Extended Nonholonomic Double Integrator.

Abstract

This paper derives hybrid control laws for the extended nonholonomic double integrator (ENDI). A new logic-based hybrid controller is proposed that yields global stability and convergence of the trajectories of the closed loop system to an arbitrarily small neighborhood of the origin. The problem of practical stabilization of the ENDI system under input saturation constraints and in the presence of small input additive disturbances is also posed and solved. Stability and convergence proofs are presented. Simulations illustrate the performance of the controllers derived.

1 Introduction

Control of nonholonomic systems has been the subject of considerable research effort over the last few years. From a practical point of view, nonholonomic systems often arise in the form of robot manipulators, mobile robots, and space and marine robots that are either designed with fewer actuators than degrees of freedom or must be able to function in the presence of actuator failures. From a theoretical stand point, there is considerable challenge in the synthesis of control laws for nonholonomic systems since, as pointed out in a famous paper of Brockett [8], they cannot be stabilized by continuously differentiable, time invariant, state feedback control laws. To overcome the limitations imposed by the Brockett's celebrated result, a number of approaches have been proposed for the stabilization of nonholonomic control systems to equilibrium points. See [16] and the references therein for a comprehensive survey of the field. Among the proposed solutions are smooth time-varying controllers [22, 23, 24, 11, 18, 20], discontinuous or piecewise smooth control laws [9, 5, 12, 3, 4], and hybrid controllers [6, 10, 13].

Despite the vast amount of papers published on the stabilization of nonholonomic systems, the majority has concentrated on kinematic models of mechanical systems controlled directly by velocity inputs. Although in certain circumstances this can be acceptable, many physical systems (where forces and torques are the actual inputs) will not perform well if their dynamics are neglected.

As a contribution to overcome this limitation, this paper derives a hybrid control law for the so-called ENDI system under input saturation constraints, in the presence of small input additive disturbances. The ENDI system can be viewed as an extension of the so called nonholonomic integrator [8]. Its importance stems from the fact that it captures the dynamics and kinematics of a nonholonomic system with three states and two first-order dynamic control inputs, (*e.g.*, the dynamics of a wheeled robot subject to force and torque inputs).

The paper proposes a switching control law for the ENDI system. Control system design borrows from hybrid control theory and is greatly inspired by the work of Hespanha [13] for the nonholonomic integrator. The main result shows that for any bounded input additive disturbances and any initial condition, the closed loop hybrid system possesses strong practical stability with bounded control input. Control system design is done by mapping the state-space into a two dimensional closed positive quadrant space and dividing it into four overlapping regions followed by the assignment of a suitable feedback law for each region.

The paper is organized as follows: Section 2 introduces the ENDI system and discusses its controllability and stabilizability properties. Section 3 derives a hybrid controller for the ENDI. Section 4 extends the control law proposed to the bounded input case. Section 5 presents a solution to the problem of practical stability of the ENDI subject to input saturation constraints and in the presence of small input additive disturbances. Finally, Section 6 contains simulation results that illustrate the performance of the control laws derived. Concluding remarks are given in Section 7.

2 The Extended Nonholonomic Double Integrator

In [8], Brockett introduced the *nonholonomic integrator* system

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_1 u_2 - x_2 u_1,$$

where $(x_1, x_2, x_3)' \in \mathbb{R}^3$ is the state vector and $(u_1, u_2)' \in \mathbb{R}^2$ is a two-dimensional input. This system displays all basic properties of nonholonomic systems and is often quoted in the literature as a benchmark for control system design [5, 13, 3, 19].

The nonholonomic integrator captures (under suitable state and control transformations) the kinematics of a wheeled robot. However, the nonholonomic integrator model fails to capture the case where both the kinematics and dynamics of a wheeled robot must be taken into account. To tackle this realistic case, the nonholonomic integrator model must be extended. It is shown in [2] that the dynamic equations of motion of a mobile robot of the unicycle type can be transformed into the system

$$\ddot{x}_1 = u_1, \quad \ddot{x}_2 = u_2, \quad \dot{x}_3 = x_1 \dot{x}_2 - x_2 \dot{x}_1, \qquad (1)$$

where $x = \left(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2\right)' \in \mathbb{R}^5$ is the state vector and $u = \left(u_1, u_2\right)' \in \mathbb{R}^2$ is a two-dimensional control vector. In this paper, system (1) will be referred to as *the extended nonholonomic double integrator* (ENDI).

Controllability and stabilizability properties

The ENDI system fall into the class of control affine nonlinear systems with drift

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$$

where $x \in M$, M is a smooth n-dimensional manifold, $u \in \mathbb{R}^m$ and the mappings $f, g_1, \ldots, g_m$ are smooth vector fields on M. The following result summarizes the controllability and stabilizability properties of the ENDI [1]. See [14, 25, 21] for relevant background.

Theorem 1 *Consider the ENDI system described by* (1). *Let M_e be the set of equilibrium solutions corresponding to $u = 0$, that is, $M_e = \{x \in \mathbb{R}^5 : \dot{x}_1 = \dot{x}_2 = 0\}$. Then, the ENDI system satisfies the following properties:*

1. *There is no time-invariant continuously differentiable feedback law that asymptotically stabilizes the closed loop to $x_e \in M_e$.*
2. *The ENDI system is locally strongly accessible for any $x \in \mathbb{R}^5$.*
3. *The ENDI system is small time locally controllable (STLC) at any equilibrium $x_e \in M_e$.*

3 Hybrid Controller Design

This section proposes a simple piecewise smooth controller to stabilize the ENDI. The key ideas involved borrow from hybrid system theory. Hybrid systems are specially suited to deal with the combination of continuous dynamics and discrete events. The literature on hybrid systems is extensive and discusses different modeling techniques [7, 26].

In this paper, a continuous-time hybrid system Σ is defined as [13]

$$\dot{x}(t) = f_{\sigma(t)}\big(x(t), t\big), \qquad t \geq t_0 \tag{2a}$$

$$\sigma(t) = \phi\left(x(t), \sigma(t^-)\right) \tag{2b}$$

where $\sigma(t) \in \mathcal{I} \triangleq \{1, \ldots, N\}$ and $x(t) \in \mathcal{X} \triangleq \cup_{\sigma=1}^{N} \mathcal{X}_\sigma \subset \mathbb{R}^n$. Here, the differential equation (2a) models the continuous dynamics, where the vector fields $f_\sigma : \mathcal{X}_\sigma \times \mathbb{R}^+ \to \mathcal{X}$, $\sigma \in \mathcal{I}$ are each locally Lipschitz continuous maps from $\mathcal{X}_\sigma$ to $\mathcal{X}$. The algebraic equation (2b), where $\phi : \mathcal{X} \times \mathcal{I} \to \mathcal{I}$, models the state of the decision-making logic. The discrete state $\sigma(t)$ is piecewise constant. The notation t^- indicates that the discrete state is piecewise continuous from the right. The dynamics of the system Σ can now be described as follows: starting at (x_0, i) with $x_0 \in \mathcal{R}_i \subset \mathcal{X}_i$, the continuous state trajectory $x(t)$ evolves according to $\dot{x} = f_i(x, t)$. When $\phi\big(x(\cdot), i\big)$ becomes equal to $j \neq i$, (and this could only happen when $x(\cdot)$ hits the set $\mathcal{X} \backslash \mathcal{R}_i$), the continuous dynamics switches to $\dot{x} = f_j(x, t)$, from which the process continues. As in [13], the "logical dynamics" will be determined, recursively by equation (2b) with $\sigma^-(t_0) = \sigma_0 \in \mathcal{I}$ where $\sigma^-(t)$ denotes the limit of $\sigma(\tau)$ from below as $\tau \to t$ and the transition function ϕ is defined by

$$\phi(x, \sigma) = \begin{cases} \sigma & \text{if } x \in \mathcal{R}_\sigma, \\ \max_{\mathcal{I}}\{k : x \in \mathcal{R}_k\} & \text{otherwise.} \end{cases} \tag{3}$$

We now review the concept of stability of a hybrid system [13].

Definition 1 (Stability) *The equilibrium point $x = 0$ of the hybrid system Σ is Lyapunov stable if for every $\epsilon > 0$ and any $t_0 \in \mathbb{R}^+$ there exists $\delta = \delta(\epsilon, t_0) > 0$ such that for every initial condition $\{x_0, \sigma_0\} \in \mathcal{X} \times \mathcal{I}$ with $\|x_0\| < \delta$, the solution $\{x(t), \sigma(t)\}$ satisfies $\|x(t)\| < \epsilon$, for all $t \geq t_0$. If in the above definition δ is independent of t_0, i.e., $\delta = \delta(\epsilon)$, then the origin is said to be uniformly stable.*

Consider now the ENDI system (1). When the state variables x_1 and x_2 are both zero, $\dot{x}_3$ will also be zero and, consequently, x_3 will remain constant. Thus a possible strategy to steer an initial state to the vicinity of the origin is the following (see [13] where similar ideas were applied to the control of the nonholonomic integrator): *i)* first, make the state variable x_3 converge to zero while keeping x_1 and x_2 away from the axis $x_1 = x_2 = 0$; *ii)* next, freeze x_3 ($\dot{x}_3 = 0$), and force x_1 and x_2 to converge to zero.

In order to derive a hybrid controller for the ENDI, it is convenient to define the function $W(x) : \mathbb{R}^5 \to \Omega \subset \mathbb{R}^2$ as

$$\omega \triangleq (\omega_1, \omega_2)' = W(x) = \left(s^2, (x_1)^2 + (x_2)^2\right)',$$

where $s = \dot{x}_3 + \lambda x_3$ and λ is a strictly positive constant. The image of W is the two-dimensional closed positive quadrant space $\Omega = \{(\omega_1, \omega_2) \in \mathbb{R}^2 : \omega_1 \geq 0, \omega_2 \geq 0\}$. This mapping has several properties, which are listed in the following lemma.

Lemma 1 *The mapping $W(\cdot) : \mathbb{R}^4 \to \Omega \subset \mathbb{R}^2$ has the following properties:*

1. *$W(0) = 0$.*
2. *if w converges to zero as $t \to \infty$, then x also converges to zero as $t \to \infty$.*
3. *if $x_3(t_0) = 0$ and $\omega_1 \leq \epsilon$ for all $t \geq t_0$, then $|x_3(t)| \leq \frac{\sqrt{\epsilon}}{\lambda}$ for all $t \geq t_0$. For the case where $x_3(t_0) \neq 0$, the bound of $x_3(t)$ is given by $|x_3(t)| \leq e^{-\lambda(t-t_0)}|x_3(t_0)| + \frac{\sqrt{\epsilon}}{\lambda}$.*

Divide now Ω into three overlapping regions (see Figure 1)

$$\begin{aligned} \mathcal{R}_1 &= \{(\omega_1, \omega_2) \in \Omega : \omega_1 > \epsilon_1 \wedge \omega_2 \leq \gamma_2\} \\ \mathcal{R}_2 &= \{(\omega_1, \omega_2) \in \Omega : \omega_1 > \epsilon_1 \wedge \omega_2 \geq \gamma_1\} \\ \mathcal{R}_3 &= \{(\omega_1, \omega_2) \in \Omega : \omega_1 \leq \epsilon_2\} \end{aligned} \tag{4}$$

where $\epsilon_2 > \epsilon_1 > 0$ and $\gamma_2 > \gamma_1 > 0$.

Consider the following dynamical system as a candidate control law to steer the ENDI trajectories to a small neighborhood of the origin:

$$u = g_\sigma(x), \tag{5}$$

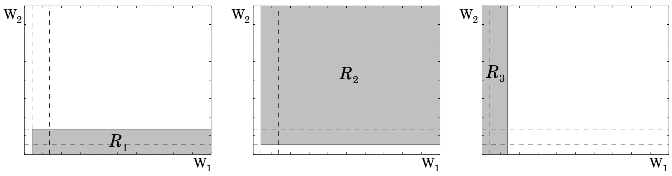

Figure 1: Definition of regions $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$.

where the vector fields $g_\sigma : \mathbb{R}^5 \to \mathbb{R}^2$, $\sigma \in \mathcal{I} = \{1, 2, 3\}$ are given by[1]

$$g_1(x) = \begin{bmatrix} -\lambda \dot{x}_1 + x_1 \\ -\lambda \dot{x}_2 + x_2 \end{bmatrix}, g_2(x) = \begin{bmatrix} -\lambda \dot{x}_1 + x_1 + x_2 s \\ -\lambda \dot{x}_2 + x_2 - x_1 s \end{bmatrix},$$

$$g_3(x) \quad = \quad \begin{bmatrix} -\lambda \dot{x}_1 - x_1 \\ -\lambda \dot{x}_2 - x_2 \end{bmatrix}. \qquad (6)$$

The control laws for each region were designed according to the following simple rules: while $\sigma = 1$, $\omega_1(t)$ must decrease or remain constant and $\omega_2(t)$ must grow without bound as $t \to \infty$; when $\sigma = 2$, $\omega_1(t)$ must decrease and reach a determined bound in finite time; and finally when $\sigma = 3$, $\omega_1(t)$ must again remain constant and $\omega_2(t)$ must converge to zero. A sketch of a typical trajectory of W is shown in Figure 2. The region that is the intersection of $\mathcal{R}_2$ and $\mathcal{R}_3$ can be seen as a hysteresis region. Its aim is to avoid the possibility of infinitely fast switching when ω_1 is near ϵ_1.

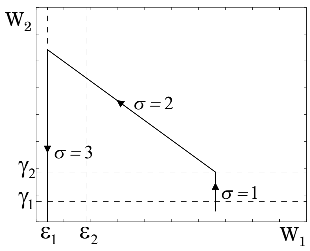

Figure 2: Image of a representative trajectory in the (ω_1, ω_2) plane.

The following theorem can now be stated.

[1]For $\sigma = 1$, if λ does not satisfy the relation $-\frac{1}{2}\big(1 + \sqrt{\lambda^2 + 4}\big)x_1(t_0) \neq \dot{x}_1(t_0) \ \vee -\frac{1}{2}\big(1 + \sqrt{\lambda^2 + 4}\big)x_2(t_0) \neq \dot{x}_2(t_0)$, then the unstable mode of the corresponded closed loop system is not excited. In this case, $g_1(x)$ has to be modified to $g_1(x) = \begin{bmatrix} -\lambda \dot{x}_1 + x_1 + \mathrm{sgn}(c_2)\,\mathrm{sgn}(s) \\ -\lambda \dot{x}_2 + x_2 - \mathrm{sgn}(c_1)\,\mathrm{sgn}(s) \end{bmatrix}$ where $\mathrm{sgn}(x) = 1$ if $x \geq 0$, $\mathrm{sgn}(x) = -1$ if $x < 0$, $c_i = \frac{\dot{x}_i(t_0) - x_i(t_0)s_1}{s_2 - s_1}$, $i = 1, 2$, and $s_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\lambda^2 + 4}$.

Theorem 2 *Consider the hybrid system Σ described by (1), (2b), (3), and (5)-(6). Let $\{x(t), \sigma(t)\} = \{x : [t_0, \infty) \to \mathbb{R}^5, \sigma : [t_0, \infty) \to \mathcal{I}\}$ be a solution to Σ. Then, the following properties hold.*

1. *Given an arbitrary pair $\{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$ (initial condition), there exists a unique solution $\{x(t), \sigma(t)\}$ for all $t \geq t_0$ such that $\{x(t_0), \sigma^-(t_0)\} = \{x_0, \sigma_0\}$.*
2. *For any set of initial conditions $\{x(t_0), \sigma^-(t_0)\} = \{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$, there exists a finite time $T \geq t_0$ such that for $t > T$ the state variables $x_1(t)$, $\dot{x}_1(t)$, $x_2(t)$, and $\dot{x}_2(t)$ converge uniformly exponentially to zero and $\omega_1(t) \leq \epsilon_2$, where $\epsilon_2 > 0$ is a controller parameter that can be chosen arbitrarily small.*
3. *The origin $x(t) = 0$ is a Lyapunov uniformly stable equilibrium point of Σ.*

Proof. See [1, 2]. □

4 Boundedness of control inputs

In this section the hybrid control law proposed is extended to the bounded input case, that is, when the ENDI system is subject to the input constraints $|u_1(t)| \leq \bar{u}_1$ and $|u_2(t)| \leq \bar{u}_2$, where $\bar{u}_1$, $\bar{u}_2$ are positive constants. It will be shown that by adding one more region associated with an extra control field the control problem can be solved. The objective of the extra region is to bring the state variables (if they have large amplitude values) to a region where the inputs constraints are satisfied.

Consider the function $W_B(x) : \mathbb{R}^5 \to \Omega \subset \mathbb{R}^2$, mapping the state-space x into the two-dimensional closed positive quadrant space Ω, $\omega \triangleq (\omega_1, \omega_2)' = W_B(x) = \big(s^2, (x_1)^2 + (x_2)^2 + (\dot{x}_1)^2 + (\dot{x}_2)^2\big)'$, where $s = \dot{x}_3 + \lambda x_3$ and λ is a strictly positive constant. Notice that $W_B(\cdot)$ satisfies the same properties as $W(\cdot)$ described in Lemma 1.

Consider also the following four overlapping regions (see Figure 3)

$$\mathcal{R}_1 = \{(\omega_1, \omega_2) \in \Omega : \omega_1 > \epsilon_1 \wedge \omega_2 \leq \gamma_2\},$$
$$\mathcal{R}_2 = \{(\omega_1, \omega_2) \in \Omega : \omega_2 \geq \gamma_2\},$$

$$\mathcal{R}_3 = \{(\omega_1,\omega_2) \in \Omega : \omega_1 > \epsilon_1 \wedge \gamma_1 \le \omega_2 \le \gamma_4\},$$
$$\mathcal{R}_4 = \{(\omega_1,\omega_2) \in \Omega : \omega_1 \le \epsilon_2 \wedge \omega_2 \le \gamma_5\} \cup \{(\omega_1,\omega_2) \in \Omega : \omega_1 > \epsilon_1 \wedge \gamma_3 \le \omega_2 \le \gamma_5\},$$

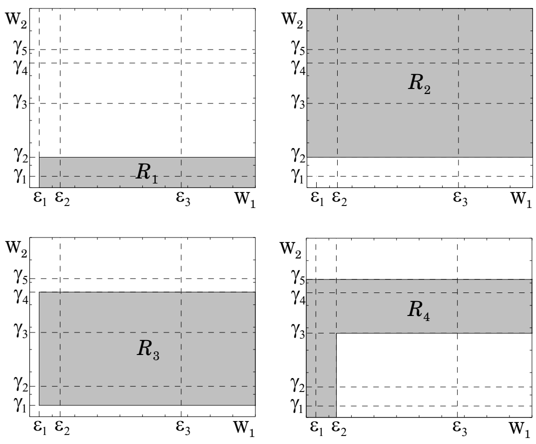

Figure 3: Definition of regions $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_4$.

where $\epsilon_{i+1} > \epsilon_i > 0$ for $i = 1, 2$ and $\gamma_{i+1} > \gamma_i > 0$ for $i = 1, 2, 3, 4$. The new control law is given by[2]

$$u = g_\sigma(x), \tag{7}$$

$$g_1(x) = \begin{bmatrix} -\lambda\dot{x}_1 + x_1 \\ -\lambda\dot{x}_2 + x_2 \end{bmatrix}, \tag{8}$$

$$g_2(x) = \begin{bmatrix} -k_{11}\ \mathrm{sat}(k_{12}x_1 + \dot{x}_1) - k_{12}\ \mathrm{sat}(\dot{x}_1) \\ -k_{21}\ \mathrm{sat}(k_{22}x_2 + \dot{x}_2) - k_{22}\ \mathrm{sat}(\dot{x}_2) \end{bmatrix}, \tag{9}$$

$$g_3(x) = \begin{bmatrix} -\lambda\dot{x}_1 + x_1 + k_1 x_2\,\mathrm{sgn}(s) \\ -\lambda\dot{x}_2 + x_2 - k_2 x_1\,\mathrm{sgn}(s) \end{bmatrix}, \tag{10}$$

$$g_4(x) = \begin{bmatrix} -\lambda\dot{x}_1 - x_1 \\ -\lambda\dot{x}_2 - x_2 \end{bmatrix}, \tag{11}$$

where $\sigma \in \mathcal{I} = \{1, 2, 3, 4\}$, k_i, k_{i1}, and k_{i2} are positive constants that satisfy $k_{i1} + k_{i2} = \bar{u}_i$ and $k_{i2} > k_{i1}$ for all $i = 1, 2$.

Figure 4 shows a typical trajectory of W_B.

The following theorem can be proved [1].

Theorem 3 *Consider the hybrid system Σ_B described by (1), (7)-(11), and the switching logic defined in (3). Let $\{x(t), \sigma(t)\} = \{x : [t_0, \infty) \to \mathbb{R}^5, \sigma : [t_0, \infty) \to \mathcal{I}\}$ be a solution to Σ_B. Then, the following properties hold.*

[2]The term $\mathrm{sgn}(s)$ that appears in the vector field $g_3(x)$ does not introduce chattering since s will never change its signal while $\sigma = 3$.

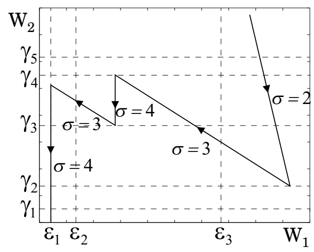

Figure 4: Image of a representative trajectory in the (ω_1, ω_2) plane.

1. *Given an arbitrary pair $\{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$ (initial condition), there exists a unique solution $\{x(t), \sigma(t)\}$ for all $t \ge t_0$ such that $\{x(t_0), \sigma^-(t_0)\} = \{x_0, \sigma_0\}$.*
2. *For any set of initial conditions $\{x(t_0), \sigma^-(t_0)\} = \{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$, there exists a finite time T such that for $t > T$ the state variables $x_1(t)$, $\dot{x}_1(t)$, $x_2(t)$, and $\dot{x}_2(t)$ converge uniformly exponentially to zero, and $\omega_1(t) \le \epsilon_2$, where $\epsilon_2 > 0$ is a controller parameter that can be chosen arbitrarily small.*
3. *The origin $x(t) = 0$ is a Lyapunov uniformly stable equilibrium point of Σ_B.*
4. *Let ϵ_3 and γ_5 be positive constants such that $\epsilon_3 \le \epsilon^\star$ and $\gamma_5 \le \gamma^\star$, where the vector $(\epsilon^\star, \gamma^\star)'$ is an admissible or feasible parameter vector, such that the inequalities*

$$\begin{aligned} \lambda|\dot{x}_1| + |x_1| + k_1|x_2| &\le \bar{u}_1, \\ \lambda|\dot{x}_2| + |x_2| + k_2|x_1| &\le \bar{u}_2, \end{aligned} \tag{12}$$

hold for all $\omega_1 \le \epsilon^\star$ and $\omega_2 \le \gamma^\star$. Then, for any arbitrary large and bounded continuous state initial conditions x_0 the control signals satisfy the constraints

$$|u_i| \le \bar{u}_i, \quad i = 1, 2 \tag{13}$$

and $\lim_{t\to\infty} u_i(t) = 0$, $\quad i = 1, 2$ along the trajectories of the closed loop system Σ_B,

5 Stability analysis under Persistent Disturbances

Consider now the ENDI system subject to magnitude limitations on their inputs and in the presence of small input additive disturbances, *i.e.*,

$$\begin{aligned} \ddot{x}_1(t) &= u_1(t) + d_1(t), \\ \ddot{x}_2(t) &= u_2(t) + d_2(t), \\ \dot{x}_3(t) &= x_1(t)\dot{x}_2(t) - x_2(t)\dot{x}_1(t), \end{aligned} \tag{14}$$

where $x = (x_1, x_2, x_3, \dot{x}_1, \dot{x}_2)' \in \mathbb{R}^5$ is a state vector and $u = (u_1, u_2)' \in \mathbb{R}^2$ is a two-dimensional control vector subject to the constraints $|u_i(t)| \leq \bar{u}_i$. The disturbance vector $d = (d_1, d_2)'$ with $d_i : [t_0, \infty) \to \mathbb{R}$ piecewise continuous in t, satisfies $|d_i(t)| \leq \delta$ for all $i = 1, 2$.

Before presenting the main result we now introduce the following definitions that extend to a hybrid control setting the usual definitions of ultimate boundedness and practical stability [17].

Definition 2 (Unif. Ultimate Boundedness) *The solutions of the hybrid system Σ (see expression (2)) are said to be uniformly ultimately bounded (with bound b) if there exist positive constants b and c, and for every $\alpha \in (0, c)$, any $t_0 \in \mathbb{R}^+$ and every initial condition $\{x_0, \sigma_0\} \in \mathcal{X} \times \mathcal{I}$ with $\|x_0\| < \alpha$ there exists a positive constant $T = T(\alpha)$ independent of t_0 such that the continuous state solution $x(t)$ of Σ satisfies $\|x(t)\| < b$ for all $t \geq t_0 + T$. The solutions are said to be globally uniformly ultimately bounded if the above condition holds for arbitrarily large α.*

Consider now the hybrid system Σ_d defined by

$$\begin{aligned} \dot{x}(t) &= f_{\sigma(t)}\big(x(t), t\big) + d(x, t), \quad t \geq t_0 \\ \sigma(t) &= \phi\big(x(t), \sigma(t^-)\big), \end{aligned}$$

where the only difference between Σ and Σ_d is the presence of a perturbation term d that for physical systems may represent input disturbances or capture unknown modeling parameters. In practice d may be an unknown function of (x, t) but it satisfies the constraint

$$\|d(x, t)\| \leq \delta, \quad \forall t \geq t_0 \, \forall x \in \mathcal{X}. \tag{15}$$

Definition 3 (Practical Stability) *Let δ be a positive constant and let $\mathcal{R} \subset \mathcal{X}$ and $\mathcal{R}_0$ be two sets where $\mathcal{R}$ is a closed and bounded set containing the origin and $\mathcal{R}_0$ is a subset of $\mathcal{R}$. Let S_D be the set of all perturbations d satisfying (15). The origin $x = 0$ is said to be* practically stable *if for each $d \in S_D$, any $t_0 \in \mathbb{R}^+$, and for every initial condition $\{x_0, \sigma_0\} \in \mathcal{R}_0 \times \mathcal{I}$, the continuous solution of Σ_d, $x(t)$ is in $\mathcal{R}$ for all $t \geq t_0$. That is, the solutions that start initially in $\mathcal{R}_0 \times \mathcal{I}$ remain thereafter in $\mathcal{R} \times \mathcal{I}$. If, in addition, each solution of the hybrid system Σ_d for each $d \in S_D$ is ultimately in $\mathcal{R}$, then one says that the hybrid system Σ (see expression (2)) has* strong practical stability*.*

To solve the problem of practical stability for Σ_d, the control law developed in Section 4 is used, together with a slight modification of the vectors fields $g_\sigma(x)$ as follows:

$$g_1(x) = \begin{bmatrix} -\lambda\dot{x}_1 + x_1 + k_1 \operatorname{sgn}(x_2) \operatorname{sgn}(s) \\ -\lambda\dot{x}_2 + x_2 - k_2 \operatorname{sgn}(x_1) \operatorname{sgn}(s) \end{bmatrix}, \tag{16}$$

$$g_2(x) = \begin{bmatrix} -k_{11} \operatorname{sat}(k_{12}x_1 + \dot{x}_1) - k_{12} \operatorname{sat}(\dot{x}_1) \\ -k_{21} \operatorname{sat}(k_{22}x_2 + \dot{x}_2) - k_{22} \operatorname{sat}(\dot{x}_2) \end{bmatrix} \tag{17}$$

$$g_3(x) = \begin{bmatrix} -\lambda\dot{x}_1 + x_1 + k_1 h(\frac{x_2}{\delta}) \operatorname{sgn}(s) \\ -\lambda\dot{x}_2 + x_2 - k_2 h(\frac{x_1}{\delta}) \operatorname{sgn}(s) \end{bmatrix}, \tag{18}$$

$$g_4(x) = \begin{bmatrix} -\lambda\dot{x}_1 - x_1 + k_1 \operatorname{sgn}(x_2) \operatorname{sat}\left(\frac{s}{\eta}\right) \\ -\lambda\dot{x}_2 - x_2 - k_2 \operatorname{sgn}(x_1) \operatorname{sat}\left(\frac{s}{\eta}\right) \end{bmatrix}, \tag{19}$$

where

$$h(x) = \begin{cases} x & \text{if } |x| \geq 1, \\ \operatorname{sgn}(x) & \text{otherwise,} \end{cases}$$

$\lambda > 0$, η is a positive constant that satisfies $\eta^2 < \epsilon_2$, $k_1 > \delta$, $k_2 > \delta$, k_{i1} and k_{i2} are positive constants such that $k_{i1} + k_{i2} = \bar{u}_i$, $k_{i2} > k_{i1} + \delta$, and $k_{i1} > \delta$ for all $i = 1, 2$. Furthermore, $\epsilon_{i+1} > \epsilon_i > 0$ for $i = 1, 2$, $\gamma_{i+1} > \gamma_i > 0$ for $i = 1, 2, 3, 4$, and γ_5 satisfies

$$\beta_1^2 + \beta_2^2 \leq \gamma_5, \tag{20}$$

where

$$\beta_i = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)} \frac{\sqrt{\lambda^2 + 4}}{\lambda} \frac{k_i + \delta}{\theta}}, \quad i = 1, 2 \tag{21}$$

for some positive constant $\theta < 1$, $\lambda_{max}(P) = \frac{\lambda}{4} + \frac{1}{\lambda} + \frac{\sqrt{\lambda^2+4}}{4}$, and $\lambda_{min}(P) = \frac{\lambda}{4} + \frac{1}{\lambda} - \frac{\sqrt{\lambda^2+4}}{4}$.

The following theorem establishes the main result of the paper.

Theorem 4 *Consider the hybrid system Σ_d described by* (7), (14) *(16)-(19) and the switching logic defined in (3). Let $\{x(t), \sigma(t)\} = \{x : [t_0, \infty) \to \mathbb{R}^5, \sigma : [t_0, \infty) \to \mathcal{I}\}$ be a solution to Σ_d. Then, the following properties hold.*

1. *Given an arbitrary pair $\{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$ (initial condition), there exists a unique solution $\{x(t), \sigma(t)\}$ for all $t \geq t_0$ such that $\{x(t_0), \sigma^-(t_0)\} = \{x_0, \sigma_0\}$.*
2. *The solutions of the hybrid system Σ_d are globally uniformly ultimately bounded.*
3. *The origin $x(t) = 0$ is practically stable relative to the positive constant δ and the two sets $\mathcal{R}$ and $\mathcal{R}_0$, where $\mathcal{R}_0 \subset \mathcal{R}$, $|d(t)| \leq \delta$,*
$$\mathcal{R}_0 = \{x \in \mathbb{R}^5 : \omega_1 \leq \eta^2 \wedge \omega_2 \leq \mu_1^2 + \mu_2^2\},$$
$$\mathcal{R} = \{x \in \mathbb{R}^5 : \omega_1 \leq \epsilon_2 \wedge \omega_2 \leq \beta_1^2 + \beta_2^2\},$$
β_i, $i = 1, 2$ is given by (21) and
$$\mu_i = \frac{\sqrt{\lambda^2 + 4}}{\lambda} \frac{k_i + \delta}{\theta}, \quad i = 1, 2$$
for some positive constant $\theta < 1$.
Furthermore, the hybrid system composed by Σ_d without the perturbation term d possesses strong practical stability.
4. *Let $\epsilon_3 \leq \epsilon^\star$ and $\gamma_5 \leq \gamma^\star$, where the vector $(\epsilon^\star, \gamma^\star)'$ is an admissible or feasible parameter vector, for which the inequalities* (12) *and*
$$\lambda|\dot{x}_i| + |x_i| + k_i \leq \bar{u}_i, \quad i = 1, 2 \qquad (22)$$
hold with $\omega_1 \leq \epsilon^\star$ and $\omega_2 \leq \gamma^\star$. Then, for any arbitrary large and bounded continuous state initial conditions x_0 the control signals satisfy the constraints $|u_i| \leq \bar{u}_i$, $i = 1, 2$.

Proof.

Existence and Uniqueness

For each $i \in \mathcal{I}$ the vector field $g_i(x)$ is globally Lipschitz. Furthermore, the distance between two points in the (ω_1, ω_2)-space where consecutive switchings can occur is always nonzero. It now follows from classical arguments [15] that the hybrid system Σ_d has exactly one solution for each initial condition $\{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$.

Global uniform ultimate boundedness

To show global uniform ultimate boundedness of system Σ_d consider first the following claims:

Claim 1 *Consider the system*
$$\ddot{x} = -k_1 \text{ sat}(k_2 x + \dot{x}) - k_2 \text{ sat}(\dot{x}) + d(t), \quad (23)$$
where the disturbance term $d(t)$ satisfies $|d(t)| \leq \delta$, and k_1 and k_2 are positive constants that satisfy $k_1 + k_2 = \bar{u}$, $k_2 > k_1 + \delta$, and $k_1 > \delta$. Then, for any initial condition $(x(t_0), \dot{x}(t_0))' \in \mathbb{R}^2$, there exist a finite time $T \geq t_0$ and a positive constant γ given by
$$\gamma = \frac{\delta}{k_1 {k_2}^2} \sqrt{4{k_1}^2 + {k_2}^2 + 4k_1 k_2 + 4{k_1}^2 {k_2}^2} \qquad (24)$$
such that for all $t \geq T$,
$$x^2(t) + \dot{x}^2(t) \leq \gamma^2. \qquad (25)$$

Proof. Let $y = (y_1, y_2)'$, $y_1 = k_2 x + \dot{x}$, and $y_2 = \dot{x}$. Then, system (23) can be rewritten as
$$\dot{y}_1 = k_2 y_2 - k_1 \text{ sat}(y_1) - k_2 \text{ sat}(y_2),$$
$$\dot{y}_2 = -k_1 \text{ sat}(y_1) - k_2 \text{ sat}(y_2) + d.$$

Clearly, for any initial condition $y(t_0) = y_0$ there exists a finite time T such that $|y_2(t)| \leq 1$ for all $t \geq T$ since
$$y_2 \dot{y}_2 \leq -|y_2|(k_2 - k_1 - \delta) < 0. \qquad (26)$$

Notice that the state $y_1 = k_2 x + \dot{x}$ satisfies
$$y_1 \dot{y}_1 \leq -|y_1| \left(k_1 \,|\text{sat}(y_1)| - \delta\right).$$

This, coupled with the fact that $k_1 > \delta$, shows that after a finite time y_1 will enter and stay in the set $|y_1| \leq \frac{\delta}{k_1}$. It then follows that y_2 will enter and stay in the set $|y_2| \leq \frac{2\delta}{k_2}$. Condition (25) holds since the set given by $\mathcal{R} = \left\{(x, \dot{x}) \in \mathbb{R}^2 : |k_2 x + \dot{x}| \leq \frac{\delta}{k_1} \wedge |\dot{x}| \leq \frac{2\delta}{k_2}\right\}$ is a subset of $\Omega_\gamma = \{(x, \dot{x}) \in \mathbb{R}^2 : x^2 + \dot{x}^2 \leq \gamma^2\}$.

Claim 2 *Consider the system*
$$\ddot{x}_1 = -\lambda \dot{x}_1 - x_1 + k + d(t) \Leftrightarrow \dot{x} = Ax + g(t), \quad (27)$$
where $x = (x_1, \dot{x}_1)'$, $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -\lambda \end{smallmatrix}\right)$, $g(t) = (0, k + d)'$, k and λ are positive constants, and the disturbance term $d(t)$ satisfies $|d(t)| \leq \delta$. Then, there exist a finite time $T \geq t_0$, a 2×2 real symmetric positive definite matrix P and finite positive constants γ, ξ, β, and $\theta < 1$ such that for any

initial condition $x_0 = (x_1(t_0), \dot{x}_1(t_0))'$, the solution $x(t) = (x_1(t), \dot{x}_1(t))'$ of (27) satisfies

$$\|x(t)\| \leq \gamma e^{-\xi(t-t_0)}\|x(t_0)\|, \quad \forall t_0 \leq t < T$$

and

$$\|x(t)\| \leq \beta, \quad \forall t \geq T \tag{28}$$

where

$$\gamma = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}},\ \xi = -\frac{1-\theta}{2\lambda_{max}(P)},\ \beta = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}\frac{\sqrt{\lambda^2+4}}{\lambda}\frac{k+\delta}{\theta}.$$

Proof. Let $V(x) = x^T P x$ be a Lyapunov function candidate, where P is a real symmetric positive definite matrix that satisfies the Lyapunov equation $PA + A^T P = -I$. The derivative of $V(x)$ along the trajectories of (27) satisfies

$$\dot{V} \leq -(1-\theta)\|x\|^2, \quad \forall \|x\| \geq \mu$$

where $\mu = \frac{\sqrt{\lambda^2+4}}{\lambda}\frac{k+\delta}{\theta}$ and $0 < \theta < 1$. Since

$$\lambda_{min}(P)\|x\|^2 \ \leq\ V(x) \ \leq\ \lambda_{max}(P)\|x\|^2,$$

$\dot{V} \leq -\frac{1-\theta}{\lambda_{max}(P)} V$ for all $\|x\| \geq \mu$ and consequently

$$\|x(t)\| \leq \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} \exp\Big[-\frac{1-\theta}{2\lambda_{max}(P)}(t-t_0)\Big]\|x(t_0)\|. \tag{29}$$

for all $\|x\| \geq \mu$. Let $\beta = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}\mu$ and Ω_β a set defined by

$$\Omega_\beta = \{x \in \mathbb{R}^2 : V(x) \leq \lambda_{min}(P)\beta^2\}.$$

The set Ω_β contains the ball $B_\mu(0) = \{x \in \mathbb{R}^2 : \|x\| \leq \mu\}$, since $\|x\| \leq \mu = \sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}(P)}}\beta \Rightarrow \lambda_{max}(P)\|x\|^2 \leq \lambda_{min}(P)\beta^2 \Rightarrow V(x) \leq \lambda_{min}(P)\beta^2$. Therefore, any solution that starts inside Ω_β cannot leave it because $\dot{V}$ is negative on the boundary. Consequently, for a solution starting inside Ω_β, the inequality (28) is satisfied for all $t \geq t_0$, since $V(x) \leq \lambda_{min}(P)\beta^2 \Rightarrow \lambda_{min}(P)\|x\|^2 \leq \lambda_{min}(P)\beta^2 \Rightarrow \|x\| \leq \beta$. If the solution starts outside Ω_β it follows from (29) that $\|x(t)\| \to 0$ as $t \to \infty$. Thus, there is a finite time T after which $\|x(t)\| \leq \mu$ and consequently the solution enter the set Ω_β in finite time. Once inside the set, the solution remains inside for all $t \geq T$.

Claim 3 *There exists a finite time $T \geq t_0$ such that for all $t \geq T$, $\sigma(t) = 4$.*

Proof. It can be easily checked that the vector fields $g_i(x)$ for $i \in \mathcal{I}$ has the following properties:

- $\sigma = 1$: $\dot{\omega}_1 < 0$ if $k_1, k_2 > \delta$, and $\omega_2 \to \infty$ as $t \to \infty$.
- $\sigma = 2$: From Claim 1, it follows that $\omega_2(t)$ reaches the boundary $\omega_2 = \gamma_2$ in finite time if its initial condition satisfies $\omega_2(t_0) \geq \gamma_2 \geq \gamma_{21}^2 + \gamma_{22}^2$, where γ_{2i}; $i = 1, 2$ is given by (24) by replacing k_1 and k_2 with k_{i1} and k_{i2}, respectively.
- $\sigma = 3$: $\dot{\omega}_1 < 0$ if $k_1, k_2 > \delta$.
- $\sigma = 4$: There exists a finite time $T > t_0$ such that for all $t \geq T$ one obtains $\omega_2(t) \leq \beta_1^2 + \beta_2^2$, where β_i is given by (21) (see Claim 2). If $k_1, k_2 > \delta$, then $\dot{\omega}_1 \leq 0$ while $|s| \geq \eta$.

Using the arguments invoked in the proof of Theorem 2, and since $\eta^2 < \epsilon_2$ and $\beta_1^2 + \beta_2^2 \leq \gamma_5$, Claim 3 can be easily established.

It now follows, from Claim 2 and Claim 3 that for each $\{x_0, \sigma_0\} \in \mathbb{R}^5 \times \mathcal{I}$ the solution of the hybrid system Σ_d is globally uniformly ultimately bounded.

Practical stability

To prove practical stability, one must show that for each initial condition $\{x_0, \sigma_0\} \in \mathcal{R}_0 \times \mathcal{I}$ and for each $d \in S_D$ (where S_D is the set of all perturbations such that $|d(t)| \leq \delta$), the continuous solution $x(t)$ of the closed loop system Σ_d must remain in $\mathcal{R}$ for all $t \geq t_0$.

From the proof of Claim 2, one can conclude that for all $t \geq t_0$ with $\sigma(t_0) = 4$

$$x_i^2(t_0) + \dot{x}_i^2(t_0) \leq \mu_i^2 \Rightarrow x_i^2(t) + \dot{x}_i^2(t) \leq \beta_i^2, \ \ i = 1, 2$$

and, thus, $\omega_2(t) \leq \beta_1^2 + \beta_2^2$. Since $\mathcal{R}_0 \subset \mathcal{R} \subset \mathcal{R}_4$, and since for $\sigma = 4$, $\dot{\omega}_1 \leq 0$ for $|s| \geq \eta$, it follows that for all $t \geq t_0$ and $x_0 \in \mathcal{R}_0$, $x(t) \in \mathcal{R}$.

Since each solution of the hybrid system Σ_d for each $d \in S_D$ is ultimately in $\mathcal{R}$, it can be concluded that the hybrid system Σ possesses strong practical stability.

Boundedness of control inputs

The boundedness of the control inputs can be easily proved by adopting the main guidelines set forth in the proof of boundedness in Theorem 3. This completes the proof of Theorem 4. □

6 Simulation Results

This section illustrates the performance of the hybrid control law developed in Section 5 when the ENDI is subject to input saturations and input additive disturbances. The control parameters were chosen as follows: $\lambda = 1.0$, $\epsilon_1 = 0.001$, $\epsilon_2 = 0.1$, $\epsilon_3 = 0.9$, $\gamma_1 = 0.1$, $\gamma_2 = 0.3$, $\gamma_3 = 0.35$, $\gamma_4 = 1.0$, $\gamma_5 = 1.1$, $k_1 = k_2 = 0.101$, $\eta = 0.05$, $k_{11} = k_{21} = 0.4$, $k_{12} = k_{22} = 0.6$, and $\delta = 0.1$. In the simulation, a disturbance vector $d(t) = [d_1(t), d_2(t)]'$ given by $d_1(t) = 0.1 \sin(t)$, and $d_2(t) = 0.1 \sin\left(t + \frac{\pi}{2}\right)$ was considered. Notice that γ_5 satisfies condition (20), where for this case $\beta_1 = \beta_2 = 0.7346$ (with $\theta = 0.99$). Notice also that for these parameters one has $\bar{u}_1 = \bar{u}_2 = 1.57$ (with $\gamma^\star = \gamma_5$ and $\theta = 0.99$).

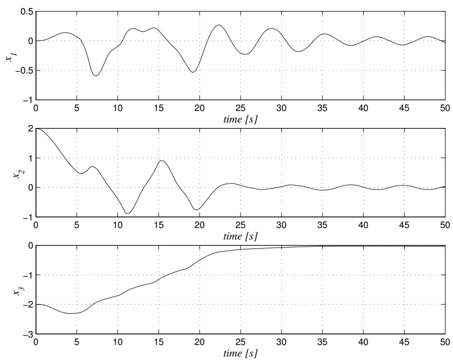

Figure 5: Time evolution of the state variables $x_1(t)$, $x_2(t)$, and $x_3(t)$.

Figures 5-7 show the simulation results for the initial condition $x(0) = [x_1, x_2, x_3, \dot{x}_1, \dot{x}_2]'(0) = [0, 2\,m, -2\,m,\ 0, 0]'$ and $\sigma(0^-) = 2$. The state x converges to a neighborhood of the origin. Notice the oscillatory behavior of $\omega_2(t)$ that is due to input saturations.

7 Conclusions

A hybrid control law was derived for the ENDI system that captures any kinematic completely

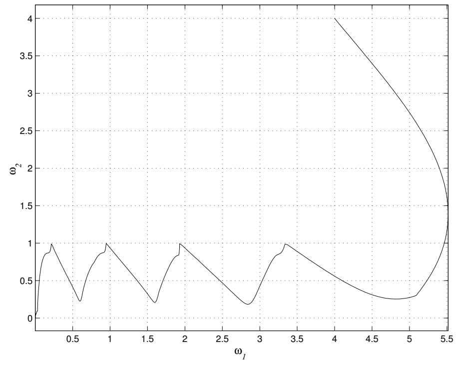

Figure 6: Trajectory evolution in Ω-space.

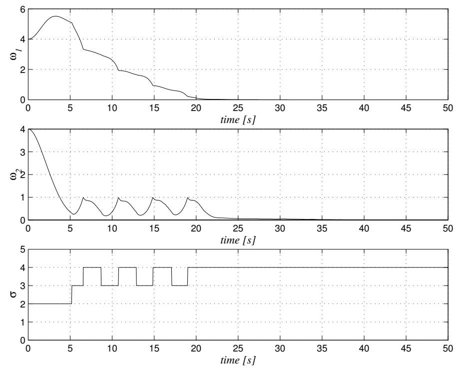

Figure 7: Time evolution of the variables $\omega_1(t)$, $\omega_2(t)$, and $\sigma(t)$.

nonholonomic model with three states and two first-order dynamic control inputs, e.g., the dynamics of a wheeled robot subject to force and torque inputs. The hybrid controller yields global stability and convergence of the closed loop system to an arbitrarily small neighborhood of the origin. An extension of the controller was done to the bounded input case. It was shown that, despite the saturations, the ENDI system can be stabilized for any initial condition and any bound (even arbitrarily small) on the inputs signals. Finally, with a slightly modification of the control law, the problem of practical stabilization was solved. The main result shows that for any input additive disturbances bounded by a constant δ, the closed loop hybrid system possesses strong practical stability with bounded control. Simulation results captured some of the features of the proposed control laws and illustrate their performance.

Acknowledgements

This work was supported in part by the EC under the FREESUB network and by the PDCTM programme of the FCT of Portugal under projects DREAM and MAROV.

References

[1] A. P. Aguiar, *Nonlinear motion control of nonholonomic and underactuated systems*, Ph.D. thesis, Dept. Electrical Engineering, Instituto Superior Técnico, IST, Lisboa, Portugal, 2002.

[2] A. P. Aguiar and A. Pascoal, *Stabilization of the extended nonholonomic double integrator via logic-based hybrid control*, SYROCO'00 - 6th International IFAC Symposium on Robot Control (Vienna, Austria), September 2000.

[3] A. Astolfi, *Discontinuous control of the brockett integrator*, European Journal of Control **4** (1998), no. 1, 49–63.

[4] A. Astolfi, *Exponential stabilization of a wheeled mobile robot via discontinuous control*, Journal of Dynamic Systems, Measurements, and Control **121** (1999), 121–126.

[5] A. Bloch and S. Drakunov, *Stabilization of a nonholonomic system via sliding modes*, Proc. 33rd IEEE Conference on Decision and Control (Orlando, Florida, USA), December 1994.

[6] A. M. Bloch, M. Reyhanoglu, and N. H. McClamroch, *Control and stabilization of nonholonomic dynamic systems*, IEEE Transactions on Automatic Control **37** (1992), no. 11, 1746–1757.

[7] M. S. Branicky, *Multiple Lyapunov functions and other analysis tools for switched and hybrid systems*, IEEE Transactions on Automatic Control **43** (1998), no. 4, 475–482.

[8] R. W. Brockett, *Asymptotic stability and feedback stabilization*, Differential Geometric Control Theory (Birkhäuser, Boston, USA) (R. W. Brockett, R. S. Millman, and H. J. Sussman, eds.), 1983, pp. 181–191.

[9] C. Canudas de Wit and O.J. Sørdalen, *Exponential stabilization of mobile robots with nonholonomic constraints*, IEEE Transactions on Automatic Control **37** (1992), no. 11, 1791–1797.

[10] C. Canudas de Wit and Harry Berghuis, *Practical stabilization of nonlinear systems in chained form*, Proc. 33rd IEEE Conference on Decision and Control (Orlando, Florida, USA), December 1994, pp. 3475–3479.

[11] J. M. Godhavn and O. Egeland, *A Lyapunov approach to exponential stabilization of nonholonomic systems in power form*, IEEE Transactions on Automatic Control **42** (1997), no. 7, 1028–1032.

[12] J. Guldner and V.I. Utkin, *Stabilization of nonholonomic mobile robots using Lyapunov functions for navigation and sliding mode control*, Proc. 33rd IEEE Conference on Decision and Control (Orlando, Florida, USA), December 1994, pp. 2967–2972.

[13] J. P. Hespanha, *Stabilization of nonholonomic integrators via logic-based switching*, Proc. 13th World Congress of IFAC (S. Francisco, CA, USA), vol. E, June 1996, pp. 467–472.

[14] A. Isidori, *Nonlinear control systems*, 2^{nd} ed., Springer-Verlag, Berlin, Germany, 1989.

[15] H. K. Khalil, *Nonlinear systems*, 2^{nd} ed., Prentice-Hall, New Jersey, USA, 1996.

[16] I. Kolmanovsky and N. H. McClamroch, *Developments in nonholonomic control problems*, IEEE Control Systems Magazine **15** (1995), 20–36.

[17] J. La Salle and S. Lefschetz, *Stability by liapunov's direct method with applications*, Academic Press Inc., London, UK, 1961.

[18] R. T. M'Closkey and R. M. Murray, *Exponential stabilization of driftless nonlinear control systems using homogeneous feedback*, IEEE Transactions on Automatic Control **42** (1997), no. 5, 614–628.

[19] K. A. Morgansen and R. W. Brockett, *Nonholonomic control based on approximate inversion*, Proceedings of the American Control Conference (San Diego, California, USA), June 1999, pp. 3515–3519.

[20] P. Morin and C. Samson, *Control of nonlinear chained systems: From the routh-hurwitz stability criterion to time-varying exponential stabilizers*, IEEE Transactions on Automatic Control **45** (2000), no. 1, 141–146.

[21] H. Nijmeijer and A. J. van der Schaft, *Nonlinear dynamical control systems*, Springer-Verlag, New York, USA, 1990.

[22] J. B. Pomet, *Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift*, Systems and Control Letters **18** (1992), 147–158.

[23] C. Samson, *Path-following and time-varying feedback stabilization of a wheeled mobile robot*, Proceedings of the ICARCV 92 (Singapore), 1992, pp. RO–13.1.1–RO–13.1.5.

[24] C. Samson, *Control of chained systems: Application to path following and time-varying point-stabilization of mobile robots*, IEEE Transactions on Automatic Control **40** (1995), no. 1, 64–77.

[25] H. J. Sussmann, *A general theorem on local controllability*, SIAM Journal of Control and Optimization **25** (1987), no. 1, 158–194.

[26] H. Ye, A. N. Michel, and L. Hou, *Stability theory for hybrid dynamical systems*, IEEE Transactions on Automatic Control **43** (1998), no. 4, 461–474.