AN ISOGEOMETRIC 2D PLATE FORMULATION BASED ON LAGRANGIAN AND B-SPLINE INTERPOLATION FUNCTIONS FOR GEOMETRIC LINEAR AND NONLINEAR ANALYSES

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ABSTRACT

In this paper, a new isogeometric plate for linear and nonlinear analysis will be proposed. Basically, plates are a specialized form of shells and they are frequently used in many metal forming processes and sciences. This paper covers the development of an efficient plate formulation by using a combination of B-splines and Lagrangian interpolation functions. The method used, keeps the geometric exact character of the isogeometric analyses and avoids locking as it is present by using low order interpolation functions. Cubic B-Splines within the plate plane guarantees the continuity across the border of the plate-patches. A simple mapping scheme will be introduced to approximate the plate geometry exactly during the whole load cycle. Lagrangian interpolation functions of third order are used in thickness direction and allow elongations, as opposed to the Kirchhoff-Love as well as Reissner-Mindlin theorie and the results show, that they are well suited for large deformation problems.

For the constitutive relation, a hyperelastic material law (finite strain theory) was used and compared with the linear elastic law for small strains. Some selected numerical examples show the limits for the linear material law. Unlike to traditional plate and shell formulations, there is no restriction on external loads, because the proposed method treats the plate as a three dimensional continuum. Beyond the tests in this paper, a lot of numerical examples are studied to demonstrate the proposed method and the numerical results are in a very good agreement with the closed-form or traditional FEM solutions.

**Keywords:** B-spline basis, Lagrange basis, Mapping techniques, 2D plate Elements, Hyperelastic deformation.

INTRODUCTION

B-spline curves and surfaces are widely used in the CAD (computer aided design) and graphics communities. Their rational representation, known as Non-Uniform Rational B-splines (NURBS) has been the de facto industrial standard over the past decades. B-splines provide a convenient set of basis functions for a control mesh of regular topology. They have been widely used in shape design, surface reconstruction, shape deformation and animation, image processing, biomedical applications and recently on isogeometric analysis.

B-Splines and NURBS basis functions that are non-interpolant are tools for engineering designs, whereas piecewise polynomial basis functions that are interpolant are shape functions for finite element analysis. Thus, there are barriers between CAD and finite element analysis that make it difficult to synthesize them together (Jae Woo, 2013). Within the framework of isogeometric analysis, a lot of methods were proposed and successfully applied to different applications under consideration of the locking phenomenon see (Beirão da Veiga, Lovadina
and Reali, 2012), (Xiaojun and Weiyin, 2014), (Schillinger and Rank 2011). Other authors uses B-Spline interpolation functions for the discretization of plate and shells according to the Kirchhoff-Love or Reissner-Mindlin theorie: (Jiawei, Xuefeng, Yumin and Zhengjia, 2006) and (Chen, Nguyen-Thanh, Nguyen-Xuan, Rabczuk, Bordas, Limbert, 2014). But most of these papers do not take into account stretches in thickness direction, which are become crucial by using thick-shells or plates. Therefore, an efficient B-Spline interpolation method will be introduced based on a two-dimensional plate to cover large deformation problems.

OBJECTIVES

The aim of this study is to show a very simple and effective method to implement a B-Spline interpolation scheme in combination with a Lagrangian interpolation scheme for a two-dimensional plate (which represents the cross-section of a three-dimensional plate). The model is based on a static-implicit total Lagrangian formulation. Following on that, a hyperelastic material law is used for a first evaluation of the patch response for the geometric linear and nonlinear case.

B-SPLINE INTERPOLATION

B-Splines are piecewise polynomial curves composed of linear combinations of B-Spline basis functions. The function coefficients are points in space, referred to as control points. A knot vector $\Xi$ is a set of non-decreasing real numbers representing coordinates of the curve in the parametric space:

$$\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\},$$

(1)

where $p$ denotes the order of the B-Spline and $n$ is the number of basis functions respectively the number of control points. The interval $[\xi_i, \xi_{i+p+1}]$ is called a patch and the interval $[\xi_i, \xi_{i+1}]$ is called a knot span. If its first and last knot are repeated $p+1$ times then it’s called an open knot vector whereas a closed knot vector otherwise. The knots of each individual basis function $N_{i,p}$ with patch index $i$ can be identified as the consecutive entries $\{i, i+1, \ldots, i+p+1\}$ in $\Xi$, see also (Schillinger and Rank 2011) respectively (Piegl and Tiller 1997). A B-Spline defined by an open knot vector interpolates both end control points while it does not interpolate with a closed knot vector. Important properties of B-Spline basis functions are as follows:

1) B-Splines fulfil partition of unity: $\sum_{i=1}^{n} N_{i,p}(\xi) = 1$

2) B-Splines are non-negative within the whole interval: $N_{i,p}(\xi) \geq 0, \ \forall \xi$

3) $C^{p-k}$ continuity where $k$ is multiplicity of knots (if internal knots are not repeated, $C^{p+1}$ continuity)

4) A B-Spline basis function of order $p$ ensures a compact support over the domain of $p+1$ knots

5) If all knots are equally spaced, the knot vector is called uniform, otherwise, non-uniform

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B-Spline basis functions $N_{i,p}$ of arbitrary polynomial degree $p$ can be generated recursively with the Cox-de Boor formula, starting from piecewise constants $N_{i,0}$.

$$N_{i,0}(\xi) = \begin{cases} 1, & \xi_i \leq \xi \leq \xi_{i+1} \\ 0, & \text{otherwise} \end{cases}$$  

(2)

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi)$$  

(3)

For the following 2D plate formulation, we use a uniform knot vector

$$\Xi = \left\{ 0, 0, 0, 0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1, 1, 1 \right\}.$$  

(4)

which delivers a cubic B-Spline patch for a polynomial degree $p = 3$ as it is shown in Fig. 1.

![Fig. 1 - Cubic B-Spline basis functions and its first derivatives formed from the open knot vector $\Xi$ in Eq. 4.](image)

Herein it is obvious that a uniform B-Spline patch with open knot vectors constitutes a suitable finite element basis, since it guarantees optimal approximation for smooth problems due to maximum continuity within the patch, but also allows for the imposition of boundary conditions by standard finite element techniques. Knot spans can be identified as finite elements in the classical sense and each knot span element should be integrated with full Gaussian integration, for instance $p + 1$ integration points. The cubic B-Spline patch in Fig. 1 consists of 6 elements, consequently for a full Gaussian integration 24 evaluation points over the whole patch are necessary.

A B-Spline curve of degree $p$ is expressed in parametric form as a linear combination of control points $p_i$ and basis functions of degree $p$. A point $c$ on a B-Spline is calculated as follows:

$$c(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi) p_i.$$  

(5)
In a similar manner, Eq. 5 can be reformulated to calculate a point on a B-Spline surface:

$$c(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(\xi) M_{j,q}(\eta) p_{i,j}. $$

(6)

Fig. 2 shows a simple B-Spline by using the knot vector Eq. 4 where Eq. 5 was used to calculate the coordinates of the curve. The red circles in Fig. 2 indicates the control points.

It is obvious that a straight line (first three control points in Fig. 2) is indicated by control points, which are also straight aligned. Since it is a fact, that a plate is straight and even in the initial configuration, all control points are aligned on the plate’s surface.

2D PLATE FORMULATION

On the basis of the principle of virtual work, the nonlinear equilibrium equations in the initial configuration can be expressed in weak form as

$$\int_{\Omega_h} S : \delta E \, dV = \int_{\Omega_h} \rho_0 \mathbf{b} \cdot \delta \mathbf{u} \, dV + \int_{\partial\Omega_h} \mathbf{t}_0 \cdot \delta \mathbf{u} \, dA$$

(7)

with $\delta \mathbf{u} = 0$ on $\partial_u \Omega_h$ $\forall \delta \mathbf{u}$

in the Total-Lagrangian form, which includes the 2nd Piola-Kirchhoff stresses $S$, the Green-Lagrangian stresses $E$, the mass density $\rho_0$ referred to the initial configuration, the volume force vector $\mathbf{b}$ and the surface stress vector $\mathbf{t}_0$ acting on the initial configuration. Furthermore, the entire plate is treated as a solid continuum similar to a solid shell element in the finite element technology. There is no split in the kinematics regarding quantities acting on the middle plane or thickness direction. Therefore, Eq. 7 in its original form represents the starting point for the further discretization.

DISCRETIZATION

The kinematics of the 2D plate formulation based on cubic B-Spline interpolation in length as well as in widthness direction and cubic Lagrangian interpolation in thickness direction as it is shown in Fig. 3.
The entire patch is defined by 9 control points in length direction and 4 nodes in thickness direction which leads finally to 72 dofs (degree of freedoms) in the two-dimensional case.

An arbitrary point at the initial (Lagrangian) and current (Eulerian) configuration within the plate continuum is expressed by the position vectors \( \mathbf{X} \) respectively \( \mathbf{x} \).

\[
\mathbf{X}(\xi, \eta) = \sum_{j=1}^{m} \sum_{i=1}^{n} N_j(\eta)N_i(\xi) \mathbf{p}_{ij} = N^\xi_j N^\eta_i \mathbf{p}_{IJ}
\]

(8)

\[
\mathbf{x}(\xi, \eta) = \sum_{j=1}^{m} \sum_{i=1}^{n} N_j(\eta)N_i(\xi) \mathbf{p}_{ij} = N^\eta_j N^\xi_i \mathbf{p}_{IJ}
\]

Furthermore, the covariant basis vectors \( \mathbf{G}_i \) at the initial configuration and the covariant basis vectors at the current configuration \( \mathbf{g}_i \) are given by

\[
\mathbf{G}_i = \frac{\partial \mathbf{X}}{\partial \theta_i} \quad \text{and} \quad \mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \theta_i}.
\]

(9)

Now the covariant components of the Green-Lagrangian strains and its variation

\[
\overline{E}_{ij} = \frac{1}{2} \left( \mathbf{g}_i \cdot \mathbf{g}_j - \mathbf{G}_i \cdot \mathbf{G}_j \right)
\]

(10)

\[
\delta E_{ij} = \frac{1}{2} \left( \mathbf{g}_i \cdot \delta \mathbf{g}_j - \delta \mathbf{g}_i \cdot \mathbf{g}_j \right)
\]

(11)

can be calculated. The total Green Lagrangian strain tensor is given by

\[
\mathbf{E} = \overline{E}_{ij} \left( \mathbf{G'} \otimes \mathbf{G'} \right)
\]

(12)

with the contravariant basis vectors \( \mathbf{G'} \) gained by inversion of the Jacobian matrix:

\[
\mathbf{J} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{G}_2 \end{bmatrix} \rightarrow \mathbf{J}^{-1} = \begin{bmatrix} (\mathbf{G}_1^T)^T \\ (\mathbf{G}_2^T)^T \end{bmatrix}.
\]

(13)
The 2\textsuperscript{nd} Piola-Kirchhoff stresses depend on the material law. Therefore, a hyperelastic material law by an underlying isotropic strain energy functional

\begin{equation}
W(J, \hat{C}) = U(J) + \hat{W}(\hat{C})
\end{equation}

consisting of the volumetric part

\begin{equation}
U(J) = \frac{K}{4} \left[ (J - 1)^2 + (\ln J)^2 \right]
\end{equation}

and the deviatoric part

\begin{equation}
\hat{W}(\hat{C}) = \frac{\mu}{2} (I_c - 3)
\end{equation}

defined in the principal strains \( \hat{\lambda}_i \), with the relationships

\begin{equation}
J = \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3
\end{equation}

\begin{equation}
I_c = \hat{\lambda}_1^2 + \hat{\lambda}_2^2 + \hat{\lambda}_3^2
\end{equation}

\begin{equation}
\hat{\lambda}_i = J^{-1/3} \lambda_i
\end{equation}

will be used (Wriggers, 2000). Herein, \( J \) describes the volumetric part of the right Cauchy-Green strains \( C \) and \( I_c \) denotes the first invariant of the deviatoric Cauchy-Green strains \( \hat{C} \). Now, the 2\textsuperscript{nd} Piola-Kirchhoff stresses

\begin{equation}
S = 2 \frac{\partial W(J, \hat{C})}{\partial C}
\end{equation}

used in Eq. 7 can be calculated.

**GLOBAL LINEAR AND NONLINEAR SYSTEM OF EQUATIONS**

A systematic expansion of Eq. 7 leads to the general form of the static-implicit system of equations of motion for the linear (Eq. 21) and nonlinear (Eq. 22) case

\begin{equation}
R = P,
\end{equation}

\begin{equation}
R(u) = P.
\end{equation}

Where \( R(u) \) describes the vector of inner forces depending on the displacements \( u \) of the nodes respectively the control points and \( P \) describes the Vektor of external forces. For the linear case, the total Green Lagrangian strain tensor was substituted by the linear strain tensor and the linear relationship between stresses and strains (Hooke’s law) was assumed. Eq. 22 is solved by the Newton-Raphson procedure. Therefore, the linearized equation system

\begin{equation}
K \Delta u = \Delta P
\end{equation}

with the global tangent stiffness matrix \( K \) is the basis for an iterative solving procedure.
RESULTS

Figure 4a shows a two-dimensional clamped plate, where the element response was tested in a static-implicit solution procedure for a hyperelastic material. For this purpose, a fine mesh with 50x10 continuum, plane strain elements (Fig. 4d) was generated in the commercial software package ANSYS® and applied to the plate. The vertical tip deflection $u$ after applying the external force $F$ is very close to the real solution and provides the basis for all further comparisons (Tab. 1, first column). Figure 4b and 4c shows the graphical solutions obtained by the self-developed 2D plate B-Spline patch for the geometric linear and geometric nonlinear case, whereas the solution of the geometric nonlinear case is pointed out in Tab. 1 in the third column. The results obtained for a coarser mesh generated in ANSYS® (Fig. 4e & Tab. 1, second column) and for the 2D plate B-Spline patch (Tab. 1, third column) demonstrate, that the count of dof’s is much less for the 2D plate B-Spline patch, even though the deflection $u$ of the plate is the same. Obviously, the effort to obtain the same solution is much less by the use of the 2D plate B-Spline patch than the effort to obtain the solution by the use of the 2D plane strain elements.

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**Figure 4** - Deformation of a clamped 2D plate
Table 1 - Elastic response of the 2D plate patch

<table>
<thead>
<tr>
<th></th>
<th>ANSYS © (Fig. 4d)</th>
<th>ANSYS © (Fig. 4e)</th>
<th>2D plate patch (Fig. 4c) (B-Spline &amp; Lagrange interpolation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50x10 elements</td>
<td>18x4 elements</td>
<td>1 patch</td>
<td>9 control points x 4 nodes</td>
</tr>
<tr>
<td>570 nodes</td>
<td>104 nodes</td>
<td>208 dof’s</td>
<td>72 dof’s</td>
</tr>
<tr>
<td>1140 dof’s</td>
<td>PLANE 182</td>
<td>PLANE 182</td>
<td>u = 2,63 m</td>
</tr>
<tr>
<td>u = 2,65 m</td>
<td>u = 2,63 m</td>
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Beyond this, a lot of tests have been performed to evaluate the 2D B-Spline patch behaviour under different load conditions. All of them turned out to be satisfactory approximations of the real solution.

CONCLUSION

The element formulation based on the combined B-Spline and Lagrangian interpolation turns out to be very useful regarding efficiency to calculate most sophisticated applications. Several systematic tests beyond the tests mentioned above, confirm the robustness and convergence to the real solution for the static-implicit case. Furthermore, the 2D plate B-Spline patch provide the basis for an extension to a 3D plate B-Spline patch formulation.

REFERENCES


