Assessment of damage indicators for structural health monitoring using a probabilistic framework

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ABSTRACT: Civil and mechanical structures are exposed to natural ageing and accidental overloading. To increase safety and reduce costs simultaneously, current classical maintenance strategies can be supported by advanced fully automatic vibration-based structural health monitoring (SHM) systems.

Such vibration-based SHM systems consist of a data acquisition system with sensors applied on a structure. From recorded time histories, damage indicators are generated and can be monitored in control charts. Sensitivity to damage and insensitivity to confounding effects are essential properties for a well-designed indicator. To design such damage indicators, experimental tests are inevitable. As such tests can be expensive with respect to time and labor costs, researchers are interested in replacing these tests by numerical simulations. Recently, a semi-analytical probabilistic framework has been proposed by the authors to determine the statistical properties of damage indicators based on the known statistical properties of a random excitation in the time domain and a finite element model. Therefore, a time consuming numerical time integration scheme, as typically applied for each sample of a sample-based stochastic structural analysis, can be avoided.

In the presented paper this probabilistic framework is applied to compare efficiently two different damage indicators based on modal filters using a numerical cracked concrete beam model. For this numerical example, the statistical properties of the damage indicators will be investigated with progressing damage. From these statistical properties, it can be determined under which conditions the distribution of the damage indicators can be assumed to be normal and, therefore, if the application of certain control charts is justified.

KEY WORDS: Uncertainty quantification, damage detection, structural health monitoring, structural dynamics, random vibrations

1 INTRODUCTION

Vibration-based structural health monitoring can be a very effective tool to support efficiently traditional maintenance strategies, and therefore reduce operational costs of civil or mechanical structures and systems. An advantage of vibration-based structural health monitoring (SHM) in comparison to other structural health monitoring systems, is the use of ambient excitation sources to introduce energy into the structure. Hence, no additional artificial excitation source is needed, which allows keeping the structure in operation during investigations.

Such a vibration-based SHM system consists of a data acquisition system with sensors applied on a structure. The recorded time histories are postprocessed and features are extracted to construct damage indicators. These damage indicators are then monitored over the life time of the structure using for example control charts, from which alerts are triggered if certain control limits are crossed. Finally, the alerts initiate the decision making. Therefore, the damage indicators are a very important part of the whole structural monitoring system. They need to be sensitive to damage and insensitive to confounding effects, like measurement noise, operational, and environmental conditions. In addition, several control charts (e.g., X-bar, Hotelling-T²) require normally distributed damage indicators to set meaningful control limits. Due to the variation of the excitation for each time interval and random measurement errors, responses and respectively damage indicators are time variant even if damage is not propagating. Consequently, the statistics of damage indicators as a function of the statistics of random excitations and measurement errors are important for the definition of control limits and the assessment of the suitability of damage indicators.

The motivation of this paper is the investigation of the statistical properties of two indicators based on modal filters, \( I_p \) and \( m_a \), proposed in [1][2][3] and applied on laboratory structures in [4][5], with respect to the time length of measured response time histories and intensity of measurement errors. Traditionally, such investigations were performed by the use of experimental data gained from expensive testing campaigns [6]. More and more, such experimental tests have been replaced by numerical simulation strategies using, for example, intrinsic Monte Carlo sampling coupled with a time integration scheme [7] or more advanced with the application of meta models [8].

In this paper, a recently proposed approach for uncertainty quantification [9][10] is applied to derive analytically the statistics of linearly combined response Fourier transforms. Based on these statistics, the statistics of damage indicators are obtained by generating and evaluating Latin hypercube samples. The numerical representation of the structure is assumed to be deterministic. To create realistic damage patterns, the results of the damage propagation calculations performed in [11] are
reused. Therein, the constitutive law parameters have been determined by an optimization-based model calibration strategy [12].

For the variation of time length and measurement error intensity, the investigations of the probability density functions of the two damage indicators show a very different behavior. By assessing the performance of the indicator with the false positive probability, strong similarities between both indicators can be observed. Finally, both indicators are equally capable to identify small damages.

2 UNCERTAINTY QUANTIFICATION OF DISCRETE RESPONSE FOURIER TRANSFORMS

This section reviews the main theoretical aspects of the probabilistic framework introduced in [10], applied in this paper.

2.1 General concept and problem description

Assuming a set of continuous response time signals \( x(t) \in \mathbb{R}^{m_r} \) for \( m_r \) degrees of freedom over time \( t \) of a structure is given by

\[
x(t) = x_f(t) + x_n(t). \tag{1}
\]

The random continuous time signal \( x_f(t) \in \mathbb{R}^{m_r} \) is the true errorless response resulting from a random continuous weakly stationary non-periodic excitation signal \( f(t) \in \mathbb{R}^{m_f} \) at \( m_f \) degrees of freedom. The random continuous time signal \( x_n(t) \in \mathbb{R}^{m_r} \) is the weakly stationary measurement error at the respective degrees of freedom. In practical applications, many pre- and postprocessing techniques are applied on the measured response signal. A typical operator is a linear combiner, which is defined as

\[
g(t) = A x(t) = A x_f(t) + A x_n(t) \tag{2}
\]

in the time domain, with \( g(t) \in \mathbb{R}^{m_r} \) and the time-invariant matrix of linear combination coefficients \( A \in \mathbb{R}^{m_r \times m_r} \). In addition, windowing in the time domain is often applied prior to a further signal processing, such as Fourier transformation.

From now on, the random continuous excitation \( f(t) \) is assumed to be a multivariate normal distribution with known expectations and covariances. By extracting a finite discrete random excitation signal \( f_j \) from \( f(t) \) with \( N \) discrete values for each excited degree of freedom \( j = 1, 2, \ldots, m_f \), the corresponding random discrete vector

\[
\bar{f} = \left[ f_1^T \quad f_2^T \quad \cdots \quad f_{m_f}^T \right]^T \tag{3}
\]

of dimension \( N m_f \) can be defined as multivariate normally distributed

\[
\bar{f} \sim \mathcal{N} \left( E(\bar{f}), C(\bar{f}, \bar{f}) \right) \tag{4}
\]

with an expectation vector \( E(\bar{f}) \in \mathbb{R}^{N m_f} \) and a covariance matrix \( C(\bar{f}, \bar{f}) \in \mathbb{R}^{N m_f \times N m_f} \), including correlations in time and space.

The random continuous signal of measurement errors \( x_n(t) \) is also assumed to be multivariate normally distributed. The finite discrete random errors \( x_n_j \) obtained from \( x_n(t) \) with \( N \) discrete values for each measured response degree of freedom \( j = 1, 2, \ldots, m_r \) are collected in a random discrete vector

\[
\bar{x}_n = \left[ x_{n1}^T \quad x_{n2}^T \quad \cdots \quad x_{nm_r}^T \right]^T. \tag{5}
\]

This random vector can be described as normally distributed

\[
\bar{x}_n \sim \mathcal{N} \left( E(\bar{x}_n), C(\bar{x}_n, \bar{x}_n) \right) \tag{6}
\]

with the expectation vector \( E(\bar{x}_n) \in \mathbb{R}^{N m_r} \) and the covariance matrix \( C(\bar{x}_n, \bar{x}_n) \in \mathbb{R}^{N m_r \times N m_r} \), which takes correlations in time and space into account. In this paper, it is assumed that measurement errors and excitations are independent from each other.

To derive the statistics of the discrete Fourier transform of a linear combiner of responses based on the known statistics of the excitation and the measurement errors, a linear deterministic operator \( \bar{Z}_f \) is derived to perform analytically the uncertainty propagation between the random excitations in the time domain and the response Fourier transforms of the linear combiner. The uncertainty propagation of the measurement errors defined in the time domain can also be realized by a linear operator \( \bar{Z}_n \).

The mean value of the discrete Fourier transform of the errorless linear combiner can be obtained by

\[
E \left( \left[ \begin{array}{c} \text{Re}(\mathcal{F}(\bar{f})) \\ \text{Im}(\mathcal{F}(\bar{f})) \end{array} \right] \right) = \left[ \begin{array}{c} \text{Re}(\bar{Z}_f) \\ \text{Im}(\bar{Z}_f) \end{array} \right] E(\bar{f}) \tag{7}
\]

and its corresponding covariance matrix by

\[
C \left( \left[ \begin{array}{c} \text{Re}(\mathcal{F}(\bar{f})) \\ \text{Im}(\mathcal{F}(\bar{f})) \end{array} \right] , \left[ \begin{array}{c} \text{Re}(\mathcal{F}(\bar{f})) \\ \text{Im}(\mathcal{F}(\bar{f})) \end{array} \right] \right) = \left[ \begin{array}{c} \text{Re}(\bar{Z}_f) \\ \text{Im}(\bar{Z}_f) \end{array} \right] C(\bar{f}, \bar{f}) \left[ \begin{array}{c} \text{Re}(\bar{Z}_f) \\ \text{Im}(\bar{Z}_f) \end{array} \right]^T \tag{8}
\]

where \( \text{Re}(\cdot) \) and \( \text{Im}(\cdot) \) are the real and imaginary part of a complex scalar, vector, or matrix. In analogy, the mean value

\[
E \left( \left[ \begin{array}{c} \text{Re}(\mathcal{F}(\bar{x}_n)) \\ \text{Im}(\mathcal{F}(\bar{x}_n)) \end{array} \right] \right) = E \left( \left[ \begin{array}{c} \text{Re}(\mathcal{F}(\bar{f})) \\ \text{Im}(\mathcal{F}(\bar{f})) \end{array} \right] \right) + E \left( \left[ \begin{array}{c} \text{Re}(\mathcal{F}(\bar{Z}_f) \bar{x}_n) \\ \text{Im}(\mathcal{F}(\bar{Z}_f) \bar{x}_n) \end{array} \right] \right) \tag{9}
\]

and

\[
C \left( \left[ \begin{array}{c} \text{Re}(\mathcal{F}(\bar{x}_n)) \\ \text{Im}(\mathcal{F}(\bar{x}_n)) \end{array} \right] , \left[ \begin{array}{c} \text{Re}(\mathcal{F}(\bar{x}_n)) \\ \text{Im}(\mathcal{F}(\bar{x}_n)) \end{array} \right] \right) = C \left( \left[ \begin{array}{c} \text{Re}(\mathcal{F}(\bar{f})) \\ \text{Im}(\mathcal{F}(\bar{f})) \end{array} \right] , \left[ \begin{array}{c} \text{Re}(\mathcal{F}(\bar{f})) \\ \text{Im}(\mathcal{F}(\bar{f})) \end{array} \right] \right) + C \left( \left[ \begin{array}{c} \text{Re}(\mathcal{F}(\bar{Z}_f) \bar{x}_n) \\ \text{Im}(\mathcal{F}(\bar{Z}_f) \bar{x}_n) \end{array} \right] , \left[ \begin{array}{c} \text{Re}(\mathcal{F}(\bar{Z}_f) \bar{x}_n) \\ \text{Im}(\mathcal{F}(\bar{Z}_f) \bar{x}_n) \end{array} \right] \right). \tag{10}
\]

A detailed description of the assembly of the vector \( \bar{g} \) and its discrete Fourier transform \( \mathcal{F}(\bar{g}) \) will be given in the following subsections. In this paper, discretization errors are assumed to be negligible by choosing a sufficiently small time step.

In the following subsections, the derivations of the linear operators, \( \bar{Z}_f \) and \( \bar{Z}_n \), are explained in detail. The time-invariant modal system properties are assumed to be deterministic.
2.2 Analytical uncertainty propagation for errorless signal

Using Duhamel’s integral for multiple inputs and multiple outputs given in [13, p.95], the linear combination of the continuous response displacements in the time domain of a proportional viscously damped linear system under continuous excitation \(f(t) \in \mathbb{R}^m\) for \(t \geq 0\) can be obtained by

\[
\mathbf{g}_t(t) = \int_0^t h(t - \tau)f(\tau)d\tau
\]

with \(\mathbf{g}_t(t) \in \mathbb{R}^m\) and the matrix of linear combination coefficients \(\mathbf{A} \in \mathbb{R}^m \times m_s\). The impulse response function \(h(t) \in \mathbb{R}^m \times m_f\) is given by

\[
h(t) = \mathbf{A} \mathbf{\Phi}_s \mathbf{d}(t) \Phi_f^T, \tag{12}
\]

where \(\mathbf{d}(t) \in \mathbb{R}^{m_f \times m_s}\) represents the time dependent diagonal matrix with diagonal elements

\[
(d(t))_{l,j} = \frac{\sin(\sqrt{\lambda_l}(1 - \zeta_l^2)\tau)}{\sqrt{\lambda_l}(1 - \zeta_l^2)} \exp(-\sqrt{\lambda_l}(\zeta_l)t). \tag{13}
\]

It is assumed that the modal properties of the system are time invariant. The modal properties for \(m_s\) considered modes are the classical undamped eigenvalues \(\lambda \in \mathbb{R}^{m_s}\), the corresponding modal damping ratios \(\zeta \in \mathbb{R}^{m_s}\), and the eigenvector matrix \(\Phi\). The mode shape matrices of response degrees of freedom \(\Phi_s \in \mathbb{R}^{m_s \times m_s}\) and of excitation degrees of freedom \(\Phi_f \in \mathbb{R}^{m_f \times m_s}\) are assembled from the mass normalized eigenvector matrix \(\Phi\).

From the continuous infinite linear combiner \(\mathbf{g}(t)\), a finite time frame is extracted through a window function \(w(t)\) with finite compact support within \(t_r < t < t_c\). The application of this window function leads to the response of the \(i\)th linear combiner of the finite time frame

\[
g_{wi}(t) = w(t) \sum_{j=1}^{m_j} \int_0^t (h(t - \tau))_{i,j} (f(\tau))_{j} d\tau. \tag{14}
\]

By introducing a time step \(\Delta t\) and defining \(t_s = (s - 1) \Delta t\) and \(t_e = e \Delta t\) with the positive integer values \(s, e \in \mathbb{Z}\) and \(0 < s < e\), the discrete form of Eq. (14)

\[
g_{wi} = w \circ \sum_{j=1}^{m_j} q_{i,j} f_j \tag{15}
\]

is derived in the time step interval \([s, e]\) with \(g_{wi} \in \mathbb{R}^{m_w}\) and \(m_w = e - s + 1\). The symbol \(\circ\) denotes the Schur product (e.g., [14]). The vector \(w \in \mathbb{R}^{m_w}\) represents the discretization of the window function \(w\) within the support \([s, e]\). The vector of the \(j\)th degree of freedom of the excitation \(f_j \in \mathbb{R}^N\) is given for all discrete time steps \([1, N]\) with \(N = e\). The matrix \(q_{i,j} \in \mathbb{R}^{m_f \times N}\) is derived from the impulse response function related to the linear combiner \(i\) and the excitation degree of freedom \(j\).

\[
\mathbf{q}_{i,j} = \Delta t \begin{bmatrix} h_{i,j,s} & h_{i,j,s+1} & \cdots & h_{i,j,s-2} & 0 \\ h_{i,j,s+1} & h_{i,j,s} & \cdots & h_{i,j,s-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{i,j,s+2} & h_{i,j,s+1} & \cdots & h_{i,j,s} & 0 \\ h_{i,j,e-2} & \vdots & \cdots & h_{i,j,e-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{i,j,e} & h_{i,j,e-1} & \cdots & h_{i,j,e-2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{i,j,e-N+1} & \vdots & \cdots & h_{i,j,e-N+2} & 0 \end{bmatrix}
\]

\forall n \leq 0 : h_{i,j,n} = 0 \tag{16}

In a next step the discrete Fourier transformation is applied to Eq. (15) through a complex matrix operator \(\mathbf{B} \in \mathbb{C}^{(m_f+1) \times m_s}\) containing the coefficients [15, p. 50]

\[
(\mathbf{B})_{k,n} = \Delta \exp(-\frac{2\pi i}{N}(k-1)(n-1)) \tag{17}
\]

with the imaginary unit \(i = \sqrt{-1}\) for \(n = 1, 2, \ldots, m_w\) and \(k = 1, 2, \ldots, \frac{m_u}{2} + 1\). This yields

\[
\mathbf{F}_{\text{gw}_i} = \mathbf{B} \left( w \circ \sum_{j=1}^{m_f} q_{i,j} f_j \right) \tag{18}
\]

with \(\mathbf{F}_{\text{gw}_i} \in \mathbb{C}^{(m_f+1) \times m_s}\). Eq. (18), which represents the Fourier transform of the \(i\)th linear combination for windowed responses, can be reformulated to

\[
\mathbf{F}_{\text{gw}_i} = \sum_{j=1}^{m_f} \mathbf{Z}_{fi,j} f_j \tag{19}
\]

using

\[
\mathbf{Z}_{fi,j} = \mathbf{B} \left( (1^{1 \times N} \otimes w) \circ q_{i,j} \right), \tag{20}
\]

where \(\otimes\) denotes the Kronecker product. The matrix \(1^{1 \times N}\) is an integer matrix of dimension \(1 \times N\) only filled with the value 1.

By combining the evaluations of Eq. (19) for all linear combinations \(i = 1, 2, \ldots, m_g\) and all excitation degrees of freedom \(j = 1, 2, \ldots, m_f\) in the random vector and linear deterministic operator,

\[
\mathbf{\mathcal{F}}_{\mathbf{R}} = \begin{bmatrix} \mathcal{F}_{\mathbf{gw}_1} \\ \mathcal{F}_{\mathbf{gw}_2} \\ \vdots \\ \mathcal{F}_{\mathbf{gw}_{m_g}} \end{bmatrix} \tag{21}
\]

and

\[
\mathbf{\tilde{Z}}_f = \begin{bmatrix} \mathbf{Z}_{f_1,1} & \mathbf{Z}_{f_1,2} & \cdots & \mathbf{Z}_{f_1,m_f} \\ \mathbf{Z}_{f_2,1} & \mathbf{Z}_{f_2,2} & \cdots & \mathbf{Z}_{f_2,m_f} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Z}_{fm,1} & \mathbf{Z}_{fm,2} & \cdots & \mathbf{Z}_{fm,m_f} \end{bmatrix} \tag{22}
\]

respectively, a simple linear relation

\[
\mathbf{\mathcal{F}}_{\mathbf{R}} = \mathbf{\tilde{Z}}_f \mathbf{\bar{f}} \quad \text{with} \quad \mathbf{\tilde{Z}}_f \in \mathbb{C}^{(m_f+1) \times N \times m_g} \tag{23}
\]
2.3 Analytical uncertainty propagation for measurement errors

The $i$th linear combiner related to the measurement errors is given by

$$g_{aw,i} = \sum_{j=1}^{m_i} (A)_{i,j} q_{w} x_{n,j}$$

(24)

with $g_{aw,i} \in \mathbb{R}^{m_w}$. The vector $x_{n,j} \in \mathbb{R}^{N}$ represents the measurement error of the $j$th degree of freedom of the response signal. The matrix $q_{w} \in \mathbb{R}^{m_w \times N}$ is defined as

$$q_{w} = [0_{m_w \times s-1}, w \mathbf{1}^{m_w}]^T,$$

(25)

where $\mathbf{1}^{m_w}$ represents the identity matrix of dimension $m_w$. The vector $w$ is assembled by the discrete values of the support values of the window function $w(t)$ as defined in Subsection 2.2. The matrix $0_{m_w \times s-1}$ is a zero valued matrix of dimension $m_w \times s-1$. The expression $q_{w} x_{n,j}$ is the windowed time frame of the measurement errors at the $j$th degree of freedom.

Similar to Section 2.2, the matrix of discrete Fourier coefficients $B$ is applied to Equation (24) to obtain the discrete Fourier transform of linearly combined windowed measurements errors

$$\mathcal{F}_{raw,i} = B \sum_{j=1}^{m_i} (A)_{i,j} q_{w} x_{n,j},$$

(26)

which can be rewritten as

$$\mathcal{F}_{raw,i} = \sum_{j=1}^{m_i} Z_{ni,j} x_{n,j}$$

(27)

with

$$Z_{ni,j} = (A)_{i,j} B q_{w},$$

(28)

where $\mathcal{F}_{raw,i} \in \mathbb{C}^{(\frac{m_w}{2}+1)}$ and $Z_{ni,j} \in \mathbb{C}^{(\frac{m_w}{2}+1) \times N}$. The expressions

$$\mathcal{F}_{zn} = \begin{bmatrix} \mathcal{F}_{raw,1} \\ \mathcal{F}_{raw,2} \\ \vdots \\ \mathcal{F}_{raw,m_g} \end{bmatrix}$$

(29)

and

$$Z_n = \begin{bmatrix} Z_{n_{1,1}} & Z_{n_{1,2}} & \cdots & Z_{n_{1,m_i}} \\ Z_{n_{2,1}} & Z_{n_{2,2}} & \cdots & Z_{n_{2,m_i}} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{n_{m_g,1}} & Z_{n_{m_g,2}} & \cdots & Z_{n_{m_g,m_i}} \end{bmatrix}$$

(30)

can be derived by evaluating Equation (27) for all possible linear combinations $i = 1, 2, \ldots, m_g$ and all possible response degrees of freedom $j = 1, 2, \ldots, m_s$.

Finally, a linear combination

$$\mathcal{F}_{zn} = Z_n x_n \quad \text{with} \quad Z_n \in \mathbb{C}^{(\frac{m_w}{2}+1)m_g \times N_{m_s}}$$

(31)

can be found that relates the random measurement errors of the responses indicated by $x_n \in \mathbb{R}^{m_{ls}}$ defined in Equation (5) to the discrete Fourier transform of the linearly combined windowed measurement errors summarized in $\mathcal{F}_{zn} \in \mathbb{C}^{(\frac{m_w}{2}+1)m_g}$.

3 DAMAGE DETECTION WITH PEAK INDICATORS

3.1 General concept

As shown in [7], the application of modal filters combined with peak indicators is very efficient to detect early damage using measured vibration data. The modal filters are designed with respect to a certain mode by defining the coefficients of a linear combiner matrix $A$ in order to eliminate the remaining modes within a certain frequency range of the Fourier transforms. Hence, only one significant peak is visible in the power spectral densities of the undamaged structure. The linear combination coefficients are defined once for the undamaged or reference state. With increasing damage, peaks at the eigenfrequencies of the modes previously eliminated appear. This appearance can be detected for each time frame of a vibration signal by a peak indicator. Figure 1 shows qualitatively the power spectral density of linearly combined responses. The applied modal filter was designed for the second mode of a structure. Due to the presence of random measurement errors and random excitation, the peak indicators are not constant over time (i.e. not deterministic) and can be characterized by its empirical probability density function. If a damage occurs, a change of the statistical properties can be observed, which can be monitored in control charts. The exceedance of control limits indicates a possible damage and initiates further decision making processes.

In order to obtain the statistics of the peak indicator for a certain load step within a virtual testing scheme, as foreseen in this paper, the following procedure is applied. First, the approach explained in Section 2 is used to define analytically the statistics of the Fourier transforms of linearly combined responses. From this multivariate distribution, Latin hypercube sample sets are generated to compute sample sets of power spectral densities. For each sample set of power spectral densities, one sample of a peak indicator can be obtained. Using all samples of the peak indicator an estimation of the probability density function and the corresponding statistics can be obtained. The aim of this paper is to investigate the suitability of two different peak indicators, $I_p$ and $m_d$, to detect small damages.

![Figure 1: Example of a power spectral density of a linear combiner using a modal filter tuned to the second eigenmode of the system. The grey frames indicate the frequency ranges used to calculate the peak indicators.](image-url)
3.2 Peak indicator \( I_p \)

In [16] a peak indicator was proposed that interprets any function of interest \( F \) with discrete equidistant values \( ( F )_k \) defined in the interval \([k_a, k_b]\) as a discrete probability density function. In this application \( F \) corresponds to the power spectral density (see Figure 1). Refs. [2][3] applied successfully this indicator in the context of damage detection.

According to [16], the peak indicator

\[
I_p = \frac{\sigma_2^2}{(k_b - k_a)^2}
\]

is derived from the quotient between the variance \( \sigma_2^2 \) of the investigated probability density function \( F \) within the interval \([k_a, k_b]\) and the variance of a fitted uniform distribution, which is equal to \( \frac{(k_b - k_a)^2}{12} \). Consequently, if the function \( F \) is constant within \([k_a, k_b]\) the indicator will be equal to one. Otherwise, the indicator is lower than one. Investigations of [16] showed that the indicator is close to one, even if the function \( F \) is not constant in the interval \([k_a, k_b]\), but follows a monotonic trend. Higher values than one are possible, if a peak appears near the boundaries \( k_a \) or \( k_b \) of the interval.

The mean value of a discrete distribution calculated from a nonnormalized discrete probability density function \( F \) defined in the interval \([k_a, k_b]\) is given by

\[
\mu_k = \frac{k_b - k_a}{k_b - k_a + 1} \left( \sum_{k=k_a}^{k_b} (F)_k \right) \quad (33)
\]

The second centered statistical moment, which is the variance, yields

\[
\sigma_2^2 = \frac{k_b - k_a}{k_b - k_a + 1} \left( \sum_{k=k_a}^{k_b} (F)_k \right)^2 - \mu_k^2 \quad (34)
\]

3.3 Peak indicator \( m_4 \)

An alternative to the indicator \( I_p \) is the indicator \( m_4 \), which has been originally introduced in [17] and has been adapted by [3][4] for the detection of damage using modal filters. The lower limit of this peak indicator is zero. The indicator tends to increase with increasing damage.

In contrast to the indicator \( I_p \), the discrete values \( (F)_k \) of a discrete function are interpreted directly as a set of samples. Their first order central moment

\[
\mu_k = \frac{1}{k_b - k_a + 1} \sum_{k=k_a}^{k_b} (F)_k \quad (35)
\]

and respective fourth order central moment

\[
\kappa_4 = \frac{1}{k_b - k_a + 1} \sum_{k=k_a}^{k_b} ((F)_k - \mu_k)^4 \quad (36)
\]

are given. A scaling of \( \kappa_4 \) has been introduced by [2] to decrease the sensitivity with respect to different loading intensities. Finally, the normalized indicator \( m_4 \) can be written as

\[
m_4 = \kappa_4 \left( \sum_{k=k_a}^{k_b} (F)_k \Delta \omega \right)^{-1} \quad (37)
\]

with \( \Delta \omega \) being the discrete frequency step.

4 BENCHMARK STUDY: SIMPLY SUPPORTED BEAM

4.1 Static loading and damage progress

In [11] a simply supported plain concrete beam has been studied under quasi static loading with progressing damage using an implicit gradient damage law [18]. The material and damage law parameters have been obtained from a model calibration [12]. Hence, the results of this study with respect to the damage patterns are very realistic, which is the motivation to use them in the subsequent investigation of damage indicators.

The geometry of the beam and the loading configuration is presented in Figure 2. The red patterns in Figure 2 describe a predamage (50% of original tensile strength) to enable multiple cracking of the beam. The load-deflection-curve for a three-point-bending test with measured displacements in vertical direction at the position of the loading is shown in Figure 3 with indications of the load steps (LS). A first crack growth has been observed at LS 17. By means of two selected load steps, the evolution of the damage pattern with progressing damage is illustrated in Figure 4. The damage severity related to LS 50 can be considered as small.

For the numerical simulations, the structure is modeled in 2D with a plane stress formulation and 9-node rectangular elements of side length between 2.5 and 5mm.

![Figure 2: Geometry of investigated simply supported plain concrete beam with static loading.](Image)

![Figure 3: Load-deflection-curve with indication of load steps (LS).](Image)

![Figure 4: Young’s modulus distribution around the cracks using the implicit damage law for load steps (LS) 50 and 60.](Image)
4.2 Dynamic loading and response

The first three circular bending frequencies of the undamaged structure are 227.1 Hz, 874.8 Hz, and 1703.7 Hz. The modal damping coefficient are assumed to be damage invariant and are set to 2.5% for each mode, which is a typical value for concrete. For the dynamic calculation, the damaged systems at each load step are linearized by extracting the global secant stiffness matrix.

The linear systems are excited by a random excitation introduced at five positions in vertical direction as shown in Figure 5. The excitation is realized by independent and identically distributed (i.i.d.) random variables with respect to time and space following a normal distribution with mean value zero and variance $2^{18} N^2$. The unit of the (co)variances of excitation is Newton to the power of two $[N^2]$. Therefore, the random excitation can be described by

$$\tilde{f} \sim N (E(\tilde{f}), C(\tilde{f}, \tilde{f}))$$

(38)

with

$$(E(\tilde{f}))_i = 0 \quad \text{and} \quad (C(\tilde{f}, \tilde{f}))_{ij} = \delta_{ij} 2^{18} N^2 \quad \forall i, j$$

(39)

using the Kronecker delta

$$\delta_{ij} = \begin{cases} 0 & : i \neq j \\ 1 & : i = j \end{cases}.$$  

(40)

The resulting strains of the structure are measured by an embedded chain of three fiber bragg grating sensors (FBGS) with a measurement length of 5cm each. The location is given in Figure 5. It is assumed that the level of dynamic excitation is low enough to avoid a further damage of the structure, but high enough to obtain a sufficient resolution of the measured strains. The excitation leads after 2s to response strain distributions with constant properties of a zero mean value and a variance of $10^{-8}$ for the undamaged system. For load step 65 a strain variance of about $10^{-5}$ has been observed.

The measurement errors are defined as i.i.d. random variables connected directly to the three measured strains of the responses with respect to time and space. They follow a normal distribution with a zero mean value and a variance of $v^2 2.5 \cdot 10^{-11}$. The intensity factor of measurement errors $v$ is varied in Subsection 4.4 and is set to one if not otherwise stated. Consequently, the measurement errors can be described by

$$\tilde{x}_n \sim N (E(\tilde{x}_n), C(\tilde{x}_n, \tilde{x}_n))$$

(41)

with

$$(E(\tilde{x}_n))_i = 0 \quad \text{and} \quad (C(\tilde{x}_n, \tilde{x}_n))_{ij} = \delta_{ij} v^2 2.5 \cdot 10^{-11} \quad \forall i, j.$$  

(42)

The statistics of the measurement errors are identical for all damage levels. For $v = 1$ a signal-to-noise ratio between 400 (undamaged) and 500 (LS 65) can be expected. In other words, the standard deviation of the measurement error is about 5% resp. 4.5% of the standard deviation of the errorless response signal, which are realistic values.

From the generalized eigenvalue problem the mode shape vectors related to the displacements can be extracted for each linearized system. The modal strains can be easily obtained by multiplying these mode shape vectors by a transformation matrix. By tuning the modal filter to the first and second bending mode shape, the matrix of linear combination coefficients $A$ is finally given by

$$A = \begin{bmatrix} -1.4016 & 2.7383 & -1.3868 \\ -0.2012 & 0.3291 & -0.1289 \end{bmatrix} 10^3,$$

(43)

where the first and second row correspond to the coefficients of the first linear combiner (lc1) and second linear combiner (lc2), respectively. To calculate the statistics of the linear combiner of response Fourier transforms according to Section 2, the first 10 modes ($m_0 = 10$) until a circular frequency of about $10000 \frac{rad}{s}$ and a time step $\Delta t = 2^{-11}s$ are chosen. All investigations consider a time frame that is extracted by a rectangular window function after the steady state has been reached (i.e. after a start time $t_s = 2\Delta t$). In the focus of subsequent studies is the position around the first eigenfrequency of the second linear combiner. As the investigated peak indicators are designed to detect the early damage and therefore early peak appearance, only damages until load step 65 are considered.

In the following, the effects of time length and measurement error intensity are investigated on both damage indicators. The results are derived by following the virtual testing scheme explained in Subsection 3.1. The sample statistics of the damage indicators are obtained from the evaluation of 500 000 Latin hypercube samples, from which the probability density function can be estimated using kernel densities. By following the philosophy of control charts, the false positive probability (the probability of unobserved, but existing damage) is used as a criterion to assess the performance of the indicators. Based on the undamaged structure, the control limits are defined at the 2.5% and 97.5% quantiles, for $I_p$ and $m_4$ respectively. The limits are defined using the estimation of the inverse cumulative distribution function based on kernel densities. Therefore, the normality assumption for the indicators does not need to be fulfilled. Nevertheless, as standard procedures assume often a normal distribution, the obtained distributions of the indicators are discussed in the light of the normality assumption.

4.3 Influence of time length

Based on the structural system described in previous subsections, the probability density function estimates of the indicator $I_p$ with respect to the time length and the load step are calculated and shown in Figure 6a. It can be observed that for all load steps an increasing time length reduces the width (variance) of the probability density function, but keeps the position of the mean value almost constant. The probability density functions of the indicator $m_4$ are depicted in Figure 6b. A change of the width and the position of the probability density function can be

Figure 5: Geometry of investigated simply supported plain concrete beam with dynamic loading.
observed with changing time length. With increasing damage the variance and the mean value of the indicator increases significantly. Hence, the results of load step 60 are out of the range of Figure 6b.

Figure 7 illustrates the evolution of the false positive detection probability in dependency of the load step and the time length. With increasing time length both indicators become more sensitive to damage. A false positive probability of 10^{-2} or lower can be interpreted as sensitive to detect damage. While for a time length of 2s the indicator $m_4$ outperforms the indicator $I_p$, it is vice versa for a time length of 16s.

For both indicators, a large time length is beneficial for the detection of damage.

4.4 Influence of measurement errors

The results of the probability density function estimates based on the indicator $I_p$ with increasing measurement error intensity $\nu$ are presented in Figure 8a for a time length of 8s. For the load step 15 related to the undamaged structure, the measurement error intensity has no influence on the probability density functions. If damage is present (see load step 50), the probability density functions are notably different. The higher the signal-to-noise ratio, the larger the mean value shift and the lower are the variances, which is beneficial for damage detection as expected.

The probability density functions of the indicator $m_4$ are shown in Figure 8b. For this indicator, the probability density functions related to the undamaged and damaged structure depend strongly on the measurement error intensity. The higher the signal-to-noise ratio is, the larger the mean value and the larger the variance of the indicators.

By assessing the false positive probability shown in Figure 9, it can be observed that the evolution of both indicators is almost identical with respect to the variation of the measurement error intensity.

4.5 Normality assumption

In practice, control charts with certain control limits are applied to monitor damage indicators over time. Most of the control charts assume a normally distributed variable. Hence, it needs to be investigated, if the normality assumption is at least approximately fulfilled. This will be exemplarily illustrated by a representative configuration using a time length of 8s with a measurement error intensity of $\nu = 1$ for load step 50. The histogram, the probability density function estimation using kernel densities, and the probability density function related to
a fitter normal distribution are compared in Figure 10. It can be observed that the distribution of the indicator \( I_p \) deviates slightly from a normal distribution while the distribution of the indicator \( m_t \) is significantly different from a normal distribution. This observation is also valid for other combinations of time length and damage intensity. However, for both indicators, a large time length and an undamaged structure lead to distributions very close to normal.

5 CONCLUSIONS

In this paper the variation of the statistical properties of two different damage indicators based on modal filter were investigated within a virtual testing scheme with respect to damage intensity, measurement error intensity and time length of the considered vibration response time history. An implicit gradient damage law was applied to define realistic damage pattern for a simply supported beam with progressing damage. Assuming an ambient vibration experiment with known statistics of a multivariate normally distributed excitation, the statistics of the damage indicators are derived in a two step approach. First, the statistics of the dynamic response Fourier transforms of a modal filter were derived by using the approach proposed in [9][10]. This avoids the computationally very demanding numerical time integration as typical needed in standard Monte Carlo methods. Secondly, a Latin hypercube sampling has been applied to derive the sample statistics of the indicators for each load step. This operation is computationally very inexpensive, which allows generating a large number of 500 000 Latin hypercube samples.

Due to the different definitions of the indicators, the effects of the investigated influences on the probability density functions are very dissimilar. Nevertheless, if the false positive probability is used as an assessment criterion, both indicators show a similar performance to detect damage. It could be derived that the increase of the time length increases significantly the performance of both indicators. With a large time length and a typical signal-to-noise ratio, both indicators were able to detect a rather small damage.

The shape of the probability density functions has been discussed for both indicators with respect to damage intensity and time length. The normality assumption, as usually required for the application of control charts in practice, is approximately fulfilled for the indicator \( I_p \) with a sufficiently large time length. For the indicator \( m_t \) the assumption of a normal distribution was hardly justified for the investigated example, but the similarity can be improved by increasing the time length.

Further research will be related to the investigation of correlated excitations, as for example, typical for wind induced excitations.

REFERENCES


![Figure 10: Distribution fitting for a time history of length 8s, a measurement error intensity of \( \nu = 1 \), and the load step 50.](image-url)