Investigation of stochastic resonance effects in problems of wind induced vibration

Stanislav Pospíšil¹, Jiří Náprstek¹

¹Institute of Theoretical and Applied Mechanics ASCR, v.v.i., Prosecká 76, 19000, Prague 9, Czech republic
email: pospisil@itam.cas.cz

ABSTRACT: We study the response of a dynamic non-linear single degree of freedom system with cubic characteristic to a combination of additive random noise and external deterministic periodic force. We apply the principles of stochastic resonance in the field of wind engineering to investigate vibration of a slender prismatic beam in a cross flow with a turbulence component. The aim of the study is to find such parameter combinations which should be avoided in practice to eliminate response amplitude increase due to the effect of the stochastic resonance. We assume the non-linear oscillator (beam) with one generalized degree of freedom in the divergence-like regime. It is described by the version of the Duffing equation. We conduct the theoretical investigation with the use of relevant Fokker-Planck equation together with a verification by numerical simulation of corresponding stochastic differential system. Real characteristics of the beam sectional model investigated in the wind tunnel are employed. Complex tunnel testing combining generated noisy background of variable intensity (additive noise) with harmonic excitation produced by the linear motor attached to the experimental stand. The special appliance related with the stand allows the snap-through effect to be respected.

KEY WORDS: Nonlinear aero-elasticity; Stochastic resonance; Turbulence; Aeroelastic divergence; Wind induced vibration; Wind tunnel experiments

1 INTRODUCTION

The response of various types of non-linear dynamic systems under random excitation has been studied during many years, as this mathematical model is inherent in a number of areas of physics including wind engineering and aero-elasticity, see e.g. [1], [2]. Both latter branches deal very often with the vibration of slender structures under the excitation of deterministic (pseudo-steady wind) wind load with the combination of noisy (turbulence) additive force. It is still often assumed that contribution of the noise in the non-resonant part of the frequency spectra is decreasing the overall response of the structure, because it is acting in the disordering manner. It seems however, that concurrently acting additive deterministic and random excitation can produce the response process containing several effects well applicable in practice not only in the beneficial, as in the case of signal amplification, but also in the negative meaning which may be the case of undesired mechanical vibration and amplification of the response of a structure.

The counter-intuitive property of the noise, its ability to induce ordering in non-linear non-equilibrium systems, has been demonstrated in the effect of the phenomenon called stochastic resonance (SR). This effect has proven to be general and has been found in many natural systems [3]. Stochastic resonance is a phenomenon, which has been surmised in physical chemistry in early forties, see e.g. [4]. Many years later several branches in theoretical and experimental physics identified this phenomenon and applied this one in optics and plasma physics, see e.g. [5], or review paper [6]. Hundreds papers have been published until now, including also a couple of monographs, for instance [7].

Basically, SR occurs when the dynamical systems are perturbed by noise and in particular, when the system is subdued to an external periodic force applied together with some noise. The response of a bistable system heads towards the periodic switching between the semi-stable states. The dependence of the response on the additive noise intensity has a resonant form, hence an optimal (in the structural engineer’s point of view - undesired) value of the noise intensity can be found. Moreover, in [8] it has been shown that a high-frequency periodic force can work as a noise and amplify the response to the low-frequency periodic signal in bistable systems.

In this paper, the response of a dynamic non-linear single degree of freedom (SDOF) system with cubic characteristic to a combination of additive random Gaussian white noise and external deterministic periodic force is studied. It is motivated by the most important and dangerous phenomenon of aeroelastic post-critical state - divergence - occurring at a prismatic slender beam in a cross-flow. This phenomenon manifests by stable periodic hopping between two nearly constant limits perturbed by random noises. This jumping between wells of the bistable elastic potential can occur under certain circumstances. The frequency of this hopping can be for certain combination of input parameters nearly constant and corresponding to external periodic force frequency. This state leads to high ratio of periodic component amplitude and intensity of the random component of the response process. In such a case the system can be used as a harmonic selective resonator detecting the useful weak harmonic signal in the high noisy background. The aim of this paper is to investigate whether the undesired - harmfull - SR can be achieved in the field of structures.
loaded by the turbulent wind. For this purpose we consider and model a beam vibration subjected to a weak periodic forcing with a double well potential and certain potential barrier. The wind force consists of a low frequency harmonic part and the turbulence modelled as a white noise. When the harmonic force (low frequency pseudo-static loading) and turbulence synergy occur, the response of the beam goes beyond the potential barrier and the stochastic resonance take place. The signal-to-noise ratio is studied for this system.

2 MATHEMATICAL MODEL AND BASIC CONSIDERATIONS

Let us assume the nonlinear mass-unity oscillator with one degree of freedom, where \( \xi \) stands for a Wiener process and represents the noise level.

\[
\begin{align*}
\dot{u} &= v; \\
\dot{v} &= -2\omega_0 v - V'(u) + P(t) + \xi(t).
\end{align*}
\]  

The potential energy \( V(u) \) being introduced in a form corresponding with the Duffing equation

\[
V(u) = -\frac{\alpha_2}{2}u^2 + \frac{\gamma}{4}u^4
\]

and

\[
V'(u) = dV(u)/du = -\alpha_2 u + \gamma u^3
\]

\( \xi(t) \) - Gaussian white noise of intensity \( 2\sigma^2 \) respecting conditions:

\[
E\{\xi(t)\} = 0; \quad E\{\xi(t)\xi(t')\} = 2\sigma^2 \delta(t-t')
\]

\( P(t) = P_0 \exp(i\Omega t) \) - external harmonic force with frequency \( \Omega \). Amplitude \( P_0 \) should be understood per unit mass.

Symbols \( \omega_0 \) and \( \omega_b \) have a usual meaning of the circular eigen-frequency and circular damping frequency of the associated linear system. The linear part of the \( V'(u) \) is negative making the system metastable in the origin, while the cubic part acts as stabilizing factor beyond a certain interval of displacement \( u \). The system is drafted in the Fig. 1 in two versions: (a) system with symmetric potential typical by an equivalent energy needed for hopping from the left into the right potential well and in opposite direction (b) system with asymmetric potential due to the supplementary linear spring which could be able (when rising its stiffness) to bring the oscillator to monostable type.

If the intensity \( \xi \) is small enough, it will oscillate around either of the stable points with very small probability of switching to the other. If we increase the noise amplitude, then there is some higher probability that the the response will jump from one basin to the other. If the noise level is just right, then the periodic forcing leads to oscillation between the wells with period \( \Omega \). In more general terms, there is stochastic resonance whenever adding noise to a system improves its performance or increases its signal-to-noise ratio. Note that the noise amplitude cannot be too large or the system can become completely random.

For the symmetric potential \( (\Delta V_- = \Delta V_+) \) the approximate frequency of escape from one well into the second (bistable system) is given by the following estimate published in the comprehensive study, see [4]:

\[
\omega_e = \omega_0 \cdot \exp(-\Delta V/D),
\]

where \( \Delta V \) is height of the barrier separating potential minima and \( D \) is proportional to \( 2\sigma^2 \) introduced in Eq. (4. The Eq. (5) is a result of theoretical and empirical investigation motivated by problems of nonlinear optics. However, it is widely used and works very well. It should be noticed that also other types of non-linearity can produce a significant periodic part of the response when excited by additive random noise. See for instance [10], where influence of bistable nonlinear damping is discussed.

Being reciprocal of Kramer’s frequency, the periodicity or waiting time of the stochastic transition between two noise-induced inter-well transition is given by \( T_0(D) = 1/r_\xi \). This stochastic synchronization happens if the mean waiting time satisfies the time-scale matching requirement [6]

\[
T_0 = 2T_0(D),
\]

where \( T_0 \) is the period of the input periodic forcing term. Response of the system defined by Eq. (1) can be significantly enhanced introducing an optimal amount of additive random noise.

3 FOKKER-PLANCK EQUATION - STARTING POINT

Taking into account that random noise in Eq. (1) has an additive character, the appropriate Fokker - Planck (FP), e.g. [11] equation with the diffusion and drift coefficient can be easily written out:

\[
\begin{align*}
\kappa_u &= v \\
\kappa_v &= -2\omega_0 v - V'(u) + P(t) \\
\kappa_w &= 2\sigma^2
\end{align*}
\]

\[
\frac{\partial p(u,v,t)}{\partial t} = -\frac{\partial p(u,v,t)}{\partial u} + \frac{\partial}{\partial v}[2\omega_0 v + V'(u) - P(t)]p(u,v,t) + \sigma^2 \frac{\partial^2 p(u,v,t)}{\partial v^2}
\]

together with boundary conditions:

\[
\lim_{u,v \to \pm \infty} p(u,v,t) = 0; \quad p(u,v,0) = \delta(u,v)
\]

If the external excitation is only due to random component, the Eq. (8) admits the closed form stationary solution of the Boltzmann type. For its various aspects and details, see e.g. [12], [13], [14] and other papers and monographs:

\[
\exp\left(-\frac{2\omega_0}{\sigma^2}H(u,v)\right)
\]

where \( H(u,v) \) represents the Hamiltonian function of the basic system. In particular:

\[
H(u,v) = \frac{1}{2}v^2 + V(u) = \frac{1}{2}v^2 - \frac{1}{2}\omega_0^2 u^2 + \frac{3}{4}\gamma^4 u^4
\]

It is evident that solution of the (10) type can be provided for any arbitrary symmetric/non-symmetric potential including
cases passing the system into monostable type. Probability density of the response has significantly non-Gaussian form, see Figure 2.

When excitation force consists of both component, stationary solution of FPK Eq. (8) no more exists. In order to approximate its solution, the formula (10) can be used as a basic part which should be multiplied by a space and time dependent series. The probability distribution function (PDF) can be expected periodic or cyclic-stationary in a certain meaning of the term in time coordinate for $t \to \infty$ and the Floquet theorem or the maximum entropy principle, e.g. [12] could also be alternatively used as a basic tool for the solution.

With respect to the linearity of the FPK equation the periodicity of the PDF should be corresponding to the frequency of the deterministic excitation component $\Omega$ and to its entire powers. Therefore the series can be written in the following form:

$$p(u,v,t) = p_o(u,v) \sum_{j=0}^{\infty} q_j(u,v) \exp(ij\Omega t)$$

(12)

We try to estimate several first functions $q_j(u,v)$ in (12) using the generalized method of stochastic moments as it can be found e.g. in [11]. For some special details dealing with non-Gaussian closure, see [15]. The generalization means that moments are introduced with polynomials and not only with factors $u^k v^l$. The choice of suitable orthogonal polynomials makes possible to simplify several further steps. Physical interpretation, however, could be some times a bit unclear. Therefore some compromise should be selected.

Let us introduce the expression (3) into Eq. (8) and multiply the right and left sides of (8) by the factor $\alpha(u,v)$, being a polynomial in variables $u,v$, as mentioned above. Subsequent integration over the whole phase space $u,v$ including several per-partes operations and employing that the probability density function (PDF) approaches to zero together with all derivatives for $u,v \to \pm \infty$, yields:

$$d\mathbb{E}\{\alpha(u,v)\} =$$

$$\mathbb{E}\{\frac{\partial \alpha(u,v)}{\partial u} \cdot \mathbb{E}\{2\omega_b \cdot v + V'(u) - P(t)\}\} +$$

$$\mathbb{E}\{\sigma^2 \frac{\partial^2 \alpha(u,v)}{\partial v^2}\}$$

(13)
\textbf{E} \{ \cdot \} \text{ represents the operator of mathematical mean value with respect to PDF } p(u,v,t). \text{ As the series (3) contains unknown } q_j(u,v), \text{ it is convenient to express Eq. (13) with reference to the known PDF } p_\alpha(u,v,t) \text{ given by (10). Taking into account (3), following general expression can be obtained:}

\[
E\{A(u,v)\}(t) = \int_{-\infty}^{\infty} A(u,v) \cdot p_\alpha(u,v,t) du dv = 
\int_{-\infty}^{\infty} A(u,v) \cdot p_\alpha(u,v) \sum_{j=0}^{\infty} q_j(u,v) \cdot \exp(ij\Omega t) du dv = 
\sum_{j=0}^{\infty} E_\alpha \{ A(u,v) \cdot q_j(u,v) \} \exp(ij\Omega t) \quad (14)
\]

Let us adapt Eq. (13) using formula (14) and apply further per partes operations similarly as before. Resulting equation is dependent on time as corresponds with (3). As this equation must hold in every time and factors \exp(ij\Omega t) are independent, the identity must be valid separately for every individual \( j \) producing the following system of time-independent equations:

\[
j \Omega E_\alpha \{ \alpha q_j \} + \frac{\sigma^2}{2\alpha_b} E_\alpha \left\{ \frac{\partial^2}{\partial v^2} q_j - \frac{\partial}{\partial u} \frac{\partial}{\partial v} q_j \right\} + \sigma^2 E_\alpha \{ \frac{\partial^2}{\partial v^2} q_j \} = p_\alpha E_\alpha \{ q_{j-1} \} \quad (15)
\]

where \( \alpha = \alpha(u,v), q_j = q_j(u,v) \) for brevity has been introduced and \( q_{-1} \equiv 0 \) chosen. As it has been mentioned, the function \( \alpha(u,v) \) should have a form of a product of two polynomials separately in \( u, v \). The series of applicable orthogonal polynomials starts: \( R_s(x) = 1, P_l(x) = x, \ldots \). Therefore it holds \( q_{0s}(u,v) = 1 \).

Respecting the above notices and taking into account that the system (15) involves not only one but a set of systems for various \( \alpha(u,v) \) representing a hierarchy of generalized stochastic moments, we can write the functions \( \alpha(u,v) \) and \( q_j(u,v) \) in a form:

\[
\alpha(u,v) = \alpha_{rs}(u,v) = r^j \cdot H_j(\beta v); r = 0, \ldots, R; s = 0, \ldots, S \quad (16)
\]

\[
q_j(u,v) = \sum_{k=0}^{R} q_{jk} u^k \cdot H_j(\beta v) \quad (17)
\]

where \( H_j(\beta v) \) are Hermite polynomials and \( \beta = \sqrt{\alpha_b/\sigma} \). Substitution of series (16), (17) into Eq. (15) and employing orthogonality of Hermite polynomials yields after laborious adaptations a linear algebraic system for unknown coefficients \( q_{jk} \):

\[
2\beta (ij\Omega + 2\alpha_b x) A_{ijk} = 2(s+1)C_{j+1}B_{jk} + B q_{j-1} \]

\[
= 2\beta^2 p_\alpha A_{j-1,k} \quad (18)
\]

\[
q_{jk} = [q_{j1}, q_{j2}, s, \ldots, q_{jR_s}]^T \quad - \text{ column vector (R components) and } A, B, C \ - \text{ square arrays (R \times R components) containing moments:}
\]

\[
A_{jk} = \int_{-\infty}^{\infty} u^{k+1} \Phi(u) du; \quad B_{jk} = \int_{-\infty}^{\infty} ku^{k+1} \Phi(u) du; \quad C_{j,k} = \int_{-\infty}^{\infty} ru^{k-1} \Phi(u) du; \quad \Phi(u) = \exp(\beta u^2 - \frac{1}{2} \gamma^2 u^4)
\]

(19)

Let us deal with the main idea of the solution method from other point of view. The Eq. (8) represents a linear partial differential equation of the second order with non-symmetric operator defined on an infinite domain, with homogeneous boundary conditions and parametric "excitation". It can be shown, that the weak solution exists and can be found using variational methods. Nevertheless some properties of the operator (non-symmetry, etc.) are limiting in a choice of methods. For instance Ritz method cannot be used due to operator non-symmetry. The modified Galerkin method seems to be acceptable. The presented solution process corresponds in principle to this method. The basic series employed is given by the formula (3) together with (17). Because \( q_j(u,v) \) are polynomials and \( p_\alpha(u,v) \) can be interpreted as a weight function, every term of the series (3) separately fulfills the boundary conditions requested for the unknown \( p(u,v,t) \) and its derivatives in \( u, v \). The series (3) with respect to (10), (11) is then substituted in fact into the Eq. (8) and then the orthogonality to the function \( \alpha(u,v) \) for any arbitrary \( r,s \) is requested. The process of orthogonalization corresponds to the application of mathematical mean value operator with respect to the PDF (10). Equivalence appeared ascertains the convergence of the approximate formula (17), see e.g. [16]. Therefore the PDF can be reached with any prescribed accuracy. Due to structure of Eq. (3) the same holds also for stochastic moments which are to be used studying various physical effects related with PDF.

4 RESPONSE PROPERTIES

4.1 Basic parameters

The most representative parameter characterizing the response as a result of combined random and harmonic excitation seems to be the first moment of the displacement \( u \). The structure of the series (3) makes possible to investigate each harmonic component of the PDF and appropriate moments separately in a form of their amplitudes. The mutual dependency of individual components can be estimated analysing the general structure of the algebraic system (18) and accompanying formulae (19). It is evident that the only independent part \( j = 0 \) corresponds to state with no-harmonic excitation component as it is given by the Boltzmann’s type solution (10). The part \( j = 1 \) being influenced by \( j = 0 \) implies the basic part of the PDF reflecting the combined random/harmonic excitation. Higher harmonics, if any, are described by \( j > 1 \). The dependency tends upwards only, i.e. the parts for \( j \leq j_0 \) are not influenced by parts for \( j > j_0 \). As regards subscripts \( r,k \) they control evaluation of displacement moments, while \( A, B, C \) are independent from \( s,l \). The pair of indices \( s,l \) reduces to \( s \) only due to orthogonality of Hermite polynomials. It is useful to put \( R = S = \) or to introduce
the same upper limit of $r,k,s$. The algorithm of $q_{j,s}$ evaluation is now obvious.

To assess the individual parts of the response the first moment (mathematical mean value) of the displacement should be analyzed. The most important among them is the basic harmonic part, i.e. $j = 1$. Respecting orthogonality of Hermite polynomials it follows from (10):

$$U_1 = E_0\{uq_1(u,v)\} = \sum_{k,j=0}^{R,R} E_0\{q_{1,k,l} \cdot u^k \cdot H_l(\beta v)\} =$$

$$= \sum_{k=0}^{R} q_{1,k,0} E_0\{u^{k+1}\}$$  \hspace{1cm} (20)

Some illustrative numerical results are outlined in the following subsection. We apply the numerical simulation on the system sketched in Figure 3 to test the sensitivity of the stand to the SR.

4.2 Example-harmonically excited beam under influence of turbulence noise

The response of a beam, loaded by the wind with turbulent component, known as the aeroelastic divergence, initiated the idea to use the theory of stochastic resonance for the explanation of hopping of the beam in between two meta-stable positions, see [9]. This kind of response has been observed during the wind tunnel measurement focused on the self-induced vibration with the large amplitudes in the non-linear range using the special experimental stand. It represents the working mechanism sensitive to the excitation by the wind. The stand and the experiments are described more in detail in the papers [17] and [18].

In order to arrange the clear modulation signal at the system, the excitation mechanism is attached which is able to excite harmonically the system with desired amplitude and the driving frequency. The SR mechanism works well with the almost over-damped system, however, low damping is not excluded from the analysis, because the SR mechanism is sensitive to any optimal combination of many parameters.

The system Eq. (1) for the unknown response $u = \psi \cdot r$ can be rewritten:

$$\psi = -\alpha \cdot \psi - \beta \psi - \gamma \psi^3 + A \sin(\Omega \cdot t) + \xi(t)$$  \hspace{1cm} (21)

with: $\alpha = 2\zeta \omega_0$, $\beta = \omega_0^3 - k_p d_0/(m r) < 0$ and $\gamma = k_p r/(2mL)$. The values of the parameters are the following:

- damping of the system $\zeta = 0.05$;
- system frequency $\omega_0 = 2.0$ [s$^{-1}$];
- driving amplitude $A = 0.15$;
- mass $m = 5$ [kg]; pre-stressing stiffness $k_p = 820$ [N/m];
- initial deformation $d_0 = 0.05$ [m] of the pre-stressing spring;
- lever length $r = 0.5$ [m];
- length of the pre-stressing spring $L = 0.25$ [m].

The equation has been solved numerically for the time period 1000s with the time step $\Delta t = 0.01$s.

The results of the stochastic resonance analysis are illustrated by the Figure 4, which presents the results (Fourier spectra) of the numerical simulations using the basic system (21). In the individual spectral lines it can be seen the influence of rising white noise intensity $\sigma^2$, which acts together with a harmonic force onto the system. For very low level of the noise the harmonic component is hardly able to overcome the interwell barrier and therefore only seldom irregular jumps between stable points occur. In local regimes the system response is relatively small and nearly linear. Optimal ratio of the noise intensity and the amplitude of the harmonic force results for its certain frequency in the system response containing a visible spectral peaks (amplification) corresponding with the frequency of the external harmonic modulation. Another important graph is shown at Figure 5. It represents the signal-to-noise ratio (SNR) as the function of the noise intensity expressed by $\sigma^2$. The single peak (in the case of coloured noise more peaks may appear) and thus the "optimal" noise strength can be identified.
Generally, the response is not harmonic and contains many higher harmonics. However its basic frequency is stable making possible to reconstruct the original harmonic component hidden in the background. The Figure 6 demonstrates the state of large superiority of the noise. Increasing the noise level can counteract the aforementioned process and thereby the stochastic resonance effect can vanish. This figure shows also the sensitivity of the system. The curves (a), (b), (c) and (d) correspond to harmonic component frequency $\Omega = 0.09$ and noise intensity $\sigma^2 = 0.1, 0.5, 1.0$ and 2.0 respectively.

The influence of the noise is clearly shown also in Figures 7 and 8. When the noise strength is small, the mass remains in one well (say in the upper part of the response, or in the divergent position). Increasing the noise strength - case b) and case c) - the response is jumping from one well to another with clear periodic pattern. Further increase of the noise intensity - case d) - leads to the more random jumping between the meta-stable positions of the mass.
5 CONCLUSIONS

The response of a dynamic non-linear single degree of freedom system with cubic characteristic to a combination of additive random noise and external deterministic periodic force were studied. The principles of the stochastic resonance has been applied investigating the response of the system represented by a prismatic beam subjected to the turbulent wind.

The effect of stochastic resonance has been described firstly theoretically using the FPK equation and method of generalized stochastic moments for its approximate solution. The convergence of this method can be expected with respect to theoretical properties of the solution process being similar to Galerkin method applied to the equations with non-symmetric operator. The convergence has also been observed when numerical evaluation of formula and algorithms obtained has been done. The non-stationary character of the response PDF with the basic and the higher harmonics has been verified and used for periodic component detection in the input signal containing both random and deterministic parts. Therefore adding noise to a system can improve its "ability of the information transfer".

The paper describes also the numerical simulation of the experimental stand used for the aeroelastic testing of profiles, before it will be tested in the turbulent flow. It shows, that under certain "optimal" value of the parameters, the signal-to-noise ratio of the response increases and the resonant-like peak occurs in the amplitude spectra. This makes an optimistic perspective for the experimental analysis, which together with the analytical and numerical ones should continue to obtain better insight into the general tendencies when individual parameters of the system and the input signal are changed.

Moreover possibilities of other analytical solution procedures should be also investigated (Floquet theorem, maximum entropy principle, FEM, etc.). Also other types of non-linearities in the system stiffness (especially non-symmetric) should be carefully studied. Other application areas like post-critical processes in aero-elasticity or vehicle dynamics should be thoroughly examined from the point of view of the adverse effects of the response amplification due to stochastic resonance effects.

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