ABSTRACT: In terms of safety and environment, reduction of the noise generated by tire vibrations on a road is very significant. In order to study the vibration properties of vehicle tires, various methods have been presented in literature. In most of these methods, the global structure of tires has been modelled as circular ring, orthotropic plate, periodic or full 3-dimensional models. A brief review of the characteristics of these models and comparison of their dynamic behaviour are the main purpose of the current study. The tire is supposed to be subjected to an excitation caused by contact of the tire and road. Study of vibrational responses demonstrates that the validity of each model is limited to a certain frequency range. To employ the circular ring and orthotropic plate models, first, we require to estimate some structural and material data associated to the nature of these models. To this end, the vibrational response of a 3D model is considered to determine some properties such as radial and tangential stiffnesses in circular ring model or bending, foundation stiffness, and tension in orthotropic plate model. Furthermore, the effect of inflation pressure on the dynamic behavior of the mentioned models is examined. For verification, the dynamical behaviour of a tire is studied experimentally. Application of the present study can be contemplated in the prediction of rolling noise and rolling resistance.

KEY WORDS: Tire models; Circular ring; Orthotropic plate; Periodic FEM.

1 GENERAL GUIDELINES

A lot of research on the vibration properties of vehicle tires has been done during the last decades. In this purpose, one of the simplest methods is the model of rotating ring on the elastic foundation. Due to the completeness and simplicity, since the 1960s, this method has drawn the attractions of numerous researchers. Development of the method is pioneered by Clark [1], Tielking [2], and Bohm [3] who presented a method for calculating the dynamic behavior of a loaded pneumatic tire modeled as an elastically supported cylindrical shell. In these works, the tire sidewall effects were modeled by the radial springs. Pacejka [4] modeled the tire as a circular ring under pressure. By considering the circumferential springs for the elastic foundation, he developed models for the lateral vibration. The effect of structural damping on the study of dynamic response of the classical model, the ring on the foundation for the first time was considered by Padovan [5] in 1976. Later, Potts et al. [6] studied the vibration of a rotating ring on an elastic foundation in terms of the material and geometric properties of the tire. In order to study the free vibration of a circular ring tire located on an elastic foundation, a finite element method was presented by Kung et al. [7] in 1987. Huang [8,9] studied the response of a rotating ring subjected to the harmonic and periodic loadings. In 2008, Wei [10] proposed an analytical approach to analyze the forced transient response of the tires modeled based on the ring on the elastic foundation. The next method to model the tire structure is the model of Timoshenko beam. In the recent decade, Pinnington and Briscoe [11] developed a one-dimensional wave model to describe the tire dynamics. The tire belt is represented as a tensioned Timoshenko beam in order to derive arbitrary sidewall impedance. The waves, which propagate along the tire, take shear and rotational effects into account. The validity of the circular ring models is limited to the frequency less than 400Hz, when the wavelength is large enough compared to the width of the tire.

In order to study the vibrational tire properties in higher frequencies in 1989 Kropp [12] proposed the orthotropic plate model on a winkler foundation, where the tire belt is modeled as a finite plate which has different tangential and lateral properties. The foundation represents the effect of sidewalls as well as the inflation pressure. Also, the external tension forces due to the inflation pressure are considered in this model. In 2001, Hamet [13] proposed an analytical approach to study the impulse response of a tire modeled by a thin orthotropic plate under tension supported by an elastic foundation. Later, Larson and Kropp [14] developed a double-layer tire model including the tangential motion and the local deformation of the tread. Their model is appropriate for the modeling of radial and tangential vibrations at the high-frequency range. However, the model of the orthotropic plate is completely dependent on the results of experiments. In order to estimate the structural properties of the orthotropic plate tire model, Perisse et al. [15] presented a procedure for the experimental modal testing of a smooth tire in low and medium frequency.

However, applying finite element (FE) approaches, one can model more accurately the structural features of a tire. Thus, especially during the past decade, a considerable number of studies have been focused to investigate the wave propagation in a symmetrical periodic element of a tire based on FE
techniques. Brillouin [16] and Mead [17] applied the Floquet’s principle or the transfer matrix to study the wave propagation in a 3-dimensional (3D) periodic structure. In 2003, Houillon et al. [18] and Mace et al. [19] developed a FE method to determine the propagation constants and wave modes. Their works were focused to obtain dispersion relations and their application in energetic methods. Recently, Duhamel et al. [20] employed a similar method to calculate dispersion relations but for point force responses. Their approach was called the Waveguide Finite Element (WFE). Furthermore, this technique was applied by Waki et al. [21] to predict the free wave propagation and the forced response of a tire. The results obtained by using WFE method is similar to those obtained with the classical FE approach. But, the computational cost of the WFE method is very low compared to the usual FE. To this end, one can easily use this model to analyze structures with complex geometries and material distributions. In addition, applying a reduction technique, the number of degree of freedom (DOF) in a periodic element can be greatly reduced so that it can significantly shorten the computation time.

In the current work, we mainly review the above mentioned models of tire with a focus on their vibrational response. For this purpose, we calculate the dynamic behavior of these models, discuss their assumptions and limitations, and compare their results with the experimental results. In case studies, the effects of inflation pressure in tire is studied. In order to verify the results given by periodic 3D model, a full 3D tire model is analyzed numerically via a FE technique.

2 BRIEF REVIEW ON TIRE MODELS

In this section, we summarize the existing models of tire proposed in literature and discuss the assumptions, limitations, and drawbacks of each model. The models can be classified into five categories; rotating ring, Timoshenko circular beam, orthotropic plate, periodic 3D and full 3D models. The properties of these models are demonstrated in detail at the following.

2.1 Rotating Ring Model

The rotating ring model is one of the simplest methods used to model a tire. In this model, it is supposed that the automobile tire is composed of two main parts: the belt band (tread) and the sidewalls. Based on the inflation pressure of the tire, the sidewall may provide three-way (radial, tangential, and lateral) elastic foundation for the belt. Here, the tread is modeled as a rotating ring and the elastic properties of sidewall are modeled by the distributed springs, \( k_r \) and \( k_o \) in radial and circumferential directions, respectively, i.e. the lateral stiffness is ignored (Fig. 1). It is assumed that there is a punctual contact between the ring and the road. In addition, the slip between the ring and the road surface is ignored. Considering \( u_r \) and \( u_\theta \) as the displacements in radial and tangential directions, respectively, for the rotating tire with no contact, the equations of motion are written as [9]

\[
\frac{EI}{R^4} (u_{rr} - u_\theta') + \frac{EA}{R^2} (u_r + u_\theta') + \frac{pb}{R} (u_r + 2u_\theta - u_r) + k_r u_r + \rho A \Omega^2 (2u_\theta - u_r') + \rho A (\ddot{u}_r - 2\Omega \ddot{u}_\theta) = q_r
\]

\[
\frac{EI}{R^4} (u_{r\theta}'' - u_\theta) - \frac{EA}{R^2} (u_r' + u_\theta') + \frac{pb}{R} (u_\theta - 2u_r' - u_\theta) + k_\theta u_\theta - \rho A \Omega^2 (2u_r' + u_\theta') + \rho A (\ddot{u}_\theta + 2\Omega \ddot{u}_r) = q_\theta
\]

where, \( R, b, h, \) and \( \rho \) are mean radius, tread with, averaged thickness, and density of the tire, respectively. \( EI \) and \( EA \) indicate the bending and the membrane stiffness, respectively. \( I \) and \( A \) is moment inertia and surface of ring cross-section and \( E \) corresponds to the Young’s modulus of tire. \( p \) denotes the internal pressure and \( \Omega \) indicates the rotational velocity. \( q_r, q_\theta \) applies the external load of the system. In these equations, primes and dotes indicate the differentiations with respect to theta and t, respectively. The displacements, \( u_r \) and \( u_\theta \), can be obtained by applying modal analysis as the following

\[
(u_r, u_\theta) = \sum_{n=-\infty}^{n=+\infty} A_n (1, iC_n e^{i(n\theta + \omega_n t)}),
\]

\[
C_n = \frac{n^3 EI + n^2 EA + 2n pb + 2\rho A \Omega(n \Omega - \omega_n)}{n^2 (EI + EA) + pb(n^2 + 1) + k_\theta + \rho A (n^2 \Omega^2 - \omega_n^2)}
\]

where, \( \omega_n \) indicates the natural frequencies of the system and \( A_n \) and \( C_n \) are constants. During the rotation, the circular ring is subjected to the two external loadings; the first one is corresponding to the weight of vehicle and the other is the excitation due to the contact between the ring and the road. In the current study, the weight of vehicle is ignored. Generally, the equation of motion in the time domain can be expressed as

\[
M \ddot{u}(t) + C \dot{u}(t) + K u(t) = q(t)
\]

where, \( M, C, \) and \( K \) are the mass, damping and stiffness matrices, respectively, while \( u \) and \( q \) denote the displacement and force vectors. Substituting Eq.(3) into Eq.(2) and considering the harmonic function \( e^{i\omega t} \), one can obtain matrix \( k_n \) as

\[
\begin{bmatrix}
-d_1 - 2\Omega \omega_n & d_2 - \omega_n^2 \\
-d_3 - \omega_n^2 & i(d_1 - 2\Omega \omega_n)
\end{bmatrix}
\]

where,

\[
d_1 = \frac{1}{\rho A} \left[ n^3 EI + n^2 EA + 2n pb + 2\rho A \Omega^2 \right]
\]

\[
d_2 = \frac{1}{\rho A} \left[ n^2 EI + n (EA + pb(n^2 + 1)) + \rho A n^2 \Omega^2 + k_\theta \right]
\]

\[
d_3 = \frac{1}{\rho A} \left[ n^3 EI + n^2 EA + pb(n^2 + 1) + \rho A n^2 \Omega^2 + k_\theta \right]
\]

If the determinant of \( k_n \) equals zero, the natural frequencies of the system are easily computed as the roots of this
equation

\[ \omega_n^4 - (4\Omega^2 + d_2 + d_3)\omega_n^2 + 4d_1\Omega\omega_n + d_2d_3 - d_1^2 = 0 \]  

(5)

In the frequency domain, we have

\[ u(\omega) = G(\omega)Q(\omega), \]

where \( G \) is the Green’s function which can be approximated by a linear combination of \( N_m \) modes as

\[ G(\omega) = \left[ -\omega^2 M + i\omega C + K \right]^{-1} \]

\[ = \sum_{n=1}^{N_m} \left( -\omega^2 + 2i\xi_n\omega + \omega_n^2 \right)^{-1} \varphi_n\varphi_n^t, \]

where, \( \omega_n \) and \( \xi_n \) indicate the natural frequency and damping of each mode and \( \varphi_n \) are the mass-normalized mode shape for mode \( n \).

2.2 Timoshenko Circular Beam Model

In Timoshenko circular beam model, the belt is modeled as a Timoshenko beam to accommodate bending, shear, and the rotary inertia effects that are significant at high frequencies. Similar to the circular ring model presented in the previous section, the sidewall of the tire is replaced by the radial and tangential springs, \( k_r \) and \( k_\theta \). Considering \( R \) as the mean radius of the tire, in the linear case, one can obtain the equations of motion of the Timoshenko circular beam as the form of

\[ \frac{G A}{R} + \frac{G I}{R^3} \left( \frac{u_r - u_\theta}{R} + \alpha' \right) - \frac{E A}{R} \frac{u_r + u_\theta}{R} - \frac{E I}{R^3} \left( \frac{u_r + u_\theta}{R} + \alpha' \right) + p - k_r u_r + p A \Omega^2 \]

\[ (-R + u_r - 2u_\theta - u_r) + p A (\bar{u}_r - 2\Omega \bar{u}_\theta) = q_r \]

\[ \frac{E A}{R} \left( \frac{u_r + u_\theta}{R} \right) + \frac{E I}{R^3} \left( \frac{u_r + u_\theta}{R} - \alpha' \right) + \frac{G A}{R} + \frac{G I}{R^3} \]

\[ \left( \frac{u_r - u_\theta}{R} + \alpha \right) - k_\theta u_\theta - p A \Omega^2 (\bar{u}_\theta + 2u_r - u_\theta) \]

\[ + p A (\bar{u}_\theta - 2\Omega \bar{u}_\theta) = q_\theta \]

\[ \frac{E I}{R^2} \left( \frac{u_r + u_\theta}{R} + \alpha' \right) - \frac{G A}{R} \left( \frac{u_r - u_\theta}{R} + \alpha \right) - p A \Omega^2 \alpha'' \]

\[ = 0 \]

(8)

where, \( G \) and \( \nu \) indicate the shear modulus and Poisson’s coefficient, respectively, and \( \alpha \) denotes rotation along the \( z \) axis. The general format of the equation of motion in time domain is referred by Eq.(3). For the current model, in order to obtain matrix \( K \), a numerical approach based on the finite difference approximation technique accompanying by the Newton-Raphson method is applied. The finite difference operators are defined as

\[ u'(\theta_i) = \frac{u(\theta_{i+1}) - u(\theta_{i-1})}{2h} \]

\[ u''(\theta_i) = \frac{u(\theta_{i+1}) - 2u(\theta_i) + u(\theta_{i-1})}{h^2} \]

(9)

where, \( \theta = (\theta_1 = 0, \theta_2, \ldots, \theta_N = 2\pi(N-1)) \), \( N \) is the number of points considered on the circular beam. Note that \( u(0) = u(2\pi) \).

After calculating \( K \) matrix, we can determine the frequency response function of the system by applying the modal analysis.

2.3 Orthotropic Plate Model

In this model, the tire is simulated by the three dimensional plate as the tire belt, which has different tangential and lateral properties, and the sidewalls modeled by a thin plate under tension on an elastic foundation (due to the inflation in the tire). Fig. 2. Based on the Kirchhoff hypothesis for thin plates, the equation of motion can be written as

\[ [-T_{0x} \frac{\partial^2}{\partial x^2} - T_{0y} \frac{\partial^2}{\partial y^2} + B_x \frac{\partial^4}{\partial x^4} + 2\sqrt{B_x B_y} \frac{\partial^2}{\partial x^2 \partial y^2} + B_y \frac{\partial^4}{\partial y^4} + s + m'' \frac{\partial^2}{\partial y^2}] u(x, y, t) = F''(x, y, t) \]

(10)

where, \( T_{0x} \) and \( T_{0y} \) are membrane tensions caused by the inflation air in the tire. \( B_x, B_y, \) and \( B_{xy} \) are the longitudinal bending, transversal bending, and the cross stiffness of the belt, respectively. For the orthotropic plates \( B_{xy} \approx \sqrt{B_x B_y} \). \( s \) indicates the stiffness of the elastic support of the belt. \( m'' \) and \( F'' \) are the mass of plate and the acting force per unit area, respectively. In the analysis, only harmonic motion is considered and the common factor \( e^{i\omega t} \) is omitted. Material losses are introduced by adding an imaginary part to the bending stiffness, tension and the stiffness of the foundation. Note that only, the radial motion of tire (the vertical motion of the plate) is considered. The corresponding boundary conditions are defined as

\[ u(x + l_x, y, t) = w(x, y, t), \text{ i.e. the belt is circular} \]

\[ u(x, y, t) = 0 \text{ at } y = 0, y = l_y, \text{ i.e. the plate is simply supported at the sides} \]

and, \( \frac{\partial^2 u}{\partial y^2}(x, y, t) = 0 \text{ at } y = 0, y = l_y. \)
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The constant $\epsilon$ where $\Omega$ calculating the corresponding Green's function in terms of wavenumbers $k$ are the eigenfrequencies associated to the vibration of the system. 

Using Green’s function technique the solution of the equation can be expressed as following

$$u(x, y, t) = \int \int F''(x_0, y_0, \tau) G(x, y, t) dx_0 dy_0 d\tau$$  \hspace{1cm} (11)

where $G(x, y, t)$ is solution of

$$[- T_{0x} \frac{\partial^2}{\partial x^2} - T_{0y} \frac{\partial^2}{\partial y^2} + B_x \frac{\partial^4}{\partial x^4} + 2 \sqrt{B_{xy}} \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} + B_y \frac{\partial^4}{\partial y^4} + s + m \frac{\partial^2}{\partial t^2} ] G(x - x_0, y - y_0, t - \tau)$$

\hspace{2cm} $= \delta(x - x_0) \delta(y - y_0) \delta(t - \tau)$  \hspace{1cm} (12)

Modeling the circular tire by an infinitely long strip and calculating the corresponding Green’s function in terms of the superposition of normal modes one can obtain [13]

$$G(x, y, t) = \frac{4}{l_x l_y} \frac{1}{m} \sum_{n=1}^{\infty} \frac{\sin(k_{ym} y_0) \sin(k_{yn} y)}{\sin(k_{ym} y_0) \sin(k_{yn} y)} \sum_{m=0}^{\infty} \epsilon_m \cos[k_{xm}(x - x_0)] \sin[\Omega_{nm}(t - \tau)] \frac{\Omega_{nm}}{\sqrt{\Omega_{nm}}} e^{-\eta \omega \Omega_{nm}(t - \tau)} H(t - \tau)$$  \hspace{1cm} (13)

where $\Omega_{nm}$ are the eigenfrequencies associated to the wavenumbers $k_{xm} = \frac{2\pi m}{l_x}$ and $k_{yn} = \frac{n \pi}{l_y}$. $\Omega_{nm}$ is defined as

$$\Omega_{nm} = Re[(k_{xm}^2 B_x + 2k_{xm}^2 k_{yn} B_{xy} + k_{yn}^2 B_y)$$

\hspace{2cm} $+ (k_{xm}^2 + k_{yn}^2) T_0 + s^2 \frac{1}{l''})$$  \hspace{1cm} (14)

The constant $\epsilon$ has the values $\epsilon_0 = \frac{1}{2}$ and $\epsilon_{m \neq 0} = 1$. $\eta$ is damping for each mode which it is calculated as

$$\eta = \frac{1}{\Omega_{nm}} Im[(k_{xm}^2 B_x + 2k_{xm}^2 k_{yn} B_{xy} + k_{yn}^2 B_y)$$

\hspace{2cm} $+ (k_{xm}^2 + k_{yn}^2) T_0 + s^2 \frac{1}{l''})$$  \hspace{1cm} (15)

2.4 Periodic 3D Model

A symmetrical periodic element of the tire, as shown in Fig. 3, is considered. The equation for time harmonic motion of a periodic section can be written as

$$Du = q$$  \hspace{1cm} (16)

where, $D = K + i\omega C - \omega^2 M$, is the dynamic stiffness matrix, $u$ and $q$ denote nodal DoFs and force vector, respectively. $K$, $C$, and $M$ are stiffness, viscous damping and mass matrices which are obtained from conventional FE methods. If $D$ is decomposed into left (L) and right (R) boundaries, interior degrees of freedom are indicated by $I$, and also it is assumed that there are no external forces on the interior nodes, the equation of motion may be expressed as

$$\begin{bmatrix}
D_{LL} & D_{LR} & D_{LI} \\
D_{RL} & D_{RR} & D_{RI} \\
D_{IL} & D_{IR} & D_{II}
\end{bmatrix} \begin{bmatrix}
u_L \\
q_L \\
u_R
\end{bmatrix} = \begin{bmatrix}
u_L \\
q_L \\
u_R
\end{bmatrix}$$  \hspace{1cm} (17)

By eliminating the interior degrees of freedom this equation can be condensed, also in order to decrease the number of degrees of freedom, a reduced basis can be used [20]. After calculation of equivalent mass and stiffness of the system, one can apply modal analysis to obtain the frequency response function.

2.5 Full 3D Model

The last model considered is full 3-dimensional tire model. This model is devoted solely to the verification of the results obtained by periodic 3D model. For this purpose, the tire is analyzed numerically by the finite element techniques. Abaqus software is applied to model and analyze the tire.

3 PREDICTION OF STRUCTURAL PROPERTIES REQUARED IN CIRCULAR RING AND ORTHOTROPIC PLATE MODELS

As described in the previous sections, in order to study the dynamic behavior of tires, two models of circular ring and orthotropic plate models, besides of 3D models, are employed. In order to apply these models, first, we require to estimate structural and material data associated to the nature of these models. For this purpose, a homogeneous grooved tire is modeled by Abaqus software. We consider that the inflation pressure in the tire is equal to 2 bars. The mechanical and structural properties of the tire are
summarized in Table 1. In the present study, a proportional damping system is considered, so that the viscous damping matrix $C$ is defined as

$$C = \alpha M + \beta K$$

where, $M$ and $K$ are mass and stiffness matrices of the model respectively. $\alpha$ and $\beta$ are damping factors whose values are computed based on an experimental measurement on the tire of Michelin R13/165/65 in which, $\alpha = 14.4$ and $\beta = 8 \times 10^{-5}$. The computed radial driving point mobility is shown in Fig. 4.

![Fig. 4- The computed radial driving point mobility](image)

Substituting the first and second resonance frequencies of the given results into Eq.(5), one can determine the stiffness of the radial and tangential springs $k_r$ and $k_\theta$ as defined in circular ring model. Table 2 displays the results.

In case of orthotropic plate model, based on the method described by Andersson et al. [22], foundation stiffness, $s$, tension, $T_{0x}$ and $T_{0y}$, and bending stiffness, $B_x$, $B_y$, and $B_{xy}$, and their corresponding damping values can be estimated. The results are presented in Table 3.

### 4 RESULTS AND DISCUSSIONS

To compare the various models of tire, several numerical studies will be addressed in this section. Furthermore, the effect of inflation pressure is addressed. In all the studies presented in this section, the models are subjected to a punctual excitation force and the reported results are corresponding to the excitation point.

In the first study, the radial point mobility calculated by the models of circular ring, orthotropic plate and periodic 3D are compared in Fig. 5. As it is expected, the models of rotating ring and Timoshenko circular beam are in a good correspondence with the periodic 3D model at low frequencies whereas, at high frequencies, the point mobility given by orthotropic plate model is very similar to the corresponding results of periodic 3D model. In order to verify the results given by tire models, an experimental measurement is done. The results displayed in Fig. 5 approve that there is a good agreements between the calculated and measured mobilities.

Second, the results of a full 3D and a periodic 3D models are compared. Fig. 6 demonstrates the point mobilities pertinent to each model. As it is seen, the results of two models have a good agreement with each other, whereas, the computation time for the periodic 3D model is significantly less than the full 3D model.

Finally, the effect of inflation pressure is examined. The point mobilities given by the periodic 3D model at pressures 0, 1, and 2 bars are plotted at Fig. 7. It is observed that, at the low frequencies enhancement of the pressure causes the mobilities decline while the resonance frequencies increase. But, at the high frequencies the

![Fig. 5- Comparison of the radial driving point mobility obtained by the numerical and experimental models of tire](image)

| Table 1: The mechanical and structural properties of the homogeneous tire |
|------------------|------------------|
| Internal radius  | 165.1 mm         |
| Width of tread   | 165 mm           |
| Height of sidewall| 115.5 mm         |
| Young modulus    | 80 MPa           |
| Poisson coefficient | 0.49             |

| Table 2: Structural and material properties required in circular ring model |
|------------------|------------------|
| $k_r$             | 1.273e6 N/m²     |
| $k_\theta$       | 9.643e5 N/m²     |

| Table 3: Structural and material properties required in orthotropic plate model |
|------------------|------------------|
| $T_{0x}$         | 4.8e4(1 + 0.09i) N/m |
| $T_{0y}$         | 8.3e4(1 + 0.09i) N/m |
| $b_x$            | 2.3(1 + 0.3i) N/m  |
| $b_y$            | 1.2(1 + 0.3i) N/m  |
| $b_{xy}$         | 1.6(1 + 0.3i) N/m  |
| $s$              | 3e4(1 + 0.01i) N/m³ |
Fig. 6 - Comparison of the radial driving point mobility obtained by the full and periodic 3D models of tire
variation of the inflation pressure has no essential influence on the mobility.

5 CONCLUSION

In summary, we outlined the characteristics of various models of tire as circular ring, orthotropic plate and 3D models. When the tire is subjected to an excitation, we compared the vibrational responses of the models. Generally, the results approve that the circular ring models are valid for low frequencies while the orthotropic plate model provides reasonable result in high frequencies. In addition, the effect of the inflation pressure was studied. It is observed that there is a significant discrepancy between the calculated point mobility and also resonance frequency for different pressures at low frequency but by increasing the frequency the discrepancy fades away. Moreover, the results given by the models of tire were verified by the experimental results. The presented study may find potential applications in study of rolling noise and rolling resistance.

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