Updating Frequency Response Functions of Linear Structures With Localized Nonlinearities

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ABSTRACT: A methodology in order to obtain the nonlinear frequency response of structures, whose nonlinearities can be considered as localized sources, is presented. The work extends the well-known Structural Dynamic Modification Method (SDM) to a nonlinear set of modifications, and allows getting the Nonlinear Frequency Response Functions (NLFRFs), through an ‘updating’ process of the Linear Frequency Response Functions (LFRFs).

In the linear case, the Transfer Function (TF), defined, for each degree of freedom, as the ratio between the output acceleration and the input force, is an invariant of the system. But, if the structure acts nonlinearly, the Transfer Function depends on magnitude of the input force. It means that the constitutive equations of the nonlinear lumped elements need to include a dependency on the relative displacement experienced by the elements.

It implies that the time-domain and the frequency-domain need to be opportunely mixed in order to obtain the Nonlinear Frequency Response. A brief summary of the analytical concepts is given, starting from the linear formulation and understanding what the implications of the nonlinear one, are. The nonlinear case requires an algorithm to get the convergence. It is explained step by step, taking as example a bilinear stiffness element. The nonlinear SDM Method has been implemented in MATLAB.

For testing the strength and robustness of the method on a real industrial application, the Finite Element (FE) Model of the Auxiliary Power System (APU) Suspension System, is considered. The nonlinear behavior of this system is characterized by the rubber mounts, which present a dynamic stiffness with softening effect. In the FE Model these elements are simulated by linear springs. First the LFRFs are extracted by the FE run; secondly the LFRFs are updated through the nonlinear SDM Method, introducing in the system nonlinear springs instead of the linear ones. The NLFRFs are obtained and the nonlinear dynamic behavior of the structure is analyzed to check the coherency with the type of nonlinearity introduced in the system.

The conclusions highlight a feasible and robust procedure, which allows a quick estimation of the effect of the localized nonlinearities on the dynamic behavior. As it is shown in the example, the method is particularly powerful when most of the FE Model can be considered acting linearly and the nonlinear behavior is restricted to few degrees of freedom. The procedure is very attractive also from a computational point of view because the FE Model needs to be run just once in order to obtain the LFRFs. After that, the estimation of the nonlinear behavior in frequency domain is obtained without any need to re-run the model, being just an updating, by means of the nonlinear SDM Method, of the linear one.

KEY WORDS: Frequency Response, Nonlinear dynamics, Structural Dynamic Modification, Softening Effect, Rubber

1 INTRODUCTION

Finite Element Models (FEMs) are widely used in order to study and predict the dynamic properties of structures and usually, the prediction can be obtained with much more accuracy in the case of a single component than in the case of assemblies. For example as the number of components in the assembly increases, the calculation quality declines because the connection mechanisms between components are not represented sufficiently. Especially for structural dynamics studies, in the low and middle frequency range, most of complex FEMs can be seen as assemblies made by linear components joint together at interfaces. From a modelling and computational point of view, these types of joints can be seen as localized sources of stiffness and damping and can be modelled as lumped spring/damper elements, most of time, characterized by nonlinear constitutive laws.

Most of finite element programs are able to run nonlinear analysis in time-domain. They treat the whole structure as nonlinear, even if there is one nonlinear degree of freedom (DOF) out of thousands of linear ones, making the analysis unnecessarily expensive from computational point of view. The situation is even more complicated if the nonlinear results are required in frequency-domain because frequency response solutions are based on linear modal decomposition approach. Hence, if NLFRFs are required, one possible approach can be run as many time-domain runs as many frequency points are required. Evidently the harmonic input force has a different frequency of excitation for each time-domain run.

After that, the output response at steady-state conditions should be considered and finally the frequency nonlinear response can be built-up.

From the above considerations, the need to make these types of calculations more efficient is really deemed. The strategy for challenging the problem takes advantages from the following concepts:
Localized nonlinearity: the nonlinear behavior is restricted to few degrees of freedom out of thousands of linear ones. It can be simulated by the use of nonlinear lumped elements, which connect pairs of DOFs. Hence the nonlinear properties can be expressed in terms of stiffness and damping nonlinear properties.

Steady-state conditions: the frequency response analysis, for definition, considers the response of the structure in steady-state conditions. In literature it is also called ‘harmonic response’, because the response is expected to be harmonic and at the same frequency of the harmonic excitation. It means that the response in the transient period can be avoided during the calculation and only the steady-state phase is of interest.

Modal Decomposition Approach: as consequence of the first two concepts, it can be said that the NLFRFs can be obtained updating the LFRFs, because the ‘nonlinear modifications’ don’t affect significantly the linear modal base.

The methodology presented in this work extends the SDM Method, usually applied to a linear set of lumped modifications [1], [2], to the nonlinear field.

2 LINEAR SDM METHOD.

The linear SDM has been widely discussed in many papers even if, few of them deal with the method and implement it numerically in order to face industrial applications. A very good theoretical approach can be found in the work [1]. Significant effort and work can be found in the publications [2], [3]. In this paragraph a quick view to the linear SDM is given, because the understanding of the nonlinear algorithm requires the deep understanding of the linear one.

SDM is defined as the procedure which permits one to evaluate the impact of a set of changes on the structural dynamic behavior, without the need to continuously re-run the FEM Model. The modified dynamic behavior can be expressed as function of the baseline FEM Dynamic Database and the set of modifications. Many authors have formulated and completed the theoretical problem, highlighting that the method becomes particularly efficient if into the modification lumped elements are involved. Lumped modifications consist of whatever relationship between two degrees of freedom, both of the structure or one degree of freedom belonging to the structure and another one belonging to an external fixed point. Usually the relationship is expressed as a combination of lumped masses, spring and damper elements.

The baseline FEM Dynamic Database can be expressed in terms of Modal Base [3] or LFRF [1, 2]. In this study the LFRF Database is considered, in which case the ‘baseline’ Transfer Function Matrix (TFM) needs to be available.

By means of SDM one can expresses the modified TFM as follow:

\[ H_{\text{mod}}(\omega) = \left[ I + H_0(\omega) \Delta B(\omega) \right]^{-1} H_0(\omega) \]  \hspace{1cm} (1)

Where \( H_0(\omega) \) is the baseline TFM,

\[ \Delta B(\omega) = \left[ -\omega^2 \Delta M(\omega) + j\omega \Delta C(\omega) + \Delta K(\omega) \right] \]  \hspace{1cm} (2)

\( \Delta B \) the Modification Matrix, being \( M, C, K \), the Mass, Damping and Stiffness Matrices of the Baseline Finite Element Model and \( \Delta M, \Delta C, \Delta K \) the increments of these terms due to the modification.

It is clear that, the knowledge of the baseline TFM and the definition of the modification matrix, allow the direct calculation of the modified TFM without any need to re-run the FEM. Seeing at the equation (1), it is worth pointing out that the linear SDM formulation allows dealing with frequency-dependent lumped elements. The use of frequency dependent Modification Matrix, in frequency domain, is still a linear problem. Also it is to be notice that the size of the FEMs, in terms of DOFs, is not a matter, in fact, the baseline TFM can be restricted to few DOFs. This set must, at least, include the DOFs where the change is required and those ones where the dynamic response needs to be evaluated. More details on this last point are given in references [1], [2], [3].

3 NONLINEAR SDM METHOD.

The frequency-dependency of the modification matrix doesn’t affect the linearity of the problem; therefore nonlinear lumped elements are considered those elements, whose constitutive equations, express dependency on the relative amplitude. In the linear case, the Transfer Function (TF), defined, for each degree of freedom, as the ratio between the output acceleration, or displacement, and the input force, is an invariant of the system. But, if the structure acts nonlinearly, the TF depends on magnitude of the input force. Hence in the nonlinear case, even if, the same symbols are still used, it is more correct to speak about NLFRFs Matrix instead of TF Matrix. The NL TF Matrix can be obtained later dividing the NLFRFs Matrix by the input force value. Evidently the NL TF Matrix is not unique and is dependent on the magnitude of the input force. The equations (1) and (2) are still valid but if we define ‘\( u \)’ as the relative displacement experimented by the lumped elements, the Modification Matrix, in the nonlinear case looks like:

\[ \Delta B(\omega, u) = \left[ -\omega^2 \Delta M(\omega, u) + j\omega \Delta C(\omega, u) + \Delta K(\omega, u) \right] \]  \hspace{1cm} (3)

Hence the amplitude dependency of the Modified FRFs Matrix is obtained:

\[ H_{\text{mod}}(\omega, u) = \left[ I + H_0(\omega) \Delta B(\omega, u) \right]^{-1} H_0(\omega) \]  \hspace{1cm} (4)

This nonlinear equation, where all the nonlinear terms are included in the Modification Matrix, requires an iterative algorithm in order to be solved.

The algorithm is explained by the following example. Considering a bilinear stiffness element:

\[ F \rightarrow k_1 \rightarrow u_a \rightarrow n_a \rightarrow u_b \rightarrow k_2 \rightarrow \Delta \]

The Modification Matrix associated with this element is:
\[
\Delta B_{\text{elem}}(\omega, u) = \begin{bmatrix}
\Delta K_{\text{elem}}(\omega, u) & -\Delta K_{\text{elem}}(\omega, u) \\
-\Delta K_{\text{elem}}(\omega, u) & \Delta K_{\text{elem}}(\omega, u)
\end{bmatrix}
\]

Being the constitutive equation of the element:

\[
\begin{cases}
\Delta K_{\text{elem}}(\omega, u) = k_1 - k_0 & \text{if } u \leq u_A \\
\Delta K_{\text{elem}}(\omega, u) = k_{\text{equ}} - k_0 & \text{if } u > u_A
\end{cases}
\]

\[
k_{\text{equ}} = k_1 \cdot u_A + k_2 \cdot (u - u_A)
\]

Even if not strictly required, it is assumed that the underline linear element has the stiffness value \(k_0\), which is not so far from the nonlinear values. For instance \(k_0\) could be the mean value between \(k_1\) and \(k_2\).

Fixed a point in the frequency-domain and assuming that the first condition of the equation (6) happens, the Modified FRFs Matrix is obtained by mean of equation (5). Clearly, this is a trial value because the initial hypothesis on the relative displacement needs to be confirmed. Therefore, the next step transforms the displacements of both DOFs, from frequency-domain to time-domain. The period of time depends on the frequency value, being the inverse of this, and the number of points in time period has to be sufficient to get properly the peak of the oscillatory response.

The Inverse Fourier Transform is:

\[
\{u(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{\mathcal{F}(u)\} e^{i\omega t} \, d\omega
\]

\[
\mathcal{F}(u) = \{u(t)\} = \left\{ \begin{array}{l}
\text{abs}(H_{a,lc}(\omega)) \exp(i(\omega t + \text{phase}(H_{a,lc}(\omega)))) \\
\text{abs}(H_{b,lc}(\omega)) \exp(i(\omega t + \text{phase}(H_{b,lc}(\omega))))
\end{array} \right\}
\]

Where:
- \(\text{abs}(H_{a,lc}(\omega))\) and \(\text{phase}(H_{a,lc}(\omega))\) are, respectively, the module and the phase of the displacement at the DOF ‘a’ of the lumped element, obtained from the Modified TFM and in the load case of interest ‘lc’.
- \(\text{abs}(H_{b,lc}(\omega))\) and \(\text{phase}(H_{b,lc}(\omega))\) are, respectively, the module and the phase of the displacement at the DOF ‘b’ of the lumped element, obtained from the Modified TFM and in the load case of interest ‘lc’.

The relative displacement is:

\[
u(t) = u_b(t) - u_a(t)
\]

The convergence is required in each point of the time period and is achieved when:

\[
\frac{u(t)_{\text{step},N+1} - u(t)_{\text{step},N}}{u(t)_{\text{step},N}} \times 100 \leq \text{toler} \%
\]

The tolerance value is usually in the range [0.5% - 1%].

This process is clearly required for each point in frequency-domain. At the end of each frequency step the maximum values of the time-domain displacements, at all the DOFs involved in the LFRFs Baseline Matrix, will be stored in order to built-up the NLFRFs Matrix.

If more than one lumped element is involved into the modification, the Modification Matrix is:

\[
\Delta B_{BB}(\omega, u) = \sum_{p=1}^{N_{\text{elem}}} \Delta B_p(\omega, u)
\]

The index ‘BB’ specifies the DOFs of the baseline FRFs Matrix affected by the modifications [1], [2].

In this case the convergence criteria need to be satisfied for all the relative displacements of the nonlinear lumped elements. Alternately, an objective function, involving all the relative displacements, can be defined, to speed-up the convergence of the problem.

\[
\text{OF}(t) = \sqrt{\sum_{p=1}^{N_{\text{elem}}} (u_{b,p}(t) - u_{a,p}(t))^2}
\]

And coherently:

\[
\frac{\text{OF}(t)_{\text{step},N+1} - \text{OF}(t)_{\text{step},N}}{\text{OF}(t)_{\text{step},N}} \times 100 \leq \text{toler}
\]

4 NONLINEAR SDM APPLIED TO THE APU SUSPENSION SYSTEM FE MODEL.

The focus of this study is on the interfaces between the Auxiliary Power Unit (APU) and its Suspension System. The APU is installed in the A346 Tail-Cone by its Suspension System, which has a double purpose: to sustain the inertia loads at which the APU is submitted and to isolate the airframe from the APU’s vibrations.

The Suspension System consists of 3 principals subassembly called:
- Left-Hand: 3 rods, 3 APU lugs on structure side, 1 Rubber Mount
- Right-Hand: 2 rods, 2 APU lugs on structure side, 1 Rubber Mount
- Aft-Hand: 2 rods, 2 APU lugs on structure side, 1 Rubber Mount

Each rubber mount is done by a steel isolator housing with an elastomeric inside.
In the FE Model the rubber mounts can be modelled by mean of spring/damper elements connecting pair of nodes. Each rubber mount is described by three lumped spring/damper elements, one for the axial direction and other two for the radial one. In this work the focus is on the stiffness properties, the damping is assumed to be fixed and its values is calculated from the area of the hysteretic loop. It is introduced in the FE Model as localized structural damping.

![Figure 2 – APU Suspension System](image1)

Experimentally, it is seen that the dynamic stiffness of the rubber mounts depend, both, on the frequency and on the dynamic amplitude. The following figure shows an example of the trend of the dynamic stiffness when the rubber mount is stretched by an axial force:

![Figure 3 – Dynamic Stiffness of Rubber Mount](image2)

![Figure 4 – Rubber Mount Axial Test](image3)

In the FE Model the rubber mounts can be modelled by mean of spring/damper elements connecting pair of nodes. Each rubber mount is described by three lumped spring/damper elements, one for the axial direction and other two for the radial one. In this work the focus is on the stiffness properties, the damping is assumed to be fixed and its values is calculated from the area of the hysteretic loop. It is introduced in the FE Model as localized structural damping.

![Figure 5 – Hysteretic Loop of Rubber Mount](image4)

![Figure 6 – A340-600 FE Model](image5)

The model used is the overall A340-600 model. The overall model is used in order to show that the methodology doesn’t present any limitation concerning the size of the model. In the figure (6), (7), (8) some details can be seen. NASTRAN is used for the linear FEM. The linear behavior of the rubber mounts is introduced by CBUSH cards. As highlighted, the frequency dependent behavior in frequency-domain is still a linear problem. It can be directly considered in NASTRAN by the combined use of PBUSH and PBUSHT cards.
Summarizing, the book-case presents the following:

- Overall linear A340-FE Model is used
- Constraints are applied to nose and main landing gear.
- The input load is applied to the APU Center of Gravity in vertical (Z+) direction, see figure (9). The APU is modelled like a heavy mass, using the CONM2 card of NASTRAN.
- The rubber mounts in the linear NASTRAN model are modelled via CBUSH elements.
- The results are evaluated at the DOFs of the rubber mounts and at the DOFs of the interfaces between the suspension system and the structure of the Tail-Cone, see figure (9).
- The linear TF Matrix, at the DOFs of interest, is obtained running the SOL111 of NASTRAN code, considering the overall aircraft.

Nonlinear ‘update’:

- The dynamic stiffness of the rubber mounts has been considered according a polynomial function of the type in Table (1). The parameters which define this type of function come from a fitting of the rubber mount experimental data, as illustrated in figure (3). Normally each rubber mount has its own fitting but, in this case, mean values are considered and applied to the three isolators.
- 3 load cases are considered: 250 N, 500 N, and 800 N. The LFRFs database, for each load case, is obtained multiplying the linear TF Matrix for the respective input.
- The NL TFs are expected to be dependent on the load case. They are obtained by mean of the nonlinear SDM Method.

### Table 1 – Nonlinear Stiffness

<table>
<thead>
<tr>
<th>Relative Displacement</th>
<th>Stiffness</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u(t) \leq u_{\text{min}}$</td>
<td>$K_{\text{NL}}(u(t)) = \frac{K_{\text{max}}}{u_{\text{min}}} u(t)$</td>
</tr>
<tr>
<td>$u(t) &gt; u_{\text{min}}$</td>
<td>$K_{\text{NL}}(u(t)) = K_{\text{max}} + \frac{(K_{\text{max}} - K_{\text{min}}) * (u(t) - \delta)}{u(t) - \delta}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{\text{max}}$</td>
<td>$1.4 * K_0$</td>
</tr>
<tr>
<td>$K_{\text{min}}$</td>
<td>$0.5 * K_0$</td>
</tr>
<tr>
<td>$u_{\text{min}}$</td>
<td>0.0006 [mm]</td>
</tr>
<tr>
<td>$\delta$</td>
<td>$\approx -0.03$</td>
</tr>
</tbody>
</table>

The dynamic behavior, evaluated at the DOFs of interest, is expected to change according the type of nonlinearity introduced in the system. Having a softening effect, some peaks of the TF are expected moving to lower frequencies.

In spite of the size of the FE Model, the LFRFs Matrix need to contain only the DOFs involved into the modifications, the DOFs where the input load is applied, and those where the dynamic response is required. The DOFs involved into the modification are 18 (3 rubber mounts x 3 spring/rubber x 2 DOFs/spring), 1 where the load is applied and 21 at the interfaces with the structure.

According the equations (3) and (4), and following the iteration procedure, the NLFRFs, for each of the load cases, are obtained and divided by the respective input load. This allows evaluating the NL TF Matrix for each case.

The following figures show the results for the DOF ‘27’ in terms of output accelerations divided by input force. The frequency range of interest is between 10 and 20 Hz where the APU Suspension System Modes are. Experimentally it is also known that the rubber mount dynamic behavior affects the response above the 12-14 Hz.

The effect of the localized nonlinear stiffness is very important:

- Figure (10) shows that comparing the black dashpot line (linear TF) against the other ones (NL TFs), particularly above the 14 Hz, when the rubber mount behavior plays a very important role, the TF changes completely. In this case for instance it can be seen how
the linear behavior overestimated the dynamic amplification in the range 10 – 18 Hz.

- Figure (11) shows a detail of the TF’s peaks for different load cases. As expected the higher the load, the lower is the frequency of the peak.

5 CONCLUSION

A novel approach to update the linear FRFs with localized nonlinearities has been presented. It extends the well-known SDM Method, based on FRFs updating, to a set of nonlinear modifications by mean of an iterative procedure. The linear method is reviewed and the nonlinear one is explained through a bilinear stiffness element.

The procedure is very attractive from a computational point of view because the FE Model needs to run just once in order to obtain the LFRFs Matrix. No matter about the size of the model because the LFRFs Matrix is required only to contain the DOFs involved into the Modification Matrix, those ones where the loads are applied and those ones where the dynamic response is required. After that, the estimation of the nonlinear behavior in frequency domain is obtained without any need to re-run the model, being just an updating, by means of the nonlinear SDM Method, of the linear one.

The approach is very practical and theoretically whatever type of nonlinear lumped element can be implemented. Both dependency, on frequency and dynamic amplitude can be considered at the same time. The method has been successfully implemented in MATLAB and an industrial case is presented. The APU Suspension System is studied, because its dynamic behavior is affected by the nonlinearities introduced in the path load through the rubber mounts.

This is a clear case where most of the FE Model can be considered acting linearly and the nonlinear behavior is restricted to few degrees of freedom. In these cases the method is particularly powerful. Comparing the linear and the nonlinear behavior, the importance of taking into account the nonlinearities in the frequency response is highlighted: in the frequency range where the effect of the nonlinearities is dominant, the dynamic response of the system changes completely. Comparing different load cases, with increasing level of load, the TF’s peaks move to lower frequencies, as expected by the type of nonlinearity introduced in the system.

Finally, even if in this paper only the nonlinear stiffness is treated, the same concept can be extended to the damping properties of the elements, allowing in a pretty easy and straight way, dealing with structural dynamics behavior, most of times, very complex to simulate.

REFERENCES