A Framework for Actuator Placement in Large Scale Power Systems: Minimal Strong Structural Controllability

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Abstract—This paper addresses the problem of minimal placement of actuators in large-scale linear time invariant (LTI) systems, such as large-scale power systems, for dynamic controller design. A novel sufficient and necessary condition to ensure a strong structurally controllable (SSC) system is proposed. Specifically, the paper addresses the problem of obtaining the minimal number of dedicated inputs, i.e., inputs which actuate only a single state variable, and the respective state variables they should be assigned to, such that the LTI system is SSC. In addition, an efficient and scalable algorithm, with polynomial implementation complexity, to achieve such minimal placement of dedicated inputs is proposed. An illustration of the proposed design methodology is provided on the IEEE 5-bus test system, thereby identifying the minimal number of physical state variables to be actuated for ensuring strong structural controllability.

I. INTRODUCTION

The last decade has witnessed the introduction of novel generation technologies in large-scale power systems. Of particular interests are the unconventional renewable generation resources, such as wind and solar farms, given their high volatility and low degree of controllability. To compensate for their low (or lack of) controllability, a fundamental challenge for the power industry is to determine the optimal placement, size and type of controllers. For example, large-scale batteries, cluster of electric vehicles, an array of flywheels, pumped hydro storage. Primarily deployed in large-scale power systems for frequency regulation, these storage devices require high initial investment. The objective is to ensure real-time dynamic control, an improved modeling approach is essential to preserve the structural properties of the system. For example in [2], a structure preserving modeling approach is proposed where the aggregate loads are modeled as dynamic components. Nevertheless, the fundamental question regarding actuator placement in a large-scale power system remains unresolved, i.e., where the storage devices be placed to ensure system controllability. In general, finding such minimal placement to ensure controllability or to achieve pre-specified control performance is an NP-hard problem, see [3]. Alternative approaches that lead to efficient and scalable algorithms, i.e., with polynomial time complexity, have been proposed in [4]. The proposed approaches are based on structural systems reformulation (see [5]) and provide optimal placement of actuators to ensure structural controllability of the system. It involves analyzing only the sparsity (zero/non-zero pattern) of the dynamical interaction placement configurations and controllability is ensured in a structural sense. Specifically, such structural system theory based methods ensure structural controllability, i.e., provide

actuator placement configurations that ensures system controllability for almost all numerical realizations of the system parameters. This approach is especially suitable for power systems where the exact values of the system parameters are not available in general due to, either, numerical inaccuracy resulting from the linearization or the unknown numerical parameters of the system components. However, the claim that controllability is ensured for almost all realizations of the system parameters, does not rule out the possibility of those realizations for which the system is uncontrollable. Moreover, such uncontrollable realizations are likely to occur while modeling large-scale power systems. It is due to the similar modeling of the system components, such as generators and loads, as well as the coupling induced by energy conservation laws (the algebraic network constraints). Consequently a set of numerically similar parameters may be observed, thereby resulting in an uncontrollable realization, even though the system may be structurally controllable. This motivates the requirement for stronger notions of controllability, namely that of strong structural controllability [6] (see also [7] for survey), which seeks conditions under which all numerical realizations of the system are controllable, see also [8] for an application. Specifically, given the structural pattern of the dynamical system, in this paper, we are interested in obtaining minimal dedicated controller placement configurations that ensure such strong structural controllability. Note that the design of the dedicated controllers (storage devices) supplements the conventional governor control of the generators.

In this paper, we focus on a more fundamental problem: given a dynamic system’s structure, find the minimal subset of state variables that require dedicated inputs (i.e., inputs that are assigned only to a single state variable) to ensure a strong structurally controllable system. Formally, we consider the following problem.

Problem Statement

Let a linear time-invariant system be modeled as

$$\dot{x} = Ax,$$

where $x \in \mathbb{R}^n$ denotes the system state and $A$ the system matrix governing the autonomous dynamics. Let $A$ denote the structural matrix encoding the sparsity pattern of $A$, i.e., the entries of $A$ are either $0$'s or $1$'s according to whether the corresponding entries of $A$ are non-zeroes or zeroes. The objective is to design the input structural matrix $\hat{B} \in \{0, 1\}^{n \times p}$ with the minimal number of columns $p$ where each column of $\hat{B}$ has exactly one non-zero entry $\times$, such that the pair $(\hat{A}, \hat{B})$ is strongly structurally controllable.\(^1\)

\(^1\)Note that by restricting $\hat{B}$ to contain exactly one non-zero entry per column, we are considering the dedicated input design problem, i.e., in which each control input may only control a single state variable. Further, the design objective is to obtain the $\hat{B}$ with the minimal number of columns $p$, i.e., in other words, we are interested in the minimal placement of dedicated inputs such that the pair $(\hat{A}, \hat{B})$ is SSC.

\(^2\)We say that the structural pair $(\hat{A}, \hat{B})$, where $\hat{A} \in \{0, 1\}^{n \times n}$ and $\hat{B} \in \{0, 1\}^{n \times p}$, is strongly structurally controllable, if and only if the linear dynamical system

$$\dot{x} = Ax + Bu$$

is controllable for all numerical realizations $(A, B)$ with the same structural pattern as $(\hat{A}, \hat{B})$.

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Most recently, in [9], the problem of the minimum number of inputs selection (non necessarily dedicated) that ensure strong structural controllability has been shown to be NP-complete if an initial set of possible inputs is provided. This is so, because the problem reduces to the analysis of a constrained maximum matching, that incurs in the existence of an order between the edges in some maximum matching.

The main contribution of this paper is twofold: first, we derive a simplified necessary and sufficient condition for strong structural controllability. Second, based on this new characterization we propose an efficient algorithm (with cubic complexity in the number of state variables). The rest of this paper is organized as follows. In Section II we introduce standard terminology in structural systems theory, followed by the main results in Section III. Finally, we provide an illustration on the IEEE 5-bus test system, where we identify the minimal placement of dedicated inputs to achieve strong structural controllability and discuss the physical implications of such design.

II. Preliminaries and Terminology

In this section we introduce some terminology and recall some basic results from linear algebra. Given a matrix $M$ of dimension $m_1 \times m_2$, we refer to $m_1$ and $m_2$ as the height and length of the matrix, respectively. A permutation matrix is a square matrix that is obtained by permuting some rows/columns of the identity matrix. Recall that for a given matrix, multiplication on the left by a permutation matrix permutes its rows, whereas the multiplication on the right, permutes the columns. Also, recall that the rank and determinant of a matrix are invariant to row-column permutations; these properties will be used throughout the paper, often without explicit mention. A lower (upper) triangular matrix is a matrix that has only zeros above (below) the diagonal, whereas, a lower (upper) block-triangular matrix is one that has only zeros above (below) the block-diagonal. We now introduce the notion of stair matrix as follows, see [11].

**Definition 1 (Stair matrix):** Let $M$ be a $m_1 \times m_2$ matrix with entries that are either 0 (zero), $\times$ (non-zero) or $\otimes$ (an arbitrary value, including zero). The matrix $M$ is said to be a stair matrix if it is of the form

$$
\begin{pmatrix}
S_{01}^{1} & 0_{n_1 \times (m_2-n_1)} \\
S_{12}^{2} & 0_{n_2 \times (m_2-n_2)} \\
\vdots & \vdots \\
S_{k-1,k}^{k} & 0_{n_{k-1} \times (m_2-n_{k-1})}
\end{pmatrix}
$$

where 0 denotes the zero matrix of appropriate dimensions and each $S_{ij}^{i}$ matrix $S_{ij}^{i}$ denotes the $i$-th step $(i = 1, \ldots, k)$ such that $n_2 < n_{i+1}^{k}$ for $i = 1, \ldots, k-1$. In addition, when designating the submatrices $S_{ij}^{i}$ of a stair matrix $M$, we assume that $M$ is in the maximal stair form, i.e., there exist no permutation matrices $P_{1}$, $P_{2}$ such that $P_{1}MP_{2}$ has more steps or zero matrices with larger length in any given step than $M$.

Remark: That the steps in a stair matrix $M$ are ordered by length, from the smallest $S_{01}^{1}$ to the largest $S_{k}^{k}$. Now, given a stair matrix we introduce the notion of step difference.

**Definition 2 (Step difference):** Let $M$ be a stair matrix with $k$ steps. A step difference, denoted by $\Delta_{i+1}$, between two adjacent steps $S_{i}^{i}S_{i+1}^{i+1}$ for $i = 1, \ldots, k-1$, is the submatrix of $S_{i}^{i+1}$ comprising the same rows of $S_{i}^{i+1}$ and only the columns from $n_{i}^{i+1}$ to $n_{i+1}^{i+1}$, i.e.,

$$
\begin{pmatrix}
0_{n_1 \times (m_2-n_1)} \\
\Delta_{i+1} \\
0_{n_{i+1} \times (m_2-n_{i+1})}
\end{pmatrix}
$$

and by definition $\Delta^1 = S_{12}^{1}$.

Additionally, consider the following characterization of step differences.

**Definition 3 (Pivot in a step difference):** Given a stair matrix, a pivot is a non-zero entry in the left-top most entry of a step difference. A step difference $\Delta_{i+1}$ of a stair matrix $M$ has a pivot if there exist two permutation matrices $P_{1}$, $P_{2}$ such that $P_{1}MP_{2}$ has $\times$ as its left-top most entry, i.e., the entry in the first column and row of $\Delta_{P}$ is non-zero.

Moreover, since there exists at most one pivot in each step difference, we can order (and name) the pivots by the induced order of the steps. Specifically, we say that two pivots $k_1$, $k_2$ are consecutive if there exists no other pivot $k'$ such that $k_1 < k' < k_2$.

**Definition 4 (Normalized forms):** We say that a step difference is in its normal form if it has a non-zero in its left-top most entry. Moreover, we say that a stair matrix is in its normal form if all step differences with pivots are in their normalized form.

Finally, we introduce the notion of a ramp matrix.

**Definition 5 (Ramp matrix):** A ramp matrix $M \in \{0, \times, \otimes\}^{m_1 \times m_2}$ is a stair matrix with $m_1$ steps where each step difference has a pivot.

Remark 1: First note that, by definition, each step of a ramp matrix $M$ is a row vector. Definition 5 also implies that, for a ramp matrix $M$, there exists a lower-triangular sub-matrix with non-zero entries in its diagonal, given by the columns of $M$ that have the pivots of the normalized step differences.

In this section we state the main results of this paper. First, we introduce a (new) simplified necessary and sufficient condition for strong structural controllability, that, in particular, relies on the satisfiability of a single criterion, instead of two as in [7], [9]. Second, we present an algorithm to obtain an input matrix that corresponds to the minimum dedicated input assignment ensuring strong structural controllability. In addition, from the existence of possible pivots to the step differences, follows that several solutions to our problem are possible (see Section IV for further discussion).

**Theorem 1 (SSC Theorem):** The structural pair $(\bar{A}, \bar{B})$, is strongly structurally controllable if and only if for each $\lambda \in \mathbb{C}$ the matrix $[\bar{A} \lambda \bar{B}]$ with $\bar{A} = A - \lambda I$ and $I$ denoting the identity matrix of appropriate dimensions, can be transformed into a ramp matrix.

Theorem 1 motivates our design approach, Algorithm 1, where, given the system structure $\bar{A}$, the design of the optimal input structural matrix $\bar{B}$ is essentially achieved, by introducing the minimum number of columns (in $\bar{B}$) with only a single non-zero entry such that $[\bar{A} \lambda \bar{B}]$, for all $\lambda \in \mathbb{C}$, is transformable to a ramp matrix. The correctness and complexity of Algorithm 1 is analyzed as follows:
**Algorithm 1**: Compute a minimal placement of dedicated inputs that achieve strong structural controllability

**Input**: Dynamic matrix structure $\hat{A}$

**Output**: Input matrix $\hat{B}$ representing the minimal placement of dedicated inputs that achieve strong structural controllability

1) Find permutation matrices $P_1$ and $P_2$ such that $M = P_1 \hat{A}^2 P_2$ (for all $\lambda \in \mathbb{C}$) is a stair matrix with $k$ steps, in its normalized form. In addition, let $p_{1}, \ldots, p_{k'}$ correspond to the $k'$ pivots of the step differences, and the row and column entry of the $\alpha$-th pivot in $M$, be denoted by $p_{\alpha}^r$ and $p_{\alpha}^c$, respectively.

2) Let

$$\hat{B}^{P_1} = \hat{B} P_1,$$

where $\mathcal{J} = \{1, \ldots, n\} \setminus \bigcup_{\alpha=1}^{k'} p_{\alpha}^c$ corresponds to the set of indices of rows without pivot for the step differences, $\hat{I}_{n \times n}$ is the diagonal matrix with non-zero entries and $\hat{I}_{n \times n}$ is the matrix resulting from only keeping the columns of $\hat{I}_{n \times n}$ with non-zero entry in the rows indexed by $\mathcal{J}$.

3) Set $\hat{B} = P_1^{-1} \hat{B}^{P_1}$. 

**Theorem 2 (Correctness of Algorithm 1)**: Algorithm 1 is correct, i.e., the output of Algorithm 1 is a structural input matrix $\hat{B}$ corresponding to the minimum number of dedicated inputs such that the pair $(\hat{A}, \hat{B})$ is SSC.

**Theorem 3 (Complexity of Algorithm 1)**: Algorithm 1 has computational complexity $O(n^3)$, where $n$ denotes the dimension of the state-space, i.e., the number of state variables.

## IV. ILLUSTRATION: A 5-bus power system

The power system in Fig. 1 consists of five dynamical components interconnected through transmission lines. These are two coal-based generators $C_1$, $C_2$, one gas-based combustion turbine $G_3$ and the aggregate loads $D_4$, $D_5$.

![Fig. 1. IEEE 5-bus test system (1,2,3,4,5 represents the bus number)](image)

The generators $C_1$, $C_2$ and $G_3$ are modeled as linearized governor control, see Appendix-A of [12] for details. The aggregate loads $D_4$, $D_5$, modeled as dynamic components, are represented by swing equations [2], [13]. Subsequently, the generators and the loads are electrically coupled through the differentiated linearized real-power flow equations

$$\lambda \frac{d^2}{dt^2} \mathbf{x}(t) + \lambda \frac{d}{dt} \mathbf{x}(t) + \mathbf{Q}(\mathbf{x}, \mathbf{u}) = \mathbf{P}(\mathbf{x}, \mathbf{u})$$

with permutation matrices

$$P_1 = \left[ \begin{array}{cccc} \varepsilon_2 & e_1 & e_1 & e_1 \\ \varepsilon_2 & e_1 & e_1 & e_1 \\ \varepsilon_2 & e_1 & e_1 & e_1 \\ \varepsilon_2 & e_1 & e_1 & e_1 \\ \varepsilon_2 & e_1 & e_1 & e_1 \end{array} \right],$$

$$P_2 = \left[ \begin{array}{cccc} \varepsilon_3 & e_1 & e_1 & e_1 \\ \varepsilon_3 & e_1 & e_1 & e_1 \\ \varepsilon_3 & e_1 & e_1 & e_1 \\ \varepsilon_3 & e_1 & e_1 & e_1 \\ \varepsilon_3 & e_1 & e_1 & e_1 \end{array} \right],$$

written in terms of vectors $e_i \in \{0, \times, \otimes\}^{15}$ with non-zero $i$th entry and zero elsewhere.

The rectangles and squares in $M_{5\text{bus}}$ denote the step differences. Each has a pivot corresponding to a non-zero entry in the top-left most entry. For the blue rectangles, only one normalized step difference is possible. It must be noted that there exist no pivot in lines 10, 14, 16. This implies that we need to keep the corresponding columns in $P_1$ to obtain $\hat{B} = [e_{13} e_{15} e_{16}]$.

We now demonstrate that the minimal solution is not unique and specifically, alternative minimal placements may be obtained by considering different permutation matrices that transform $A_{5\text{bus}}$ to ramp forms. As an example, consider the
squared step difference highlighted with red in $M_{3bus}$. A new normalized stair matrix is obtained with the permutation of rows corresponding to red-marked rectangles. This results in an alternative input matrix that ensures strong structural controllability for the 5-bus power system. In more detail, considering

$$P_1' = [e_2 \ e_3 \ e_5 \ e_6 \ e_4 \ e_0 \ e_{12} \ e_{13} \ e_7 \ e_{10} \ e_{14} \ e_{15} \ e_{11} \ e_{16}],$$

where the bold face canonical vectors highlight $P_1'$'s dissimilarity to $P_1$, an alternative input matrix that ensures strong structural controllability is given by $B' = [e_{13} \ e_{14} \ e_{16}]$ associated with the permutation pair $(P_1', P_0)$. In the same fashion we could also obtain $B'' = [e_{12} \ e_{15} \ e_{16}]$ and $B''' = [e_{12} \ e_{14} \ e_{16}$ as other candidates for the minimal placement design.

Discussion of results: As per the four alternative input matrices achieving the minimal design, the state variables to be actuated are subsets of the nodal power injections/consumptions. The states $x_{16}$ always require a dedicated input, i.e., the power consumption by the aggregate load $D_5$. The second actuator can be placed at bus-4 ($x_{15}$), or, at bus-3 ($x_{13}$). The third actuator can be placed at bus-4 ($x_{15}$), or, at bus-3 ($x_{13}$). All possible input matrices are physically feasible. However, based on performance objective, for example cost of actuators, some of the input matrices to achieve the condition of strong structural controllability for the 5-bus power system may be more suitable.

V. CONCLUSIONS

In this paper we have provided a systematic method with polynomial implementation complexity, in terms of number of state variables, to obtain the minimal placement of dedicated inputs ensuring strong structural controllability of a given LTI system. We have shown that our method yields the globally optimal dedicated input placement. By duality, the results extend to the corresponding strong structural observability output design under similar constraints. Additionally, we have illustrated our design approach by providing the optimal placement of actuators in an IEEE 5-bus test system.

APPENDIX

Proof of Theorem 1: \(\implies\) Let $(A, B)$ be a pair (with complex entries) such that $(A, B)$ is a numerical realization of $(\bar{A}, \bar{B})$. Consider for each $\lambda \in \mathbb{C}$, the matrix $[A^\lambda \ B]$ which, by hypothesis, can be transformed into a ramp matrix. By Remark 1, it then follows that the resulting ramp matrix contains a lower triangular submatrix with non-zero diagonal entries, and, hence,

$$\text{rank}[A^\lambda \ B] = n, \quad \forall \lambda \in \mathbb{C}.$$ 

Thus, by the Popov-Belevitch-Hautus (PBH) test for controllability [14] it follows that $(A, B)$ is controllable; since, the above holds for all numerical realizations of $(\bar{A}, \bar{B})$ we conclude that $(\bar{A}, \bar{B})$ is SSC.

\(\implies\) On the contrary, suppose that the matrix $[A^\lambda \ B]$ cannot be transformed into a ramp matrix for all choices of $\lambda \in \mathbb{C}$. Hence, any matrix composed of $n$ columns of $[A^\lambda \ B]$ can be rearranged as a stair matrix where there is at least one step difference, with size at least $2 \times 2$. Such a stair matrix is clearly block lower triangular. Now considering the diagonal block which contains the step difference of size at least $2 \times 2$, it follows that there exists a numerical parametrization (realization) of the entries that makes the determinant of that block equal to zero. In other words, there exists a numerical realization $(\bar{A}, \bar{B})$ and $\lambda \in \mathbb{C}$ such that any matrix composed of $n$ columns of $[A^\lambda \ B]$ is rank deficient, which implies that $\text{rank}[A^\lambda \ B] < n$. Thus, by the PBH test for controllability [14] it follows that the realization $(A, B)$ is not controllable. This contradicts with the hypothesis that $(\bar{A}, \bar{B})$ is SSC and the assertion follows.

Proof of Theorem 2: Let $C_{p_i}$ denote the column in $M$ containing the $\alpha$-th pivot. First, remark that $B$ is a feasible solution, since the matrix $[\bar{A} - \lambda \bar{B}]$ can be transformed in a ramp matrix, where the matrix

$$[C_{p_i}^T \ J^\prime(p_i, p_j^0) \ C_{p_j} \ \cdots \ C_{p_{i+n-1}}^T \ J^\prime(p_{i+n-1}, p_{i+n}) \ C_{p_i}^T]$$

where $J^\prime(p_i, p_j^0) = \{i \in \mathbb{N} : p_i < j < p_i^0\}$, is a lower-triangular matrix. Second, the minimality is obtained by noticing that a step matrix leads to the maximum number of steps, and the subset of columns associated with the pivots corresponds to the maximum number of independent columns with respect to all possible parameterizations (which follows by similar reasoning used in the proof of Theorem 1). Finally, remark that there are as many columns in $B$ as necessary to complete the rank $n$. In addition, these columns are linearly independent since they correspond to a single non-zero entry (in different positions) columns.

Proof of Theorem 3: Step 1 can be implemented in $O(n^3)$, see for instance [15] for a discussion on obtaining two permutation matrices, such that: first, the maximum number of zeros are shifted to the top-right to ensure a lower-block triangular matrix structure, and second, to move the maximum number of zeros in each diagonal block to the top-right, which leads to a stair matrix. Step 2 can be implemented with linear complexity, as well as Step 3, since $P_{\mathcal{L}}^{-1} = P_1^T$.

REFERENCES


