TOPOLOGY OPTIMIZATION OF CONTINUUM STRUCTURES WITH UNCERTAINTY IN LOADING DIRECTION

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ABSTRACT

Uncertainty in applied loads is a vital factor that needs to be considered in the design of the engineering structures. This paper concerns the minimization of mean compliance for continuum structures subjected to directional uncertain applied loads. Due to the uncertain behaviour of this type of the optimization problem, the existed deterministic topology optimization methods are not able to solve such problems. The loading directional uncertainty is described by directional interval variables which are divided into many small intervals, and then the uncertain small interval variables are approximated by their deterministic midpoints. In doing so, the uncertain topology optimization problem is transformed into deterministic multiple load case one. The optimization problem is then formulated as minimizing the mean compliance under multiple load cases, subject to material volume constraint. A soft-kill bi-directional evolutionary structural optimization (BESO) method is developed to solve the problem, which requires very few changes to the BESO computer code. The results reported in this work show that the proposed methodology suits engineering design and represents an improvement over existing topology optimization methods.

Keywords: topology optimization, directional uncertainty, interval, multiple load case, Bi-directional Evolutionary Structural Optimization (BESO).

INTRODUCTION

Structural topology optimization is a powerful tool for discovering new solutions to engineering design problems. Recent years have witnessed extensively investigations in this area, among which topology optimization of continuum structures (Bendsøe, 1988; Xie, 1993; Nishiwaki, 1998; Wang, 2003) is by far the most challenging technically and at the same time the most rewarding economically (Huang, 2010). Among various numerical methods for topology optimization including homogenization method, solid isotropic material with penalization (SIMP), evolutionary structural optimization (ESO) (Xie, 1993; Xie, 1994; Xie, 1997) and its improved algorithm – bi-directional evolutionary structural optimization (BESO) (Huang, 2007; Huang, 2009)and level set method (Wang, 2003), BESO has been extensively studied by many researchers around the world and applied to a wide range of structural as well as multidisciplinary design problems.

However, when uncertainties are present in the design parameters during the engineering design innovation process, such as applied loads, dimensions and material properties, uncertain topology optimization should be performed to determine the optimal topology. The final result obtained from solving an uncertain design problem varies with deflection values. Among the various uncertainties, uncertainty in applied loads is often a governing issue.
Formal incorporation of loading uncertainty into the design optimization framework has recently become a strong focus of the topology optimization community. Jung and Cho (Jung, 2004) developed a reliability-based topology optimization of geometrically nonlinear structures under consideration for loading uncertainty. A Bi-level program formulation for confidence structural robust optimization was explored for the design problem with load uncertainties by Guo et al. (Guo, 2009). Zhao and Wang (Zhao, 2014) reported efficient approaches to solving robust topology optimization problem of structures under loading uncertainty, including uncertainties in concentrate loading magnitude and direction and random field loading magnitude. Guest and Igusa (Guest, 2008) presented a method for solving structural topology optimization problems with uncertainty in the magnitude and location of loading. The structural topology optimization problem involved uncertainty of applied loads was investigated utilizing a level set based topology optimization method by Dunning and Kim (Dunning, 2011). In the above studies, probabilistic distributions are employed to represent the variations in the applied loads with the assumption that the distributions are precisely known. Nevertheless, the applied loads may not be precisely represented due to some factors in engineering practices, such as lack of sufficient data, data with fuzziness and unknown or non-constant reproduction conditions, where random variables are no longer proper to represent uncertainties. It is therefore advisable to use uncertain-but-bounded variables to describe the those uncertainties without considering their actual distributions. Inspired by the aforementioned idea, interval variables are employed to represent the uncertainties of the applied loads in this paper; in addition, compared with uncertainty in loading direction, uncertainty in loading magnitude may have smaller influence on the final topology for a geometrically linear structure (Huang, 2011; Zhao, 2014). Thus, this paper reports a simple method to handle structural design problem with loading directional uncertainty by utilizing interval variables. The directional uncertainties are firstly described by intervals without considering their distributions, which are divided into many small intervals, and then the uncertain small interval variables are approximated by their deterministic midpoints respectively. Finally, we transform the uncertain optimization topology problem into a deterministic multiple load case one, where load cases are derived to accurately compute the expected mean compliance with material volume constraint.

**BRIEF REVIEW OF SOFT-KILL BESO METHOD**

A soft-kill BESO method (Huang, 2009; Huang, 2010) is a deterministic topology optimization method, which is usually used to obtain an optimal topology with the maximum stiffness of a structure by simultaneously adding and removing elements per iteration. Unlike the conventional ESO/BESO, elements in the soft-kill BESO method are directly interpreted as solid/void with no intermediate values.

To obtain a solid-void design with maximum stiffness is equivalent to minimizing the strain energy, i.e., the mean compliance for a given volume of material. Hence, the topology problem can be formulated as the following:

\[
\begin{align*}
\text{Minimize} : & \quad C = \frac{1}{2} f^T u \\
\text{Subject to} : & \quad V^* - \sum_{e=1}^{N} V_e x_e = 0, \\
& \quad Ku = f \\
& \quad x_e = x_{\min} \text{ or } 1
\end{align*}
\]
where \( f \) and \( u \) are the applied load and displacement vectors, respectively. \( V' \) is the prescribed volume of the final structure and \( V_e \) is the volume of the \( e \)th element. \( N \) is the number of total elements of a structure. \( x_{\min} \) denotes a small value, and \( x_e \) is an indicating number, which determines the existence \( (x_e = 1) \) or void \( (x_e = x_{\min}) \) of each element.

In the soft-kill BESO method, material interpolation scheme is employed to steer the solution to nearly solid-void designs. To achieve that goal, Young’s modulus of the intermediate material is interpolated as a function of the individual element:

\[
E(x_e) = E_0 x_e^p
\]

where \( E_0 \) denotes Young’s modulus of the solid material, \( p \) denotes penalization exponent.

Then, the global stiffness matrix \( K \) can be expressed as follows:

\[
K = \sum_e x_e^p k_e^0
\]

where \( k_e^0 \) denotes the elemental stiffness matrix of the solid element.

In the soft-kill BESO method, we optimize a structure by the use of discrete design variables. That is to say that only two bound materials are found in the design. The analytical sensitivity number used in the soft-kill BESO methods can be defined by the relative ranking of the sensitivity of an individual element as

\[
\alpha_e = \Delta C_e = \begin{cases} 
\frac{1}{2} u_e^T k_e^0 u_e & \text{when } x_e = 1 \\
\frac{x_{\min}^{p-1}}{2} u_e^T k_e^0 u_e & \text{when } x_e = x_{\min}
\end{cases}
\]

where \( u_e \) and \( k_e \) denote the displacement vector and the stiffness matrix of the \( e \)th element, respectively.

In order to improve the convergence of the soft-kill BESO method, the sensitivity number with its historical information is averaged as the following:

\[
\alpha_e = \frac{\alpha_e^k + \alpha_e^{k-1}}{2}
\]

where \( k \) is the iteration number. In this way, the updated number includes the whole history of the sensitivity information in the previous iterations.

The soft-kill BESO usually starts from a full design and reduces the structural volume iteratively by switching element status. Before elements are removed from or added to the current design, the next iteration volume \( V_{k+1} \) has to be defined as

\[
V_{k+1} = V_k (1 \pm ER)
\]

where \( ER \) is the evolutionary volume ratio.

If the target volume \( V_{k+1} \) is larger than the current volume \( V_k \), addition elements are needed, and those elements are defined by sorting the sensitivity numbers. Thus, the number of addition elements is decided by the virtual target volume \( V_{k+1} \) as follows:

\[
V_{k+1} = V_k (1 + ER)
\]
In order to add elements which have smaller sensitivity numbers than those in the current elements, the threshold sensitivity number \( \alpha_{th} \) will be decided by the maximum sensitivity number of the removal elements. The addition elements are defined as the following condition:

\[
\alpha_e \leq \alpha_{th}
\]  

(8)

where \( \alpha_e \) denotes the sensitivity number of the removal element. Note that, if the number of the removal elements is larger than that calculated by the volume addition ratio \( AR \), the threshold sensitivity number \( \alpha_{th} \) will be revised.

However, when the target volume \( V_{k+1} \) is smaller than the current volume \( V_k \), removal elements are needed, and the number of removal elements will be decided by the virtual volume \( V_{k+1} \) as follows:

\[
V_{k+1} = V_k (1 - ER)
\]  

(9)

Void elements may have larger sensitivity numbers than those of removal elements. They will be added if

\[
\alpha_e > \alpha_{th}
\]  

(10)

where \( \alpha_e \) denotes the sensitivity number of the addition element. In that case that the number of the addition elements is larger than that calculated by the volume addition ratio \( AR \), the threshold sensitivity number \( \alpha_{th} \) will be revised.

The soft-kill BESO method is performed by adding and removing elements until the objective volume is reached and the following convergence criterion is satisfied:

\[
\text{error} = \frac{\sum_{e=1}^{N} (C_{k+1}^{e} - C_{k-N+1}^{e})}{\sum_{e=1}^{N} C_{k+1}^{e}} \leq \tau
\]  

(11)

where \( k \) is the current iteration number, \( \tau \) is a allowable convergence tolerance and \( N \) is an integer number. Normally, \( N \) is set to be 5 which implies that the change in the mean compliance over the last 10 iterations is acceptably small.

**UNCERTAINTY IN DIRECTION OF LOADING**

Notice that, compared with uncertainty in loading direction, uncertainty in loading magnitude may have smaller influence on the final topology for a geometrically linear structure as mentioned above. In this section, we derive a simple approach for calculating expected mean compliance in the presence of uncertainty in direction of loading.

The situation of loading directional uncertainty in two-dimensional space is studied in this paper, which can be easily extended to the design problem that subjected to loading with directional uncertainty in three-dimensional space. Without generality, we define directional uncertainty of loadings in \( x-y \) space in terms of their angles of application, \( \mathbf{\theta} = (\theta_1, \theta_2, \ldots, \theta_s, \ldots, \theta_s) \), where \( \theta_s \) is regarded as a interval vector without considering its distribution in the interval and \( s \) denotes the total number of external loading. In this paper, we define that \( \theta_s \) is positive when uncertain load goes counterclockwise around the deterministic load; otherwise, \( \theta_s \) is negative. The more detailed explanation of \( \theta_s \) can be seen
in Fig. 1, where one example is illustrated. In Fig. 1, \( f \) is in the range from \( f^L_i \) to \( f^R_i \), which described by the interval \( [\theta^L_i, \theta^R_i] \); as defined before, \( \theta^L_i \) is a negative number, while \( \theta^R_i \) is positive. Thus, we can also obtain the following equation, as

\[
\theta^i = [\theta^L_i, \theta^R_i]
\]

(12)

where \( \theta^L_i \) and \( \theta^R_i \) denote the lower bound and upper bound of the interval vector, respectively; the superscript \( i \) represents the interval number or vector.

Fig. 1 - Description of loading directional uncertainty in two-dimensional space.

The interval \( [\theta^L_i, \theta^R_i] \) which represents the loading directional uncertainty is first divided into \( t \) small intervals:

\[
\theta^i = [\theta^L_i, \theta^R_i] = (\theta^i_1, \theta^i_2, \ldots, \theta^i_j, \ldots, \theta^i_t), \quad \theta^j = [\theta^j_1, \theta^j_2], \quad j = 1, 2, \ldots, t
\]

(13)

Based on interval mathematics (Hu, 2010), an arbitrary interval deviation and midpoint can be respectively defined as

\[
\theta^j = \frac{\theta^R_j - \theta^L_j}{2}, \quad \theta^c_j = \frac{\theta^R_j + \theta^L_j}{2}
\]

(14)

Then, the \( j \)th small interval and its midpoint are respectively expressed as

\[
\theta^j = [\theta^j_1, \theta^j_2] = \left[ \theta^i + \frac{\theta^j}{t}, \theta^i + \frac{j \theta^j}{t} \right], \quad \theta^c_j = \theta^i + \frac{(2j - 1)\theta^j}{t}
\]

(15)

From equations (14) and (15), the following expression can be obtained approximately given that \( t \) is large enough as

\[
\theta^j \approx \theta^c_j \approx \theta^j
\]

(16)

The above equation implies that the uncertain \( \theta^j \) can be approximately substituted by its midpoint \( \theta^c_j \) with a large \( t \).

Now, the uncertain design problem subjected to the \( j \)th directionally uncertain loading can be approximated as a deterministic design one, where the loading is with a certain direction.
represented by $\theta_j^j$. In doing so, the uncertain topology optimization problem subjected to a loading with uncertain direction interval, namely $\theta_j^j$, can be transformed to a deterministic one. If the objective is to minimize the mean compliance with a volume constraint, the design problem subjected to loading $f_j$ with directional uncertainty $\theta_j^j$ can be formulated as

$$
\text{Minimize: } C_j = \frac{1}{2} f_j^T u_j \\
\text{Subject to: } V^* - \sum_{e=1}^{N} V_e x_e = 0, \\
Ku_j = f_j \\
x_e = x_{\text{min}} \text{ or } 1
$$

(17)

where $u_j$ denotes the displacement subjecting to the loading $f_j$.

In order to minimize the complementary work, we can define the elemental sensitivity number, which denotes the relative ranking of the elemental sensitivity, as

$$
\alpha_{je} = \Delta C_{je} = \begin{cases} \frac{1}{2} k_e^0 u_j^T k_e^0 u_{je} & \text{when } x_e = 1 \\ \frac{x_{\text{min}} - 1}{2} k_e^0 u_j^T k_e^0 u_{je} & \text{when } x_e = x_{\text{min}} \end{cases}
$$

(18)

Based on the above analysis, the design problem subjected to one loading $f$ with directional uncertainty $\theta^j$ as to minimize the mean compliance with a volume constraint can be approximated as multiple load cases, which can be stated as

$$
\text{Minimize: } C = \sum_{j=1}^{J} \omega_j C_j = \frac{1}{2} \sum_{j=1}^{J} \omega_j f_j^T u_j \\
\text{Subject to: } V^* - \sum_{e=1}^{N} V_e x_e = 0, \\
Ku_j = f_j \\
x_e = x_{\text{min}} \text{ or } 1
$$

(19)

where $\omega_j$ is the prescribed weighting factor for the $j$th loading with a direction represented by $\theta_j^j$, and $\sum_{j=1}^{J} \omega_j = 1$. Notice that this paper is concerned with the optimal design of structures with linear material and small deformation. Thus, Eq. (19) is reasonable. As the displacement field of one load case is independent of that of another load case, the sensitivity of the weighted objective function can be stated as

$$
\frac{dC}{dx_e} = -\frac{1}{2} p x_e^{p-1} \sum_{j=1}^{J} \omega_j \left( u_j^T k_e^0 u_{je} \right)_j
$$

(20)

Thus, the sensitivity number used in the soft-kill BESO method can be found as

$$
\alpha_e = -\frac{1}{p} \frac{dh}{dx_e} = \begin{cases} \frac{1}{2} \sum_{j=1}^{J} \omega_j \left( u_j^T k_e^0 u_{je} \right)_j & \text{when } x_e = 1 \\ \frac{x_{\text{min}} - 1}{2} \sum_{j=1}^{J} \omega_j \left( u_j^T k_e^0 u_{je} \right)_j & \text{when } x_e = x_{\text{min}} \end{cases}
$$

(21)
Note that, the above sensitivity is usually processed with the purpose to produce a mesh-independent solution, so a blurring filter (Zuo, 2015) is used in this paper.

**OPTIMIZATION PROCEDURE**

In the current BESO procedure, the design problem subjected to $s$ different loads with directional uncertainties will be approximately transformed to deterministic design topology optimization problem with $st$ load cases. Based on the optimization procedure of the BESO method, the iterative procedure for the proposed method with uncertainty in direction of loading is depicted in a flow chart as Fig. 2.

![Flow chart of the procedure of the proposed method](image_url)

Fig. 2 - Flow chart of the procedure of the proposed method
**NUMERICAL EXAMPLE**
A simple 2D example is investigated in order to illustrate the validity of the proposed method as well as the importance of considering load directional uncertainties in structural topology optimization problems.

In this example, we apply the proposed method to a simple column with design domain of 40 mm in length, 40 mm in height, and 1 mm in thickness shown in Fig. 3 (a), which is divided into 40 × 40 four node plane stress elements. The design domain is clamped on the bottom side. The soft-kill BESO parameters are \( ER = 2\% \), \( AR_{\text{max}} = 50 \% \), \( r_{\text{min}} \) = 3 mm, \( x_{\text{min}} \) = 0.001, \( p \) = 3.0, and \( \tau = 0.1 \% \); for the material properties, Young’s modulus of 1 GPa and Poisson’s ratio of 0.3 are assumed. The available material volume will cover 30 \% of the design domain. In other words, the volume fraction of final design will be 30 \%. A single -1 kN external load is applied to the top of the structure with the directional uncertain interval of \( \theta = \left[-\frac{\pi}{9}, \frac{\pi}{9}\right] \). And in the uncertain topology analysis, \( \theta \) is divided into 20 small intervals, which means that the value of \( t \) is 20.

The optimal material distributions obtained from deterministic optimization and the proposed method are compared in Fig. 3. It can be obviously seen that the final layout under directional loading uncertainty is apparently different from that of obtained by deterministic optimization; the deterministic solution is a straight column, nevertheless, the uncertain topology becomes two bars to support the directional uncertainty of the applied loads.

**CONCLUSIONS**
Based on interval mathematics, the uncertain topology optimization problem with loading directional uncertainty is approximated by a deterministic one. The optimization problem is established as minimizing the mean compliance under multiple load cases, subject to material volume constraint, which is solved by an improved BESO method. Numerical example shows that solution obtained for the uncertain problem is noticeably different from that produced for deterministic loading conditions. This study shows the importance of considering uncertainties in loading direction to design a structure.
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