An Economic Model Predictive Control scheme with terminal penalty for continuous-time systems

Andrea Alessandretti[1 2], A. Pedro Aguiar[2 3] and Colin N. Jones[1]

***Abstract*— In this paper, convergence and performance properties of a sample-data continuous time model predictive control (MPC) scheme with economic performance index are developed. In particular, we provide sufficient conditions for convergence of the close loop state trajectory to a steady state and constructive methods to design a terminal set and a terminal cost to satisfy them. Further, considering an average performance index, sufficient conditions under which the system in closed loop with the MPC controller outperforms the system operated at the economically optimal steady state are derived for the case of convergent and non convergent behaviors. Two numerical examples are presented to illustrate the different design techniques.**

I. INTRODUCTION

In a classic MPC scheme, the main goal is to drive the state of a given system to a desired steady state or state trajectory. To this end, a performance index is chosen that penalizes the distance from the current state to the desired one. Using this approach many MPC schemes have been proposed in the literature; the reader is referred to, e.g., [17], [16], [19] for an overview of methods that utilize the so-called terminal set and terminal weight, and [11] for the terminal-set-free case. Moreover, for the continuous time case we refer to, e.g, [14], [4], and [9]. In addition to the approaches mentioned above, in [1] an extra economic performance index, which is not a measure of the distance to the set-point, is used to influence the closed loop transient behavior.

In Economic MPC, the performance index captures an economic performance that we wish to optimize. Since such index can not be arbitrarily chosen, finding guarantees on the closed loop behavior associated to an Economic MPC schemes is generally a challenging task. In the works [5], [2], [3], the economically optimal steady state is precomputed and used to constrain the terminal state of the prediction with a terminal equality [5], [2], or inequality [3], constraint. Sufficient conditions for convergence to such optimal steady state are provided in [5] and later generalized by [2], [3] using a dissipativity property of the system. In [10] the terminal constraint is dropped and sufficient conditions for convergence to an arbitrarily small neighbourhood of the optimal steady state are derived. The case of changing performance index is addressed in [7]. In [6] a generalized terminal set, consisting of all the feasible steady states, is considered and the terminal cost is chosen to be the product of a constant term β and the stage cost, where the latter is enforced to be not increasing for consecutive solutions of the MPC optimization problem. Sufficient conditions for convergence to *a* steady state are provided together with an analysis of the effect of β on the asymptotic behaviour. For the same scheme, in [18] the authors exploit a time varying β to improve closed loop average performance. In [12] a given control Lyapunov function (CLF) defined over the desired region of attraction, is exploited to design a dual mode scheme where, in a first phase, the controller minimizes the economic cost enforcing the state within a level set of the CLF, and in a second phase, triggered at an arbitrarily given time, a Lyapunov decrease is enforced and convergence is achieved.

Although, these control methods are commonly applied to control continuous time systems, all the results are presented in the discrete time domain leaving it to accurate discretization procedures to close the gap between theory and practical applications. The only exception is the work in [12], although as main restriction, the method requires a given CLF defined over *the whole desired region of attraction*.

Motivated by these observations, this work proposes a continuous time economic MPC scheme where a generally local CLF is used to design a suitable terminal set and terminal cost (similarly to the discrete time version [3] but with different methods for terminal set/cost design).

The remainder of this paper is organized as follows: Section II contains the problem definition. The convergence and performance properties of the proposed scheme are discussed in Section III. Section IV presents constructive methods to design a terminal set and a terminal cost for convergence guarantee. These methods are illustrated with two numerical examples in Section V, followed by Section VI with some conclusions.

II. PROBLEM DEFINITION

Consider the continuous time dynamical system

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0 \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and the input vectors at time $t \geq 0$, respectively, and $x_0 = x(0)$ is the initial state. The system is subject to the following constraints

$$(x(t), u(t)) \in \mathcal{X} \times \mathcal{U} \subseteq \mathbb{R}^n \times \mathbb{R}^m, \qquad t \geq 0, \tag{2}$$

where the sets $\mathcal{X}$ and $\mathcal{U}$ denote the state and input constraint sets, respectively. Next, the open loop MPC optimization problem $\mathcal{P}(\cdot)$ is defined.

[1] École Polytechnique Fédérale de Lausanne (EPFL), Lausanne, Switzerland.
[2] Institute for Systems and Robotics (ISR), Instituto Superior Tecnico (IST), Lisbon, Portugal.
[3] Faculty of Engineering, University of Porto (FEUP), Porto, Portugal.

Definition 1 (Open loop MPC problem): Given $z \in \mathbb{R}^n$ and a horizon length $T \in \mathbb{R}_{>0}$, the open loop MPC optimization problem $\mathcal{P}(z)$ consists of finding the optimal control signal $\bar{u}^*([0,T])$ that solves

$$J_T^*(z) = \min_{\bar{u}([0,T])} J_T(z, \bar{u}([0,T])) \tag{3a}$$

$$\begin{aligned} \text{s.t.} \quad & \dot{\bar{x}}(\tau) = f(\bar{x}(\tau), \bar{u}(\tau)) && \forall \tau \in [0,T] \\ & (\bar{x}(\tau), \bar{u}(\tau)) \in \mathcal{X} \times \mathcal{U} && \forall \tau \in [0,T] \\ & \bar{x}(0) = z, \; \bar{x}(T) \in \mathcal{X}_{aux} \end{aligned}$$

with $J_T(z, \bar{u}([0,T])) := \int_0^T l(\bar{x}(\tau), \bar{u}(\tau))d\tau + m(\bar{x}(T))$. □
For a generic trajectory $y(\cdot)$, we denote by $y([t_1, t_2])$ the trajectory considered in the time interval $[t_1, t_2]$. Moreover, for a generic function, the dependence on the parameters is omitted whenever it is clear from the context.

The *finite horizon cost* $J_T(\cdot)$, which corresponds to the *performance index* of the MPC controller, is composed of the *stage cost* $l : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ and the *terminal cost* $m : \mathbb{R}^n \rightarrow \mathbb{R}$, which is defined over the *terminal set* $\mathcal{X}_{aux} \subseteq \mathbb{R}^n$. We denote by $k_{aux} : \mathcal{X}_{aux} \rightarrow \mathcal{U}$ a feasible *auxiliary control law* defined over the terminal set, i.e., $k_{aux}(x) \in \mathcal{U}$ for all $x \in \mathcal{X}_{aux}$. In a sample-data receding horizon strategy, the control input is computed at the discrete time instants $\mathcal{T} := \{t_0 = 0, t_1, \dots\}$, where $t_i > t_j$ for $i > j$, and the MPC control law is defined as

$$u(t) = k_{MPC}(x) := \bar{u}^*(t - \lfloor t \rfloor; x(\lfloor t \rfloor)), \tag{4}$$

where $\lfloor t \rfloor$ is the maximum sampling instant $t_i \in \mathcal{T}$ smaller than or equal to t, i.e., $\lfloor t \rfloor = \max_i \{t_i \in \mathcal{T} : t_i \leq t\}$. Next we introduce the concepts of feasible steady state set and economically optimal steady state set needed for the problem formulation.

Definition 2 (Feasible steady state set): Given the constrained system (1)-(2), the feasible steady state set $\mathcal{S}$ is defined as follows: $\mathcal{S} := \{(x,u) : f(x,u) = 0, \; (x,u) \in \mathcal{X} \times \mathcal{U}\} \subseteq \mathcal{X} \times \mathcal{U}$. □

Definition 3 (Economically optimal steady state set): Consider the constrained system (1)-(2), the feasible steady state set $\mathcal{S}$ of Definition 2, and an economic cost function $l(\cdot)$ of the state input pair. The economically optimal steady state set is defined as follows: $\mathcal{S}_e := \{(x,u) : l(x,u) = \min_{(x',u') \in \mathcal{S}} l(x',u')\}$. □
In order to asses the economic performance of the MPC scheme we introduce the following performance index.

Definition 4 (Average performance index): The average performance index associated to the closed loop system (1) with (4) is defined as follows: $l_{av}(x_0) := \limsup_{\delta \rightarrow +\infty} \frac{1}{\delta} \int_0^\delta l(x(\tau), u(\tau))d\tau$. □

Considering the constrained system (1)-(2) in closed loop with the MPC control law (4), the goals of this paper are to

- provide sufficient conditions for convergence of the closed loop state and input trajectories to a given steady state $(x_s, u_s) \in \mathcal{S} \supseteq \mathcal{S}_e$,
- investigate under which circumstances the closed loop system outperforms the system operated at a steady state, i.e., $l_{av}(x_0) \leq l(x_s, u_s)$,
- provide constructive methods to design a terminal set and terminal cost to guarantee convergence.

III. Convergence and Performance

This section is divided in two parts, which are dedicated to 1) the derivation of a set of sufficient conditions for closed loop convergence to a steady state $(x_s, u_s) \in \mathcal{S}$ and 2) average performance analysis of the closed-loop system.

The following classic assumptions from the MPC literature are used in both parts:

Assumption 1: The function $f(\cdot)$, of the system (1), is locally Lipschitz continuous in x and piecewise continuous in u in the region of interest. □

Assumption 2 (Initial feasibility): The optimization problem $\mathcal{P}(x_0)$ admits a feasible solution. □

A. Sufficient condition for convergence

In the following, for a generic function $g(\cdot)$, the terms $g_x(\cdot)$ and $g_{xx}(\cdot)$ denote the Jacobian and the Hessian, respectively, of $g(\cdot)$ with respect to the vector x.

Assumption 3 (Sufficient conditions for convergence): For a given feasible steady state $(x_s, u_s) \in \mathcal{S}$

(i) The state constraint set $\mathcal{X}$ and the terminal set $\mathcal{X}_{aux} \subseteq \mathcal{X}$ are closed, connected, and contain x_s, i.e., $x_s \in \mathcal{X}_{aux}$. Moreover, the input constraint set $\mathcal{U}$ is compact with $u_s \in \mathcal{U}$.

(ii) There exist a function $\lambda : \mathbb{R}^n \rightarrow \mathbb{R}$, which is continuously differentiable away from x_s, such that, for all $u \in \mathcal{U}$, the following holds

$$\dot{\lambda}(x) = \lambda_x(x) f(x,u) \leq s(x,u) - \alpha(\|x - x_s\|), \tag{5a}$$

$$s(x,u) = l(x,u) - l(x_s, u_s) \tag{5b}$$

for all x with $\lambda(x)$ differentiable at x and for some class-$\mathcal{K}_\infty$[1] function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$.

(iii) The function $m(\cdot)$ is continuously differentiable away from x_s.

(iv) There exists a feasible control law $k_{aux} : \mathbb{R}^n \rightarrow \mathbb{R}^m$, defined over the terminal set $\mathcal{X}_{aux} \subseteq \mathbb{R}^n$, such that, for the closed loop system (1) with $u(t) = k_{aux}(x)$ we have $x(t) \in \mathcal{X}_{aux} \subseteq \mathcal{X}$, $u(t) \in \mathcal{U}$ and

$$\dot{m} = m_x(x) f(x,u) \leq -l(x, k_{aux}(x)) + l(x_s, u_s) \tag{6}$$

for all the x with $m(\cdot)$ differentiable at x and initial conditions $x_0 \in \mathcal{X}_{aux}$. □

Similarly to [2], but for continuos time systems, the following theorem establish convergence of the closed loop state and input trajectories of (1)-(4) to the given steady state.

Theorem 1 (Convergence of Economic MPC): Consider the constrained system (1)-(2) in closed loop with (4) and suppose that Assumptions 1-3 hold. Then, the state vector $x(t)$ converges to x_s as $t \rightarrow \infty$ with region of attraction consisting of the set of states x for which $\mathcal{P}(x)$, introduced in Definition 1, admits a feasible solution. □

[1]A function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be belong to class $\mathcal{K}_\infty$, or to be a class-$\mathcal{K}_\infty$ function, if it is zero at zero, strictly increasing and radially unbounded, i.e., $\alpha(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Proof: Due to space constraints, only an outline of the proof is presented. Similarly to [5], [2], the dissipativity assumption (5) is used to define an auxiliary optimization problem $\mathcal{P}^a(\cdot)$ that shares the same optimizer of the economic optimization problem $\mathcal{P}(\cdot)$ and, in contrast to $\mathcal{P}(\cdot)$, fulfills the standard conditions required in classic MPC. Then, using standard argument from classic MPC on $\mathcal{P}^a(\cdot)$ (see, e.g., [4], [9], [1]) convergence is established. ■

Remark 1 (Classic MPC): The sufficient conditions for convergence of a classic MPC scheme can be recovered from Assumption 3 with $(x_s, u_s) = (0, 0)$, $l(0, 0) = 0$, and by choosing $\lambda(x) = c$, for any constant c. This results in $l(x, u) \geq \alpha(\|x\|)$ for all $(x, u) \in \mathcal{X} \times \mathcal{U}$, which fits the classic MPC framework where the stage cost penalizes the distance from the current state and input to the origin. □

B. Performance of the MPC controller

In this section the average performance of the proposed scheme is analyzed using the following main assumption:

Assumption 4: (Bounded performance index) For the closed loop (1) with (4), the performance index is uniformly bounded, i.e., $J_T^*(x(t)) < +\infty$, $\forall t > 0$. □

Assumption 4 can be satisfied either using an a priori bound on the state trajectory or by exploiting the closed loop properties of the convergent MPC scheme proposed in the previous section. These two scenario are represented by the following prepositions, respectively.

Proposition 1: (Bounded behaviour) Consider the closed loop (1)-(4) and let Assumptions 1-2 hold with $\mathcal{U}$ bounded. If $x(\cdot)$ is uniformly bounded, then Assumption 4 holds. □

Proof: From the boundedness of the set $\mathcal{U}$ and the state x, the functions $l(\cdot)$ and $m(\cdot)$ are bounded and, therefore, the optimal performance index $J_T^*(\cdot)$ is always bounded. ■

Proposition 2 (Convergent behaviour): Consider that Assumptions 1-3 hold, then for any $x_0 \in \mathbb{R}^n$ Assumption 4 holds. □

Proof: The proof follows from the invariance of any level set of the Lyapunov function used to proof convergence in Theorem 1 and is omitted due to space constraints. ■

in Next we show that Assumption 4 is satisfied in the case of convergent behavior, considered in Theorem 1, but also from an a priory knowledge of a bounded closed loop behaviour.

Theorem 2 (Performance of MPC scheme): Let Assumptions 1 and 4 hold, then for the closed loop system (1) with (4) the following is true: $l_{av}(x_0) \leq l(x_s, u_s)$ for all x_0 such that $\mathcal{P}(x_0)$ admits a feasible solution and for any steady state $(x_s, u_s) \in \mathcal{S}$ such that the inequality (6) holds. □

Proof: The proof can be obtained following the same steps in [3], [2], for the continuous time framework presented in this work and is omitted due to space constraints. ■

IV. Computation of the terminal set and terminal cost

In this section we present systematic procedures to compute a terminal cost and a terminal set that satisfy Assumption 3. First, an auxiliary control law that exponentially stabilizes the origin of the system and a polynomial bound of the stage cost are considered to be given. Then, if such elements are unknown, but the linearization of the system around the desired equilibrium point is stabilizable, two constructive methods are provided.

For the sake of clarity in this section we assume, without loss of generality, that $(x_s, u_s) = (0, 0)$.

A. Known auxiliary law

In the following, for a given function $g(\cdot)$ and scalar r we denote by $\mathcal{L}(g, r)$ the r-sublevel set of $g(\cdot)$ defined by $\mathcal{L}(g, r) := \{x : g(x) \leq r\}$.

Assumption 5 (Known auxiliary law): Suppose that a feasible control law $k_{aux} : \mathcal{X}_{aux} \to \mathcal{U}$, together with a certificate of *exponential* stability of the origin of the closed-loop system (1) with $u(t) = k_{aux}(x(t)) \in \mathcal{U}$ are given. Suppose also that certificate is a continuously differentiable Lyapunov function $V_{aux} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, with the positive constants k_1, k_2, k_3, and a such that

$$k_1\|x\|^a \leq V_{aux}(x) \leq k_2\|x\|^a \tag{7a}$$

$$\dot{V}_{aux}(x) = V_x(x) f(x, k_{aux}(x)) \leq -k_3\|x\|^a \tag{7b}$$

hold for all $x \in \mathcal{X}_{aux}$, for some set $\mathcal{X}_{aux} := \mathcal{L}(V_{aux}, r) \subseteq \mathcal{X}$ with $r \geq 0$. □

The use of a stabilizing terminal controller with the associated Lyapunov function is standard in MPC. The extra requirement introduced by Assumption 5 is the *exponential* stabilisability property implied by the specific bounds on the Lyapunov condition (7).

Assumption 6 (Bound on stage cost): The control law $k_{aux}(\cdot)$ from Assumption 5 and the stage cost $l(\cdot)$ satisfies

$$l(x, k_{aux}(x)) - l(0, 0) \leq \sum_{i=1}^{v} a_i \|x\|^i, \quad \forall x \in \mathcal{X}_{aux} \tag{8}$$

for some constants $v \in \mathbb{N}_{>0}$ and $a_i \in \mathbb{R}$, $i = 1, \ldots, v$. □

Assumption 6 captures a wide range of economic stage costs and auxiliary laws, e.g., any stage cost and auxiliary law with polynomial upper bound on x, tight at the origin.

In MPC, in order to achieve optimality of the MPC control law and closed loop stability, we would like to choose a terminal cost to be the so-called optimal "cost-to-go" so recovering the optimal infinite horizon control problem. Although, since this is in general not possible, this section, similarly to the quasi-infinite horizon approach [4], [8], [9] but for an economic cost, shows how to choose an upper bound of the optimal terminal cost that approximates the optimal "cost-to-go" and provides stability guarantee. Such bound is obtained by combining the exponential stabilization property of the auxiliary law from Assumption 5, together with the bound on stage cost of Assumption 6:

Proposition 3: Consider the system (1) in closed loop with the auxiliary law from Assumption 5 and let Assumption 6 hold. Then, the terminal cost function

$$m(x(t)) = \sum_{i=1}^{v} a_i \left(\frac{k_2}{k_1}\right)^{i/a} \frac{a k_2}{i k_3} \|x(t)\|^i \tag{9}$$

and the terminal set $\mathcal{X}_{aux}$, from Assumption 5, satisfy Assumption 3 (iii)-(iv). $\square$

Proof: Consider the closed-loop system (1) with $u(t) = k_{aux}(x(t))$ and $x_0 \in \mathcal{X}_{aux}$. The choosen terminal set is invariant being a sublevel set of a Lyapunov function, thus the state and the input trajectories are feasible from the inclusion $\mathcal{X}_{aux} \subseteq \mathcal{X}$ and by feasibility of auxiliary law $k_{aux}(\cdot)$, respectively. In order to define the terminal cost we first use (7) to bound the time evolution of $\|x(t)\|$ as (see. e.g. Theorem 4.10 [15]) $\|x(t)\| \le \|x(t_0)\| \left(\frac{k_2}{k_1}\right)^{1/a} e^{-k_3(t-t_0)/(ak_2)}$, which combined with the bound (8) of Assumption 6 results in $l(x(t), k_{aux}(x(t))) - l(0,0) \le \bar{l}(x(t))$ with $\bar{l}(x(t)) := \sum_{i=1}^{v} a_i \left(\frac{k_2}{k_1}\right)^{i/a} \|x(t_0)\|^i e^{-(k_3 i(t-t_0))/(ak_2)}$. Integrating the bound from t to $+\infty$ we obtain the proposed terminal cost

$$m(x(t)) = \int_t^{\infty} \bar{l}(x(\tau))d\tau = \sum_{i=1}^{v} a_i \left(\frac{k_2}{k_1}\right)^{i/a} \frac{ak_2}{ik_3}\|x(t)\|^i.$$

Note that the function $m(\cdot)$ satisfy Assumption 3 (iii) being differentiable with $x \neq 0$. Moreover computing the time derivative of $m(x(t)) = \int_t^{\infty} \bar{l}(x(\tau))d\tau$, nothing that $x(t) \to 0$ as $t \to +\infty$, we obtain $\dot{m}(x) = -\bar{l}(x(t)) \le -l(x(t), k_{aux}(x(t))) + l(0,0)$, which satisfies Assumption 3 (iv). ∎

B. Unknown auxiliary law and stabilizable linearization

Assumption 7 (Stabilizable linearization): The function $f(\cdot)$ is twice continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^m$, and the origin of the linearized system $\dot{x} = Ax + Bu$, with $A := f_x(0,0)$ and $B := f_u(0,0)$, is stabilizable. $\square$

In the following we consider the auxiliary law

$$k_{aux}(x) = Kx, \qquad A_{cl} := A + BK \tag{10}$$

where K is any matrix such that A_{cl} is Huwitz, which by Assumption 7, always exists.

For sake of clarity, we define the following operator:

Definition 5 (The operator $\mathcal{T}(\cdot)$): Let Assumption 7 hold and consider the constrained system (1)-(2) with $0 \in \mathcal{X}$ and $0 \in \mathcal{U}$, the auxiliary law (10), and a set $\mathcal{A} \subseteq \mathbb{R}^n$ with $0 \in \mathcal{A}$. For any positive definite matrix $0 \prec Q \in \mathbb{R}^{n\times n}$, vector $q \in \mathbb{R}^n$, and positive scalar $\beta \in (0, -\lambda_{max}(A_{cl}))^2$, the operator $\mathcal{T}(\mathcal{A}, Q, q, \beta)$ returns the matrix P that uniquely solves the Riccati equation

$$(A_{cl} + \beta I)'P + P(A_{cl} + \beta I) = Q, \tag{11}$$

the vector $p := -(A_{cl} + \beta I)'^{-1} q$, and the set $\mathcal{X}^* := \mathcal{L}(V, r)$ with $V(x) := x'Px$ and where r is the optimal solution of the optimization problem

$$r = \max_{\bar{r}} \bar{r} \text{ s.t.} \tag{12a}$$

$$(2x'P + p')e(x) \le 2\beta x'Px + \beta p'x, \ \forall x \in \mathcal{L}(V, \bar{r}) \tag{12b}$$

$$Kx \subseteq \mathcal{U}, \ \forall x \in \mathcal{L}(V, \bar{r}) \tag{12c}$$

$$\mathcal{L}(V, \bar{r}) \subseteq \mathcal{A} \tag{12d}$$

with $e(x) := f(x, Kx) - A_{cl}x$. We write $(P, p, \mathcal{X}^*) = \mathcal{T}(\mathcal{A}, Q, q, \beta)$. $\square$

Lemma 1 (Properties of $\mathcal{T}(\cdot)$): Let Assumption 7 hold and consider the constrained system (1)-(2), the auxiliary law (10), and a set $\mathcal{A} \subseteq \mathbb{R}^n$ with $0 \in \mathcal{A}$. For any positive definite matrix $0 \prec Q \in \mathbb{R}^{n\times n}$, vector $q \in \mathbb{R}^n$, and positive scalar $\beta \in (0, -\lambda_{max}(A_{cl}))$, let $W(x) := x'Px + p'x$ and $\mathcal{X}^*$ be such that $(P, p, \mathcal{X}^*) = \mathcal{T}(\mathcal{A}, Q, q, \beta)$. Then, the following is true:

(i) The set $\mathcal{X}^*$ is always non-empty and $\mathcal{X}^* \subseteq \mathcal{A}$.

(ii) For any $x \in \mathcal{X}^*$ we have $Kx \in \mathcal{U}$, and the following inequality holds: $\dot{W}(x) \le -x'Qx - x'q$.

If, in addition, $q = 0$, $0 \in \operatorname{int} \mathcal{A}$[3], and $0 \in \operatorname{int} \mathcal{U}$, then:

(iii) The set $\mathcal{X}^*$ is compact, positively invariant, and with non-empty interior. $\square$

Proof:

(i) Property (i) is satisfied from the fact that $r = 0$ is always a feasible solution of (12), which implies $\mathcal{X}^* = \{0\}$.

(ii) Computing the first derivative of $W(\cdot)$, in combination with (12b) and (11), we obtain

$$\begin{aligned}\dot{W} &= f(x, Kx)'Px + x'Pf(x, Kx) + p'f(x, Kx) \\ &= x'A_{cl}'Px + x'PA_{cl}x + p'A_{cl}x + 2e(x)'Px \\ &+ p'e(x) \le x'(A_{cl} + \beta I)'Px + x'P(A_{cl} + \beta I)x, \\ &+ p'(A_{cl} + \beta I)x, \ \forall x \in \mathcal{X}^* \\ &= -x'Qx - q'x, \ \forall x \in \mathcal{X}^* \end{aligned} \tag{13}$$

that, together with the constraints (12c) and (12d), satisfy the property (ii).

(iii) Note that if $q = 0$ then $p = 0$. Moreover, from $0 \in \operatorname{int} \mathcal{A}$, $0 \in \operatorname{int} \mathcal{U}$, and the continuity of Kx, there always exists an $\bar{r}$ small enough, such that (12d) and (12c) are satisfied. Moreover, from Theorem 4.7 of [15], if Assumption 7 holds, then for any $\gamma > 0$ there exists a $\delta_\gamma > 0$ such that $\|e(x)\| \le \gamma\|x\|$ for all $\|x\| < \delta_\gamma$. Thus, in order to satisfy (12b) it is enough to choose γ small enough such that the following inequality hold

$$x'Pe(x) \le \gamma\|x\|^2\|P\| \le \beta\lambda_{min}(P)\|x\|^2 \le \beta x'Px$$

for all $\|x\| < \delta_\gamma$, which concludes the proof.

∎

Note that, choosing $q = 0$, the operator $\mathcal{T}(\cdot)$ returns the standard ellipsoidal terminal set used in classic MPC, [4].

An important step in the design of the terminal cost is the derivation of an upper bound of the stage cost. We consider the following standard regularity assumption.

Assumption 8 (Regularity of the cost function): The stage cost function $l(\cdot)$ is twice continuous differentiable on $\mathbb{R}^n \times \mathbb{R}^m$.

Lemma 2 (Quadratic bound of the stage cost): Let Assumptions 7-8 hold and consider the auxiliary control law (10). Then, for any compact set $\mathcal{C}$ the optimization problem

$$\lambda^* = \min_{\lambda} \lambda \text{ s.t. } l(x, Kx) \le l_q(x), \ \forall x \in \mathcal{C}. \tag{14a}$$

$$l_q(x) := \lambda x'x + l_x(0,0)x + l(0,0) \tag{14b}$$

[2]For a given matrix A the terms $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote the minimum an maximum real valued eigenvalue of A

[3]For a generic set A we denote by $\operatorname{int} A$ the interior of A.

admits a feasible solution. □

Proof: The non-emptiness of the feasible set (14a)-(14b) follows from the feasibility, by using Assumption 8, of the bound computed in Lemma 22 and Lemma 23 of [2]. ■

Theorem 3: Let Assumptions 7-8 hold and consider the constrained system (1)-(2), with $0 \in \operatorname{int}\mathcal{X}$ and $0 \in \operatorname{int}\mathcal{U}$, and the auxiliary law (10). Then, Assumption 3 is satisfied choosing $\mathcal{X}_{aux}$ and $m(\cdot)$ as in the following algorithm :

1) Compute $(P_{lin1}, p_{lin1}\mathcal{X}_{lin}) = \mathcal{T}(\mathcal{X}, Q, 0, \beta_{lin})$ for any $\beta_{lin} \in (0, -\lambda_{max}(A_{cl}))$ and $Q \succ 0$.
2) Compute λ^* solving the optimization problem (14) with $\mathcal{C} = \mathcal{X}_{lin}$.
3) Compute the terminal cost $m(\cdot) = x'P_{aux}x + p'_{aux}x$ and the terminal set $\mathcal{X}_{aux}$ solving

$$(P_{aux}, p_{aux}, \mathcal{X}_{cost}) = \mathcal{T}(\mathcal{X}_{lin}, \lambda^* I, l_x(0,0)', \beta_{aux})$$
$$(P_{lin2}, p_{lin2}, \mathcal{X}_{aux}) = \mathcal{T}(\mathcal{X}_{cost}, Q, 0, \beta_{lin})$$

for some $\beta_{aux} \in (0, -\lambda_{max}(A_{cl}))$. □

Proof: From the property (iii) of Lemma 1 the set $\mathcal{X}_{lin}$ is compact, positively invariant, and by (12c) and (12d), the associated state and input trajectories are feasible. Solving the optimization problem (14) we compute the quadratic upper bound $\lambda^*\|x\|^2 + l_x(0,0) + l(0,0) \geq l(x, Kx)$ for all $x \in \mathcal{X}_{lin}$. From the property (ii) of Lemma 1 we have $\dot{m}(x) \leq -\lambda^*\|x\|^2 - x'l_x(0,0) \leq -l(x, Kx) + l(0,0)$, for all $x \in \mathcal{X}_{cost} \subseteq \mathcal{X}_{lin}$. Last, we solve $(P_{lin2}, p_{lin2}, \mathcal{X}_{aux}) = \mathcal{T}(\mathcal{X}_{cost}, Q, 0, \beta_{lin})$ to guarantee that the invariant set $\mathcal{X}_{aux}$ is contained in $\mathcal{X}_{cost}$. ■

In contrast to Theorem 3, the following result guarantees non-empty interior of the terminal set.

Theorem 4 (MPC with terminal inequality): Let Assumptions 7-8 hold and consider the constrained system (1)-(2), considered with $0 \in \operatorname{int}\mathcal{X}$ and $0 \in \operatorname{int}\mathcal{U}$, and the auxiliary law (10). Then, Assumption 3 is satisfied choosing $\mathcal{X}_{aux}$ and $m(\cdot)$ as in the following algorithm:

1) Compute the set $\mathcal{X}_{aux}$ as $(P_{lin}, p_{lin}, \mathcal{X}_{aux}) = \mathcal{T}(\mathcal{X}, Q, 0, \beta_{lin})$ for any $\beta_{lin} \in (0, -\lambda_{max}(A_{cl}))$ and $Q \succ 0$.
2) Compute λ^* solving the optimization problem (14) with $\mathcal{C} = \mathcal{X}_{aux}$.
3) Choose $m(\cdot)$ as in (9) with $k_1 = \lambda_{min}(P_{lin})$, $k_2 = \lambda_{max}(P_{lin})$, $k_3 = \lambda_{min}(Q)$, $a = 2$, $v = 2$, $a_1 = \|l_x(0,0)\|$, and $a_2 = \lambda^*$. □

Proof: Following the proof of Theorem 3 until step 3 we obtain a Lyapunov function $V_{lin}(x) := x'P_{lin}x$ for which the inequality $\dot{V}_{lin}(x) \leq -x'Qx \leq -\lambda_{min}(Q)\|x\|^2$ holds for all $x \in \mathcal{X}_{aux}$, and a quadratic upper bound on the stage cost. Thus, we can use Proposition 3 to compute a suitable terminal state and terminal cost. By the property (iii) of Lemma 1, the set $\mathcal{X}_{aux}$ as non-empty interior. ■

Remark 2: (Unknown dissipativity function) Since the function $\lambda(\cdot)$ is not used in the design algorithms, only its existence, and not its knowledge, is required. □

V. EXAMPLE

In the following examples MPC controllers are designed using a uniform time sampling $\mathcal{T} = \{0.05i,\ i \in \mathbb{N}_{\geq 0}\}$.

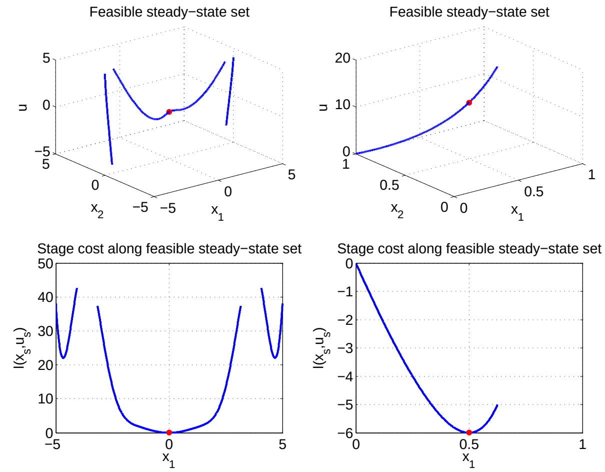

Fig. 1. The feasible steady state set is denoted with the continuous blue line in the state and input space (top) and with the associates stage cost (bottom), of the first (left) and second (right) examples. The red dot denotes the economically optimal steady state.

The MPC optimization problems are solved using ACADO Toolbox [13].

A. Example 1 : Known auxiliary law

The first example is an academic system with non-stabilizable linearization around the optimal steady state. Consider the constrained dynamical system (1)-(2) with

$$\dot{x} = f(x,u) = \begin{pmatrix} x_1 x_2 + u \\ -x_2 + \frac{1}{2}x_1\cos(x_1) \end{pmatrix}, \qquad (15)$$

$x := [x_1, x_2]'$, constrain sets $\mathcal{X} = \{x : \|x\|_\infty \leq 5\}$ and $\mathcal{U} = \{u : \|u\| \leq 5\}$, and the stage cost function

$$l(x,u) = x_1^2 x_2 + u x_1 - x_2^2 + \frac{1}{2}x_1 x_2 \cos(x_1) + \|x\|^2 + \|u\|^2. \qquad (16)$$

The steady state set $\mathcal{S} := \{(x,u) : x_2 = \frac{1}{2}x_1\cos(x_1), u = -\frac{1}{2}x_1^2\cos(x_1)\}$, computed setting (15) to zero, is displayed in Fig. 1 (left column) with the associated economically optimal steady state $(x_e, u_e) = (0,0)$. Inequality (5) holds choosing $\lambda(x) = \frac{1}{2}\|x\|^2$, in fact $\dot{\lambda}(x) = l(x,u) - \|x\|^2 - u^2 \leq l(x,u) - \|x\|^2$. For the closed loop system (15) with

$$u := k_{aux}(x) = -x_1 - x_1 x_2, \qquad (17)$$

i.e., $\dot{x} = \begin{pmatrix} -x_1 \\ -x_2 + \frac{1}{2}x_1\cos(x_1) \end{pmatrix}$, the function $V(x) = \frac{1}{2}\|x\|^2$ is a Lyapunov function that certifies the exponential stability of the origin. In fact, computing its time derivative we obtain $\dot{V} = \dot{x}_1 x_1 + \dot{x}_2 x_2 = -x_1^2 - x_2^2 + \frac{1}{2}x_1 x_2\cos(x_1) \leq -x_1^2 - x_2^2 + \frac{1}{2}\|x_1\|\|x_2\| \leq -x_1^2 - x_2^2 + \frac{1}{4}(x_1^2 + x_2^2) = -\frac{3}{4}\|x\|^2$, where the last inequality follows from $\|x_1\|\|x_2\| \leq \frac{1}{2}(x_1^2 + x_2^2)$. Considering the cost (16), the control law (17), and the Lyapunov function $V(\cdot)$, Assumption 5 is satisfied with $a = 2$, $k_1 = k_2 = \frac{1}{2}$,

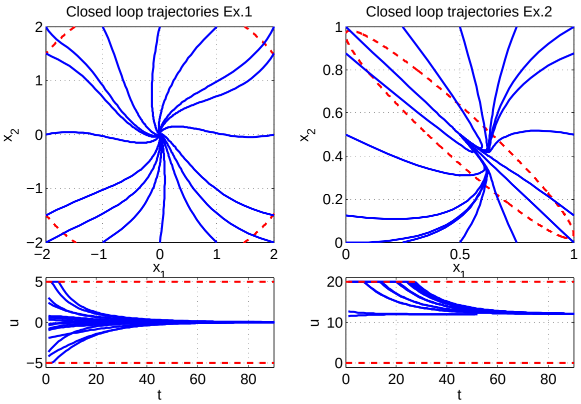

Fig. 2. The state (top) and input (bottom) trajectory for the system closed loop for example 1 (left) and 2 (right) with different initial conditions are denoted with the continuous blue line. The dotted red line represents the auxiliary constraint set (top) and the input constraints (bottom).

$k_3 = \frac{3}{4}$, and $\mathcal{X}_{aux} = \mathcal{L}(x'Px, r)$ where a suitable constant r can be obtained solving the optimization problem

$$r = \max_{\bar{r}} \bar{r} \text{ s.t. } \; k_{aux}(x) \subseteq \mathcal{U}, \forall x \in \mathcal{L}(x'Px, \bar{r}) \subseteq \mathcal{X}$$

Moreover, combining the stage cost (16) and the auxiliary law (17) we obtain $l(x, k_{aux}(x)) = \frac{1}{2}x_1 x_2^2 \cos(x_1) + x_1^2 + x_1^2 + x_1^2 x_2^2 + 2x_1^2 x_2 \leq \frac{3}{2}\|x\|^2 + 2\|x\|^3 + \|x\|^4$ which satisfies Assumption 6 with $v = 4$, $a_2 = \frac{3}{4}$, $a_1 = 0$, $a_3 = 2$, $a_4 = 1$. The associated convergent closed loop state and input trajectory are displayed in Fig. 2 (left).

Remark 3 (Global convergence): Note that, for the unconstrained case, the resulting economic MPC controller has a *global* region of attraction. □

B. Example 2 : Isothermal CSTR

The model in this example is taken from [5], where a chemical reaction in isothermal CSTR is modelled by the following two state constrained dynamic system

$$\dot{x} = \begin{pmatrix} 0.1u(1 - x_1) - 1.2x_1 \\ -0.1u\, x_2 + 1.2x_1 \end{pmatrix}, \tag{18}$$

where the state vector $x := [x_1, x_2]'$, with $x_i \in [0, 1]$, $i = 1, 2$, is a vector of molar concentrations of the materials of the reaction and $u \in [0, 20]$ is the flow through the reactor that we are allowed to control. The economic cost, associated to the production and separation costs, is defined as $l(x, u) = -(2ux_2 - 0.5u)$.

The steady state set, from Definition 2, is computed by setting the derivative of the state (18) to zero, resulting in $\mathcal{S} := \left\{(x, u) : \; x_2 = 1 - x_1, \; u = \frac{12x_1}{(1-x_1)}\right\}$, displayed in Fig. 1 (right column) with the associated economically optimal steady state $(x_e, u_e) = ([0.5, 0.5]', 12)$. Regularization terms, taken from [5], are introduced to make the system dissipative with $\lambda(x) = [10, 20]'x$, resulting in

$$l(x, u) = -(2ux_2 - 0.5u) + 0.505\|x - x_s\|^2 + 0.505\|u - u_s\|^2.$$

The associated convergent closed loop state and input trajectories are displayed in Fig. 2 (right) where the terminal set and terminal cost are designed using the algorithm of Theorem 4.

VI. CONCLUSION

An MPC scheme with terminal penalty for economic optimization is presented together with the derivation of a continuous time sufficient condition for convergence to a steady state. Under an assumption on the boundedness of the economic performance index, the closed loop system is proved to have an average performance index not worse that the one obtained operating the system at a steady state. Moreover, different approaches for a systematic design of a suitable terminal set and terminal cost to guarantee convergence to a desired steady state are presented and illustrated via numerical examples.

REFERENCES

[1] A. Alessandretti, P. Aguiar, and C. N. Jones. A Model Predictive Control Scheme with Additional Performance Index for Transient Behavior. In *52nd IEEE Conference on Decision and Control December 10-13*, pages 5090–5095, Florence, Italy, 2013.
[2] R. Amrit, J. B. Rawlings, and D. Angeli. Economic optimization using model predictive control with a terminal cost. *Annual Reviews in Control*, 35(2):178–186, Dec. 2011.
[3] D. Angeli, R. Amrit, and J. B. Rawlings. On Average Performance and Stability of Economic Model Predictive Control. *IEEE Transactions on Automatic Control*, 57(7):1615–1626, July 2012.
[4] H. Chen and F. Allgöwer. A Quasi-Infinite Horizon Nonlinear Model Predictive Control Scheme with Guaranteed Stability. *Automatica*, 34(10):1205–1217, Oct. 1998.
[5] M. Diehl, R. Amrit, and J. B. Rawlings. A Lyapunov Function for Economic Optimizing Model Predictive Control. *IEEE Transactions on Automatic Control*, 56(3):703–707, Mar. 2011.
[6] L. Fagiano and A. R. Teel. Generalized terminal state constraint for model predictive control. *Automatica*, 49(9):2622–2631, Sept. 2013.
[7] A. Ferramosca, J. Rawlings, D. Limon, and E. Camacho. Economic MPC for a changing economic criterion. In *49th IEEE Conference on Decision and Control (CDC)*, pages 6131–6136. IEEE, Dec. 2010.
[8] R. Findeisen and F. Allgöwer. The quasi-infinite horizon approach to nonlinear model predictive control. *Nonlinear and Adaptive Control*, 2003.
[9] F. A. Fontes. A general framework to design stabilizing nonlinear model predictive controllers. *Systems & Control Letters*, 42(2):127–143, Feb. 2001.
[10] L. Grüne. Economic receding horizon control without terminal constraints. *Automatica*, 49(3):725–734, Mar. 2013.
[11] L. Grune and J. Pannek. Nonlinear model predictive control: theory and algorithms. 2011.
[12] M. Heidarinejad, J. Liu, and P. D. Christofides. Economic model predictive control of nonlinear process systems using Lyapunov techniques. *AIChE Journal*, 58(3):855–870, Mar. 2012.
[13] B. Houska, H. J. Ferreau, and M. Diehl. ACADO toolkit-An open-source framework for automatic control and dynamic optimization. *Optimal Control Applications and Methods*, 32(3):298–312, 2011.
[14] A. Jadbabaie, J. Yu, and J. Hauser. Unconstrained receding-horizon control of nonlinear systems. *IEEE Transactions on Automatic Control*, 46(5):776–783, May 2001.
[15] H. K. Khalil. *Nonlinear systems (3rd Edition)*. Prentice Hall, 2002.
[16] D. Mayne, J. Rawlings, C. Rao, and P. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, June 2000.
[17] M. Morari and J. H. Lee. Model predictive control: Past, present and future. *Computers & Chemical Engineering*, 23(4):667–682, 1999.
[18] M. A. Müller, D. Angeli, and F. Allgöwer. Economic model predictive control with self-tuning terminal cost. *European Journal of Control*, 19(5):408–416, Sept. 2013.
[19] J. B. Rawlings and D. Q. Mayne. *Model Predictive Control Theory and Design*. Nob Hill Pub., 2009.