Moving Horizon Estimation with Decimated Observations

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Abstract: This paper addresses the problem of moving horizon (MH) state estimation of discrete lumped nonlinear systems. It is assumed that the measurements of the observed variables are not available at every sampling instant (decimated observations). An estimation algorithm is provided for that purpose, together with results on its convergence. It is shown that, under convenient assumptions, the estimation error is bounded, with a bound that grows with the number of samples between consecutive observations. The algorithm features are illustrated by simulations concerning the application to state estimation in a model of the HIV-1 infection. The simulations show that the MH estimator exhibits superior performance over the extended Kalman filter. This difference of performance increases with the growth of the time interval between consecutive measurements.

Keywords: Moving Horizon, Estimation, Nonlinear, HIV-1 infection, stability, decimated observations.

1. INTRODUCTION

Although the original idea is old (Y.A. Thomas (1975)), moving horizon (MH) estimation has received an increased attention since the past 15 years (Robertson et al. (1996)) up to the present (Haverbeke et al. (2009)) (see also the references in Alessandri et al. (2008)). Besides its intrinsic robustness properties, that makes MH adequate to solve estimation problems in the presence of un-modeled dynamics, a major advantage is the capacity to incorporate constraints.

In addition to the above features, the solution of estimation problems in bio-medical applications require the ability to tackle estimation problems when the measurements of the observed variables are not available at every sampling instant. This is a situation referred hereafter as “decimated observations” which is not treated in the available literature. An example is provided by the control of HIV-1 infection (Perelson and Nelson (1999)).

The main contribution of the present work consists in an MH estimation algorithm for nonlinear discrete systems that copes with decimated observations. It is shown that, under convenient assumptions, the estimation error is bounded by a bound that grows with the number of samples between consecutive observations. The algorithm features are illustrated by simulations concerning the application to state estimation in a nonlinear model of the HIV-1 infection. The simulations show that the MH estimator exhibits superior performance over the extended Kalman filter. This difference of performance increases with the growth of the time interval between consecutive measurements.

This paper is organized as follows: Section 2 formulates the state estimation problem and proposes a MH estimator to deal with decimated observations. Section 3 presents stability properties. Section 4 provides an interpretation for the selection of the cost function and Section 5 includes simulation results that compare the performance with the extended Kalman filter (EKF) estimator in relation to a model of the HIV-1 infection. Finally, section 6 draws conclusions.

Due to space limitations some of the proofs are omitted. These can be found in Barreiro et al. (2010).

2. MH ESTIMATION WITH DECIMATED OBSERVATIONS

This section formulates the state estimation problem and introduces a MH estimator for a discrete (or sampled data representation) of a nonlinear system whose measurements are not available at every sampling instant.

2.1 Process Model

Let \( M \) be the set of time instants (indexes) where measurements are available, and \( \sigma_k(i) : \mathbb{N} \rightarrow M \) the index time \( i \) of the \( k \)th measurement. We consider a dynamic system described by the discrete time equations

\[
\begin{align*}
x_{i+1} &= \phi_i(x_i, u_i, \omega_i) \\
y_k &= h_k(x_{\sigma_k}) + v_k
\end{align*}
\]

where \( x_i \in \mathbb{R} \) is the state vector of the system, \( u_i \) its control, \( y_k \) denote the measurements, and \( \omega_i \in \Theta_i \) and \( v_k \) represent input disturbances and measurement
noise, respectively. The sets $X_k$, $\Theta_k$ and $\mathcal{V}_k$ are subsets (with appropriate dimension) of the Euclidean space that incorporate the constraints associated to (1). The initial condition $x_0$ and the signals $\omega_k$, $v_k$ are assumed to be unknown. The function $\sigma_k$ represents a renumbering of the time index. It stands for the fact that observations are not available at every time instant, but only in a subset of an observation. A frequent case is when an observation is only available every $n_s$ samples. Then, $\sigma_k := n_s k$.

### 2.2 Decimated Observations

Before we describe the MH estimator algorithm, we first introduce two operators that allow us to work with a more convenient representation of (1).

The first operator, denoted by $\Sigma$, will permit to write in an appropriate way the recursive composition of a function and is defined as

\[
\Sigma \{ \phi \}_{a}^{b-1}, z, \{ \omega \}_{a}^{b-1}, a, b \ :=
\begin{cases}
\phi_a & \text{if } a = b \\
\phi_{b-1} \left( \Sigma \{ \phi \}_{a}^{b-2}, z, \{ \omega \}_{a}^{b-2}, a, b-1 \right), & \text{otherwise}
\end{cases}
\]

where $\{ \phi \}_{a}^{b}$ denotes a sequence of functions $\{ \phi_a(\cdot), \phi_{a+1}(\cdot), \ldots, \phi_b(\cdot) \}$, $z$ is the input and $\{ \omega \}_{a}^{b}$ a sequence of input disturbances. Note that the state solution of system (1) at time $i + 1$ with initial condition $x_0$ can be written as $x_{i+1} = \Sigma[\{ \phi \}_{0}^{0}, x_0, \{ \omega \}_{0}^{0}, 0, i+1]$.

The second operator, denoted by $\chi$, is defined as the difference between the evolution of state $x$ with input disturbance and the state $x$ with zero input disturbances, that is,

\[
\chi \left( \{ \phi \}_{a}^{b-1}, z, \{ w \}_{a}^{b-1}, a, b \right) := \Sigma \left[ \{ \phi \}_{a}^{b-1}, z, \{ w \}_{a}^{b-1}, a, b \right] - \Sigma \left[ \{ \phi \}_{a}^{b-1}, z, \{ 0 \}_{a}^{b-1}, a, b \right].
\]

For simplicity of notation the following abbreviations are used

\[
\begin{align*}
\Sigma[\phi, \omega, a, b] & := \Sigma \left[ \{ \phi \}_{a}^{b-1}, z, \{ \omega \}_{a}^{b-1}, a, b \right] \\
\chi[\phi, \omega, a, b] & := \chi \left( \{ \phi \}_{a}^{b-1}, z, \{ \omega \}_{a}^{b-1}, a, b \right).
\end{align*}
\]

We are now ready to introduce another representation of system (1) that will play an important role on the developments that follow.

Consider the system

\[\begin{align*}
x_{k+1} & = f_k(x_k) + w_k & (2a) \\
y_k & = h_k(x_k) + v_k & (2b)
\end{align*}\]

where $f_k(x) = \Sigma[\phi, x, 0, \sigma_k, \sigma_{k+1}]$ and $w_k = \chi[\phi, x, \sigma_k, \omega, \sigma_k, \sigma_{k+1}]$. Since $f_k(x_k) + v_k$ is equal to $\Sigma[\phi, x, \sigma_k, \omega, \sigma_k, \sigma_{k+1}]$, it is straightforward to conclude that $x_k$ in (2) is equal to $x_{k+1}$ in (1). Thus, system (2) describes how the state in (1) is transferred from a point where a measurement is available to the next one (where a measurement occurs again). Note however that in (2) $w_k$ might depend on $x_k$.

Hereafter we use the index $k$ for solutions of system (2) and $i$ for solutions of system (1).

To obtain $x_i$ from (2), we have to perform the following computation

\[
k = \max \{ k : \sigma_k \leq i \}
\]

\[
x_i = \Sigma[\phi, x_k, \omega, \sigma_k, i].
\]

### 2.3 Moving Horizon Estimator

Using the notation in Rao et al. (2003), we denote by $x(k; z, l, \{ w_j \})$ the solution of system (2) at time $k$ when the initial state is $z$ at time $l$ and the disturbance sequence is $\{ w_j \}_{j=l}^{k}$. When $w_j = 0$ we will write $x(k; z, l)$. Also $y(k; z, l, \{ w_j \}) := h_k(y(k; z, l, \{ w_j \}))$ and $y(k; z, l) := h_k(x(k; z, l))$.

The objective is to find the state sequence $\{ \hat{x}_j \}$ that is most likely to be in some sense close to the real state $\{ x_j \}$, given the sequence of observations $\{ y_{\sigma_j} \}$, the inputs $\{ u_i \}$ and the model with constraints described in (1).

To this effect, we consider the following objective function defined in the equivalent system (2),

\[
\Phi_T(x_0, \{ w_k \}) := \sum_{k=0}^{T-1} L_k(w_k, v_k) + \Gamma(x_0),
\]

where $T > 0$ is the estimation horizon, $v_k = y_k - y(k; x_0, 0, \{ w_j \})$, $L_k : \mathcal{W}_k \times \mathcal{V}_k \to \mathbb{R}_{\geq 0}$ is the running cost and $\Gamma : \mathcal{X}_0 \times \mathbb{R}_{\geq 0}$ represents a penalty on the initial condition. It is assumed that some prior information of the initial state is known, and this one is captured by $\Gamma(\cdot)$, that satisfies the following property

\[
\Gamma(\hat{x}_0) = 0, \quad \Gamma(x) > 0 \quad \forall x \neq \hat{x}_0
\]

where $\hat{x}_0 \in \mathcal{X}_0$ is the (a priori) most likely value of $x_0$.

In Section 4, we provide a better insight of how to choose $L_k(\cdot)$ and $\Gamma(\cdot)$, but the main idea is that $\Phi_T$ will penalize large $w_k$ and $v_k$, and $\hat{x}_0$ far from an initial guess $\hat{x}_0$.

The optimization problem can now be stated as follows: Find the pair $(\hat{x}_0, \{ \hat{w}_k \}_{k=0}^{T-1})$ that minimizes $\Phi_T(x_0, \{ w_k \})$ subjected to $(x_0, \{ w_k \}) \in \mathcal{X}_T$. The constraint set $\mathcal{X}_T$ is given by

\[
\mathcal{X}_T := \left\{ \begin{array}{l}
x(k; x_0, 0, \{ w_j \}) \in \mathcal{X}_k, \quad k = 0, \ldots, T \\
w_k \in \mathcal{W}_k, \quad v_k = y_k - y(k; x_0, 0, \{ w_j \}) \in \mathcal{V}_k, \quad k = 0, \ldots, (T-1)
\end{array} \right\}
\]

and it arises from the restrictions $\mathcal{X}_k$, $\mathcal{W}_k$ and $\mathcal{V}_k$, where $\mathcal{W}_k$ inherits restrictions from $\Theta_k$. The computation of the set $\mathcal{W}_k$ can be involved. However, for the particular case that $\Theta_k$ is a bounded set, Lemma 3 provides a bound for $\mathcal{W}_k$, although a smaller one might exist.

In general this optimization cannot be applied online because the computational complexity grows unbounded with increasing horizon $T$. To account for this problem and enforce a fixed dimension optimal control problem, a possible strategy is to explore the ideas of dynamical programming by breaking the summation in $\Phi_T$ as follows

\[
\Phi_T(x_0, \{ w_k \}) := \sum_{k=T-N}^{T-1} L_k(w_k, v_k) + \sum_{k=0}^{T-N-1} L_k(w_k, v_k) + \Gamma(x_0)
\]

Since the first term of the right hand side depends only on the state $x_{T-N}$ and on the sequences $\{ w_k \}_{k=T-N}^{T-1}$ and $\{ v_k \}_{k=T-N}^{T-1}$, the optimization problem can be reformulated as
Here such that for any two states $\alpha$ \( \in \alpha \) the inverse of $\alpha$ \( \alpha \) will be used in the sequel. We will freely use basic results on noise. To this effect, we first recall the following definitions that will be used in (5) converges to zero in absence of noise. To obtain the estimate of $Z$ \( Z \) by an approximation of it $\hat{Z}(z)$ for all $z$ \( z \). The above definition takes into consideration the fact that system (1) may not have measurements at each sampling instant of time. From this definition, we can conclude that if system (1) is uniformly observable then system (2) is also uniformly observable, meaning that

$$\varphi(||x_1-x_2||) \leq \sum_{j=0}^{N-1} ||y(k+j;x_1,k) - y(k+j;x_2,k)||$$

The stability results presented in this section make use of the following assumptions:

**A0** The vector fields $\phi_i(\cdot)$ and $h_k$ satisfy the following growth conditions:

a) $||\phi_i(z_1,u_k,\omega_1) - \phi_i(z_2,u_k,\omega_2)|| \leq c_{\|}(||z_1,\omega_1|| - (z_2,\omega_2)||$

b) $||h_k(z_1) - h_k(z_2)|| \leq c_{\|}||z_1 - z_2||$

**A1** $L_k(\cdot)$ and $\Gamma(\cdot)$ are left continuous in their arguments for all $k \geq 0$.

**A2** There exist $K_\infty$-functions $\eta(\cdot)$ and $\gamma(\cdot)$ such that $\eta(||(w,v)||) \leq L_k(w,v) \leq \gamma(||(w,v)||)$ for all $(w,v) \in (W_k \times V_k)$, $x, x_0 \in X_k$ and $k \geq 0$.

**A3** There exists an initial condition $x_0$, disturbance sequence $\{w_k\}_{k=0}^\infty$ such that, for all $k \geq 0$, $(x_0, \{w_k\} (k \geq 0) \in \Omega_k$.

**A4** The interval of time between two consecutive measurements is finite, i.e. $\sigma_k - \sigma_{k-1} < n_{max}$ for some $n_{max}$.

**A5** There exist $K_\infty$-function $\hat{\gamma}(\cdot)$ such that $0 \leq \hat{Z}_k(z) - \Phi_k \leq \hat{\gamma}(||z - \hat{x}_k||)$ for all $z \in X_k$.

**A6** Let $R_\tau^N = \{x(\tau; z, \tau - N, \{w_k\}) : (z, \{w_k\}) \in \Omega_\tau^N \}$ where $R_\tau^N = R_\tau$ for $\tau \leq N$. For a horizon length $N$, any time $\tau > N$, and any $p \in R_\tau^N$, the approximate arrival cost $\hat{Z}(\cdot)$ satisfies the inequality

$$\hat{Z}(p) \leq \min_{z,\{w_k\}_{k=0}^{\tau-N}} \sum_{k=\tau-N}^{\tau-1} L_k(w_k,v_k)$$

subjected to initial condition $\hat{Z}_0(\cdot) = \Gamma(\cdot)$. For $\tau \leq N$, the approximate arrival cost $\hat{Z}(\cdot)$ satisfies instead the inequality $\hat{Z}(\cdot) \leq Z(\cdot)$.

With exception of **A4** all the assumptions stated above were considered in Rao et al. (2003). Assumption **A6** loosely speaking means that the approximate arrival cost should not add “information” that is not present in the

$$\hat{Z}(\cdot) = \hat{Z}(\cdot)$$

3. STABILITY RESULTS

In this section we provide conditions under which the state estimate computed in (5) converges to zero in absence of disturbances and noise, or to a small neighborhood of the true values in the presence of bounded disturbances and noise. To this effect, we first recall the following definitions that will be used in the sequel.

**Definition 1.** A function $\alpha : R_0^+ \rightarrow R_0^+$ is a $K_\infty$-function if it is continuous, strictly monotone increasing, $\alpha(x) > 0$ for all $x \neq 0$, $\alpha(0) = 0$ and $\lim_{x \rightarrow \infty} \alpha(x) = \infty$.

We will freely use basic results on $K_\infty$ functions, such as, the inverse of $\alpha(\cdot) \in K_\infty$ exists and is a $K_\infty$ function, also if $\alpha(\cdot) \in K_\infty$ then $a \leq b \Rightarrow \alpha(a) \leq \alpha(b)$.

**Definition 2.** System (1) is uniformly observable if there exist a positive integer $N$, and a $K_\infty$ function $\varphi(\cdot)$ such that for any two states $x_1$ and $x_2$, \( \varphi(||x_1 - x_2||) \leq \sum_{j=0}^{N-1} \|y(\sigma_k + j; x_1, \sigma_k) - y(\sigma_k + j; x_2, \sigma_k)\| \), where $y(\sigma_k; z, \sigma_j) = h_k(x(\sigma_k, z, \sigma_j))$ with $x(\sigma_k, z, \sigma_j)$ being the solution of (1) without disturbances at time $\sigma_k$ when the state starts at time $\sigma_j$ with value $z$.
data. See details and strategies to choose \( \hat{\mathcal{Z}} \) in Rao et al. (2003).

With this framework adopted, the following preliminary technical results can be derived.

**Lemma 3.** If there exist positive constants \( \delta \) and \( \mathbf{d} \) such that for any sequence \( \{\mathbf{w}_i\}_{i=1}^{\infty} \), with \( b > a \)
\[ \|\mathbf{w}_i\| \leq \delta, \quad \|z_{i} - z_{2}\| \leq \mathbf{d} \]
and Assumption A0) a) holds, then
\[ \|\Sigma[\mathbf{f}(\phi, z_{1}, w, a, b)] - \Sigma[\mathbf{f}(\phi, z_{2}, 0, a, b)]\| \leq \left( \sum_{i=1}^{\infty} c_{\phi}^{i} \right) \delta + c_{\phi}^{k-a} \mathbf{d} \]

Lemma 3 provides a bound for the difference between two solutions of system (1) that depends on the difference between the initial condition of each solution and on the bounded disturbance at every step. This is an important Lemma that will be used often. Notice that in particular, this Lemma gives a bound for \( w_k \).

**Proposition 4. A0** and A4 imply that \( A0^* \)

a) \[ \|f_k(z_1) - f_k(z_2)\| \leq c_f \|z_1 - z_2\| \]
b) \[ \|h_k(z_1) - h_k(z_2)\| \leq c_h \|z_1 - z_2\| \]
holds for system (2) for some \( c_f, c_h > 0 \).

**Proof.** This is a direct consequence of Lemma 3 and the definition of \( f_k \).

The following Lemma establishes that, under reasonable assumptions, bounded noises and bounded estimates of the noises imply bounded estimation error.

**Lemma 5.** If system (2) is uniformly observable, \( N > N_o \), \( A0^* \) holds and \( V_{k:j} \cup V_{k:j} \cup V_{k:j} \cup V_{k:j} \) and \( \|w_{k+j}\| \) are all bounded by \( b \); then there exists a \( K_\infty \) function \( \zeta() \) such that \( V_{k:j} \cup V_{k:j} \cup V_{k:j} \cup V_{k:j} \) holds. Then by Proposition 4, \( \bar{x}_k \) is bounded.

The next two propositions taken from Rao et al. (2003) provide conditions for the existence of the solution to the optimization (4), and convergence of the estimation error \( \hat{x}_k - x_k \), respectively.

**Proposition 6.** (Rao et al. (2003)). If assumptions A0)-A3) and A5) hold, system (2) is uniformly observable, and \( N > N_o \), then a solution exists to optimization (4) for all \( x_0 \in \mathbb{X}_0 \) and \( T \geq 0 \).

**Proof.** The proof is given in Rao et al. (2003) (Proposition 3.3).

**Proposition 7.** (Rao et al. (2003)). If assumptions A0*, A1-A3, A5 and A6 hold, system (2) is uniformly observable, \( N > N_o \) and \( w_k, v_k = 0 \), then for all \( x_0 \in \mathbb{X}_0 \), \( \|\hat{x}_k - x_k\| \rightarrow 0 \) as \( k \rightarrow \infty \).

**Proof.** The proof is given in Rao et al. (2003) (Proposition 3.4).

We are now ready to state the main results of this section. The first one states that if there are no disturbances or noise, then the estimation error converges to zero.

**Corollary 8.** If assumptions A1-A6 hold, system (1) is uniformly observable, \( N > N_o \) and \( w_k, v_k = 0 \), then for all \( x_0 \in \mathbb{X}_0 \), \( \|\hat{x}_i - x_i\| \rightarrow 0 \) as \( i \rightarrow \infty \).

**Proof.** By Proposition 4, \( A0^* \) holds. Then by Proposition 7, \( \|\hat{x}_k - x_k\| \rightarrow 0 \) as \( k \rightarrow \infty \). Thus, by Lemma 3 we can conclude that \( \|x_i - \hat{x}_i\| \) with \( \hat{x}_i \) obtained from (5) also converges to 0 as \( i \rightarrow \infty \).

We now show under the following assumption that the estimate \( \hat{x}_i \) converges to a neighborhood of the true value.

**A7** There exists positive constants \( \delta_0 \) and \( \delta_\varepsilon \) such that \( \Theta_k \subseteq B_{\delta_0} \) and \( \mathbb{V}_k \subseteq B_{\delta_\varepsilon} \) for all \( k \), where \( B_{\delta} = \{ x : ||x|| \leq \varepsilon \} \).

**Proposition 9.** Suppose that \( A0 \), A4) and A7) hold, a solution exists to (4) for all \( x_0 \in \mathbb{X}_0 \), \( N > N_o \), and system (1) is uniformly observable. Then the estimation error \( \|\hat{x}_i - x_i\| \) for \( i \geq \sigma_{N_o} \) are bounded by \( \beta(||\delta_0 + \delta_\varepsilon||) \) where \( \beta(\cdot) \) is a \( K_\infty \) function.

**Proof.** By Proposition 4, \( A0^* \) holds. Then by A7) and Lemma 5 it follows that the error \( \|x_i - \hat{x}_i\| \) is bounded. Using Lemma 3 we can then conclude that \( \|x_i - \hat{x}_i\| \) is also bounded.

### 4. IMPLEMENTATION

This section provides an interpretation of the minimization described in Section 2.3 and proposes a scheme to select the running cost \( L_k() \). We consider the class of systems where the process noise is only additive (like system (2)). In this case \( w_k = x_{k+1} - f_k(x_k) \). Also, instead of computing \( \tilde{x}_{T+1} \) (as in section 2.3) we would like to compute \( \tilde{x}_{T+T} \), i.e. we use \( y_T \) to estimate \( x_T \). We suppose that the input disturbances \( w_k \) and measurement noises \( v_k \) are stationary zero mean white Gaussian sequences of random variables, mutually independent with covariances \( Q_k \) and \( R_k \), respectively. The initial priori information of the initial condition is also assumed to be a Gaussian random variable with covariance \( \Pi_0 \).

In this setup, as done in Goodwin et al. (2005), we would like to find the constant that maximizes the probability density \( p(x_0, x_1, ..., x_T|y_0, y_1, ..., y_T) \) given the observations, that is,
\[ \{\hat{x}_k\}_{k=0}^{T} = \arg \max_{x_0, x_1, ..., x_T} \{p(x_0, x_1, ..., x_T|y_0, y_1, ..., y_T)\} \]
Performing some straightforward computations and applying the Bayes Theorem we obtain
\[ \{\hat{x}_k\}_{k=0}^{T} = \arg \max_{x_0, x_1, ..., x_T} \{p(x_0) \prod_{k=0}^{T} p_{V_k}(y_k - h_k(x_k))p_{X_k}(x_0) \}
\times \prod_{k=0}^{T-1} p_{W_k}(x_{k+1} - f_k(x_k)) \]
where \( p_{V_k} \), \( p_{W_k} \) and \( p_{X_k} \) denote the probability density functions of \( v_k \), \( w_k \) and \( x_0 \), respectively. Applying logarithm and using the normal probability density function we get
\[ \{\hat{x}_k\}_{k=0}^{T} = \arg \min_{x_0, x_1, ..., x_T} \{ \sum_{k=0}^{T} \|y_k - h_k(x_k)\|^2_{R_k} + \sum_{k=0}^{T-1} \|x_{k+1} - f_k(x_k, u_k)\|^2_{Q_k} + \|x_0 - \hat{x}_0\|^2_{\Pi_0} \} \].
where \( ||z||_2^2 = z^TAz \). We can now conclude that this optimization is the same as the one described in Section 2.3 if \( L_k(\cdot) \) is chosen to be

\[
L_k(w, v) = w^TQ_k^{-1}w + v^TR_k^{-1}v
\]

and

\[
\Gamma(x) = (x_0 - \bar{x}_0)\Pi_0^{-1}(x_0 - \bar{x}_0)
\]

plus the term \( v^T R_T^{-1} v_T \) that arises from taking account \( y_T \) to estimate \( x_T \).

To compute the arrival cost, one strategy is to approximate it by employing a first order Taylor series approximation of the model around the estimated trajectory \( \hat{x}_k \). This strategy yields the Extended Kalman Filter (EKF) covariance update formula. Thus,

\[
\{\hat{x}_k\}_{k=T-N}^{T-N+1} = \arg \min_{\hat{x}_{T-N}, \hat{x}_{T-N+1}, \ldots} \sum_{k=T-N}^{T-N+1} \frac{1}{\Pi_{k-1}} \| y_k - h_k(\hat{x}_k) \|^2_{Q^{-1}_{k-1}}
\]

\[
+ \sum_{k=T-N}^{T-N+1} \| \hat{x}_{k+1} - f_k(\hat{x}_k, u_k) \|^2_{Q^{-1}_k}
\]

\[
+ \alpha \| x_{T-N} - f_{T-N-1}(\hat{x}_{T-N-1}) \|^2_{Q^{-1}_{T-N-1}}
\]

where \( \alpha \) is a forgetting factor and \( \Pi \) is computed using the EKF formula.

Note however that since the original process model is (1) (with decimated observations) and not (2), an extra step is needed. Assuming that \( \omega_i \) in system (1) is Gaussian with covariance \( \Sigma^\omega_i \), then we will approximate \( w_k \) of system (2) to be Gaussian with covariance \( Q_k \), as it is described in the following pseudo code.

**Pseudo Code 1**

**Initialization:**

\[
\bar{x} = \hat{x}_k;
\]

\[
\Sigma = \Sigma^\omega_k;
\]

**Use EKF to compute \( Q_k \):**

for \( i = \sigma_1 + 1, \ldots, \sigma_{k+1} - 1 \)

\[
[\bar{x}, \Sigma] = \text{ekfUpdateTime}(\bar{x}, \Sigma, \Sigma^\omega_k)
\]

end for

**Return:**

\[
Q_k = \Sigma
\]

where ekfUpdateTime(\( \bar{x}, \Sigma, \Sigma^\omega_k \)) returns one step ahead of the predicted mean and covariance using the standard EKF formulas.

The procedure to obtain the estimates is described in the following pseudo code.

**Pseudo Code 2**

**Initialization:**

for \( i = 0, 1, \ldots, N - 1 \)

compute \( \hat{x}_k \) using formula (6),

but with \( T - N \) replaced by 0 everywhere,

and the last term replaced by \( \Gamma(x_0) \);

end for

\[
\Sigma = \Pi_0;
\]

\[
\bar{x} = x_0;
\]

**Update estimates:**

for \( k = N, N + 1, \ldots \)

\[
\Sigma = \text{ekfUpdateMeasurement}(\Sigma, \bar{x}, y_{j-1}, R_j) - 1)
\]

\[
\bar{x} = \hat{x}_{j-1};
\]

for \( l = \sigma_1 - 1, \sigma_1 + 1, \ldots, \sigma_j \)

\[
[\bar{x}, \Sigma] = \text{ekfUpdateTime}(\Sigma, \bar{x}, \Sigma^\omega_k);
\]

end for

compute \( \hat{x}_k \) using formula (6)

end for

where ekfUpdateMeasurement(\( \Sigma, \bar{x}, y_{j-1}, R_j \)) implements the measurement update EKF formulas.

5. HIV MODEL

The proposed algorithm is illustrated in the estimation of the concentration of HIV-1 virus, infected T-CD4+ cells and healthy cells. Our model is the following (Perelson and Nelson (1999)):

\[
\frac{dT}{dt} = s - dT - e^{-u_1}BT\nu
\]

\[
\frac{dT^*}{dt} = e^{-u_1}BT\nu - \mu_2 T^*
\]

\[
\frac{d\nu}{dt} = e^{-u_2}kT^* - \mu_1 \nu
\]

where \( T \) is the concentration of healthy T-CD4+ cells, \( T^* \) is the concentration of infected cells and \( \nu \) is the concentration of free virus particles, all in units per \([mM] \). The quantities \( u_1 \) and \( u_2 \) are the manipulated variables related to the quantities of drugs administered. The other symbols, which are described in Table 1, are assumed to be constant parameters that are related to each individual, see references in Barão and Lemos (2007).

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<th>Par.</th>
<th>Description</th>
<th>U. Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d )</td>
<td>Mortality rate for healthy cells</td>
<td>0.02</td>
</tr>
<tr>
<td>( k )</td>
<td>Production rate of virus by infected cells</td>
<td>100</td>
</tr>
<tr>
<td>( s )</td>
<td>Production rate of healthy cells</td>
<td>10</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Infection rate coefficient</td>
<td>( 24 \times 10^{-5} )</td>
</tr>
<tr>
<td>( \mu_1 )</td>
<td>Elimination rate for the virus</td>
<td>2.4</td>
</tr>
<tr>
<td>( \mu_2 )</td>
<td>Elimination rate for infected cells</td>
<td>0.24</td>
</tr>
</tbody>
</table>

Table 1. HIV model parameters description and used values.

To apply the MHE method described before we will use the discretization with the following update formula given by Euler’s method:

\[
\phi_k(T, T^*, \nu, u_1, u_2) = \begin{bmatrix}
T + t_s(s - dT - e^{-u_1}BT\nu) \\
T^* + t_s(e^{-u_1}BT\nu - \mu_2 T^*) \\
\nu + t_s(e^{-u_2}kT^* - \mu_1 \nu)
\end{bmatrix}
\]

(8)

where \( t_s \) is the time interval between two consecutive points in the discretization.

We consider that we have discrete measurements given by

\[
y_{t_s} = h_k(T, T^*, \nu) = \nu
\]

Figure 1 shows the time evolution of the state \((T, T^*, \nu)\) and the estimated state using the Extended Kalman Filter (EKF) and the MH estimator. In the simulation, the unit of time is day, but the sampling time is \( t_s = 0.1 \). The measurement data \( y_{t_s} = \nu \) is only provided at times (in days) 0, 1, 3, 5, 7, 10, 15, 30, and 50. The values of the
Table 2. Filter and system parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma_\omega$</td>
<td>$\begin{bmatrix} 3 &amp; 0 &amp; 0 \ 0 &amp; 3 &amp; 0 \ 0 &amp; 0 &amp; 3 \end{bmatrix}^2$</td>
</tr>
<tr>
<td>$R_{\sigma_k}$</td>
<td>$200^2$</td>
</tr>
<tr>
<td>$x_0$</td>
<td>$\begin{bmatrix} 1000 \ 0 \ 5 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\bar{x}_0$</td>
<td>$\begin{bmatrix} 500 \ 0 \ 0 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\Sigma_{\bar{x}_0}$</td>
<td>$\begin{bmatrix} 1000 &amp; 0 &amp; 0 \ 0 &amp; 1000 &amp; 0 \ 0 &amp; 0 &amp; 1000 \end{bmatrix}^2$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1</td>
</tr>
</tbody>
</table>

parameters described in Section 4 are described in Table 2. From the figure it can be seen that the MH algorithm performs slightly better than the EKF (in particular at the transient phase). This fact is more relevant when the measurements are less frequent. See Fig. 2 that display the curve of the mean estimation error as a function of the size of the interval of time between measurements. The EKF performance degrades faster when the interval between measurements increases.

The algorithm features were illustrated by simulations concerning the application to state estimation in a model of the HIV-1 infection. From the simulations we could concluded that the MH estimator exhibits superior performance over the extended Kalman filter. This difference of performance increases with the growth of the time interval between consecutive measurements.

Fig. 1. Evolution of the state of the system and the estimates given by the EKF and the MHE, with $N = 3$ and measurements at times: 0, 1, 3, 5, 7, 10, 15, 30, and 50

Fig. 2. Performance comparison of the MHE versus EKF for different interval of measurements. The measurements are provided periodically with a period that ranges .5, 1, 1.5, ..., 4. The simulations end at time 15.

6. CONCLUSIONS

We considered the problem of moving horizon state estimation of discrete lumped nonlinear systems subject to constraints and whose measurements of the observed variables are not available at every sampling instant (decimated observations). We proposed an estimation algorithm together with results on its convergence. It was shown that, under suitable assumptions, the estimation error is bounded in the presence of bounded disturbances and noise. In particular, if these ones vanish, the error convergences to zero.

REFERENCES


