Railroad vehicle modelling in probabilistic vibration analysis of a railway bridge with randomly fluctuating track ballast stiffness

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ABSTRACT: The paper presents an efficient method for analysing an influence of random changes in ballast stiffness on railway bridge vibrations caused by a train passage. Three different types of train model are taken into account: a sequence of moving forces, a set of moving unsprung masses and two-level mass-spring-damper system. Ballast stiffness fluctuation appearing along the track is considered as stationary Gaussian process. Three autocorrelation types of the process are assumed, including “white-noise” and exponentially decaying function – with or without harmonic pulsation. Numerical analysis of probabilistic response characteristics is performed for a simply supported, one-track railway bridge, by applying Monte Carlo method. The results demonstrate that a simplified model of the train, being a set of moving forces, may be used in stochastic displacement analysis. In turn, the acceleration analysis requires a more realistic, fully sprung model of the vehicle. Random fluctuation of the ballast stiffness has a much greater impact on accelerations of bridge vibrations than on displacements.

KEY WORDS: Railway bridge dynamics; Vehicle modelling; Random ballast stiffness; Monte Carlo simulations.

1 INTRODUCTION

The quality of railways on high speed lines has crucial influence on safety and comfort of travel, therefore in many recent works, variability effects of geometrical and material track parameters are examined in the train – track vibration analysis. The problem is especially important when consider the railway track lying on a bridge. However, in many papers on dynamic behaviour of bridge - track - train system, much attention is paid to geometrical irregularities of rails, in contrary to the ballast stiffness effects which are quite rarely raised by the authors.

As it is commonly known, the ballast stiffness may vary along the track due to different factors (e.g. changes in ballast layer thickness, non-uniform compacting, environmental conditions or dynamic load effects). Ballast stiffness is also dependent on track condition (e.g. track alignment and profile, material breakage, cementation, biological debris). Simultaneous analysis of factors mentioned above is not possible using deterministic methods. In these circumstances, probabilistic approach is a good alternative because it allows to describe all changes in ballast stiffness by stochastic fluctuation around a specific constant value.

So far, the problem of randomly varying ballast stiffness has been considered in literature only for a railway track lying on the ground. Such a problem has been analysed by Naprstek and Frýba [1], who investigated vibrations of a beam resting on Winkler foundation with random stiffness. Oscarsson [2] performed laboratory and field experiments in order to determine probabilistic characteristics of vehicle - ballasted track system. Wu and Thomson [3] examined an influence of random irregularities of ballast stiffness and sleeper spacing on railway track vibrations. They concluded that random ballast stiffness, as opposed to sleeper spacing, has no influence on the noise level emitted by a rail during a train passage.

A certain approach to an assessment of randomly varying ballast stiffness influence on railway bridge vibrations has been presented by the authors of this paper [4]. An original model of bridge - track - train system with the ballast idealized as continuous, viscoelastic non-inertial foundation, has been formulated by Galerkin’s Finite Element Method. The effect of vibration propagation from a train entering or leaving the bridge has been taken into consideration, by addition of two track sections adjacent to the bridge. The dynamic load caused by a train passage has been modelled in a simplified manner, as a set of moving unsprung masses. Therefore, couplings between train and bridge vibrations have been omitted. The model of a bridge structure, created especially for stochastic analysis, allowed to describe along-track variations of ballast stiffness by a random function dependent on spatial coordinate measured along the track axis.

This paper is a continuation of the research described in [4] which was of initial nature and was limited to probabilistic displacement analysis of bridge girder and rails. The analysis presented in the paper is extended to cover accelerations of railway bridge vibrations which are necessary not only in ride comfort assessment but also in the track and bridge durability problem. This is based on a fact that in dynamic reliability assessment due to material fatigue much importance is attributed to the frequency of strain peaks occurrence [5]. Such frequency depends on accelerations of in-time strain variations, and in consequence, depends on acceleration of structure vibrations.

Substantial problem in acceleration analysis of railway bridge vibrations is a proper choice of a railroad vehicle model. In extensive literature concerning bridge dynamics, a train is idealized in various manner: as a series of moving forces or point masses, as a set of moving single-mass viscoelastic oscillators, or by two- or three dimensional models consisting of mass elements connected by viscoelastic
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constraints. For example Yau et al. [6] when researching the ride comfort of passengers have accepted a train model in the form of moving single-mass viscoelastic oscillators. In papers [7-9] the authors have described a train as a mass system with two-level suspension, assuming that each car has six internal degrees of freedom. Such a model, where four wheelsets are mounted in two two-axle bogies is most frequently used in literature. Xia and Zhang [10] for analysis of a beam railway bridge vibrations have presented a spatial model of a train in which each car has fifteen internal degrees of freedom.

Detailed modelling of a train is without doubt important when dynamic response of a vehicle is under analysis. However, when the bridge vibrations are considered, the question of a rail vehicle model is still open. Frýba [11] states that the detailed train description is irrelevant when the vehicle mass is considerably smaller than the mass of analysed structure. Nevertheless, many contemporary models of railway vehicles tend to be very complex, what increases computational effort while simulating bridge vibrations.

The probabilistic analysis performed by Monte Carlo method is especially time-consuming, therefore it is desired to reduce a train model as much as possible. For that reason, the main purpose of this paper is a comparison of probabilistic characteristics of bridge and track vibrations calculated for three train models: a sequence of moving forces (“force model”), moving point masses (“mass model”) and commonly used two-dimensional mass model with two-level suspension (“train model”). In order to determine the probabilistic characteristics of displacements and accelerations of railway bridge with randomly varying stiffness of the track ballast, the Monte Carlo computational method has been adopted. Correlation between ballast stiffness variations appearing along the track has been taken into account. Three different types of the correlation function has been assumed: exponentially decaying harmonic function, exponentially decaying function without harmonic pulsation and “white noise” type function. Numerical analysis has been performed for evaluating an influence of ballast stiffness fluctuation on bridge displacements and accelerations. As a final conclusion, justification of simplifications in rail vehicle models adopted for stochastic vibration analysis has been presented.

2 STOCHASTIC MODEL OF BALLAST STIFFNESS

Let’s assume that vertical stiffness of the track ballast varies randomly along the track and these variations can be described as a minor fluctuation around a certain constant value. Then, the function describing the random ballast stiffness can be written in the following form:

\[ k(x) = \bar{k} + \tilde{k}(x) \] (1)

as a sum of mean value \( \bar{k} \) and zero-mean random function \( \tilde{k}(x) \) dependent on spatial coordinate \( x \) measured along track axis [4]. Because many independent factors may affect ballast stiffness, the random stiffness fluctuation can be treated as a stationary Gaussian process, according to the central limit theorem. Power spectral density function (PSD function) of such a process can be obtained by \textit{in situ} measurements or can be analytically determined basing on an \textit{a priori} assumed correlation function. In this paper the second approach is used.

From the assumption of stationarity of the stiffness fluctuation process \( \tilde{k}(x) \), it follows that the correlation function \( K_{k_k}(\zeta) \) depends only on the distance \( \zeta = x_2 - x_1 \) between two points \( x_1 \) and \( x_2 \) on the track axis, that yields

\[ K_{k_k}(x_1, x_2) = K_{k_k}(x_2 - x_1) = K_{k_k}(\zeta). \]

Taking into consideration that the correlation between stiffness fluctuation at two points on the track decays with the increase in distance \( \zeta \), we conclude that the adopted correlation function should be a decaying function. Next, considering experimental results from paper [12], which definitely show the presence of a characteristic frequency of track stiffness variations resulting from sleeper spacing, we conclude that an essential part of PSD function of the \( \tilde{k}(x) \) process should be concentrated around this frequency.

Above requirements are satisfied by the correlation function assumed in the form:

\[ K_{k_k}(\zeta) = \sigma_\zeta^2 \exp(-\alpha|\zeta|) \cos(\theta\zeta) \] (2)

which corresponds with PSD function (see [4]):

\[ S_{k_k}(\omega) = \frac{\sigma_\zeta^2 \alpha}{\pi} \frac{\alpha^2 + \omega^2 - \theta^2}{(\alpha^2 - \omega^2 + \theta^2)^2 + 4\alpha^2\omega^2} \] (3)

where \( \sigma_\zeta^2 = K_{k_k}(0) \) is the variance of \( \tilde{k}(x) \) process. The PSD function (3) reaches maximum values around the frequency \( \omega = \theta = 2\pi / T_\theta [\text{rad/m}] \) where \( T_\theta \) is the dominant wavelength of ballast stiffness fluctuation, which is assumed to be equal to sleeper spacing. The parameter \( \alpha > 0 \) affects the correlation decaying. The larger its value is, the quicker the correlation disappears, so the process becomes more chaotic.

The assumed correlation function (2) decays exponentially with the parameter \( \alpha \) and pulsates harmonically with the frequency \( \theta \). In order to assess an influence of the assumed correlation type on standard deviations of the dynamic bridge response, two additional autocorrelation types will be examined in numerical analysis:

- the function decaying exponentially without pulsation \( \theta \):

\[ K_{k_k}(\zeta) = \sigma_\zeta^2 \exp(-\alpha|\zeta|) \] (4)

- and most chaotic process of “white noise” type:

\[ K_{k_k}(\zeta) = \sigma_\zeta^2 \delta(\zeta) \] (5)

where \( \delta(\zeta) \) stands for Dirac delta function. The corresponding density functions are as follows, respectively:

\[ S_{k_k}(\omega) = \frac{\sigma_\zeta^2 \alpha}{\pi} \frac{1}{\alpha^2 + \omega^2} , \quad S_{k_k}(\omega) = \frac{\sigma_\zeta^2}{2\pi} \] (6)

3 DYNAMIC MODEL OF A TRAIN

In this paper commonly used two-dimensional dynamic model is adopted (see e.g. in Reference [8]). Therefore, it is assumed that a running train consists of \( N_t \) identical railroad vehicles supported by two-axle bogies through two-level suspension (primary and secondary). Each vehicle comprises a car-body, two bogies and four wheelsets, joined together with linear
springs and viscous dampers modelling primary and secondary suspension (Figure 1). The car-body and bogies are modelled as rigid mass elements, each of them possesses two dynamic degrees of freedom: vertical displacement \( w_i \) and rotation \( \phi_i \), \( i = 1, 2, 3 \). Masses and central rotational mass moments of inertia of car-body and bogies are denoted by symbols \( M_c \), \( I_i \) and \( M_b \), \( J_i \). Stiffness and damping parameters of primary and secondary suspension are denoted by \( k_i \), \( c_i \) and \( k_i \), \( c_i \), respectively (see Figure 1). It is assumed that wheelsets (each of the mass \( M_w \)) roll over the track in a full contact with rails, therefore their vertical displacements \( W_i \) are the same as rail deflections in contact points.

By these assumptions, vibrations of every vehicle are described by six internal dynamic degrees of freedom, collected in the vector \( \mathbf{w}_i = [w_i, w_j, \phi_i, \phi_j, \phi_k]^\top \) and four dynamic deflections of the rail \( \mathbf{W}_i = [W_1, W_2, W_3, W_4]^\top \), where symbol \( (\cdot)^\top \) means the transpose of a matrix.

Applying Langrange equations after formulating energy expressions, we can obtain the following equation of motion of the vehicle numbered \( i \):

\[
\mathbf{B}_i \ddot{\mathbf{w}}_i + \mathbf{C}_i \dot{\mathbf{w}}_i + \mathbf{K}_i \mathbf{w}_i = \mathbf{F}_i \tag{7}
\]

where:

\[
\mathbf{w}_i = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{bmatrix}^\top, \quad \mathbf{F}_i = \begin{bmatrix} 0 \end{bmatrix}^\top
\]

and “over-dots” represent time derivatives. Mass matrix \( \mathbf{B}_i \), damping matrix \( \mathbf{C}_i \) and stiffness matrix \( \mathbf{K}_i \) are of dimension \( 10 \times 10 \). Excitation vector contains dynamic impact forces\( \mathbf{F}_i = [F_1, F_2, F_3, F_4]^\top \) acting on wheelsets (see Figure 2).

\[
\mathbf{B}_i = \begin{bmatrix} \mathbf{B}_{i1} & \mathbf{B}_{i2} & \mathbf{B}_{i3} \end{bmatrix}, \quad \mathbf{C}_i = \begin{bmatrix} \mathbf{C}_{i1} \end{bmatrix}, \quad \mathbf{K}_i = \begin{bmatrix} \mathbf{K}_{i1} \end{bmatrix}
\]

\[
\mathbf{w}_i = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{bmatrix}^\top, \quad \mathbf{F}_i = \begin{bmatrix} 0 \end{bmatrix}^\top
\]

and “over-dots” represent time derivatives. Mass matrix \( \mathbf{B}_i \), damping matrix \( \mathbf{C}_i \) and stiffness matrix \( \mathbf{K}_i \) are of dimension \( 10 \times 10 \). Excitation vector contains dynamic impact forces

\[
\mathbf{F}_i = [F_1, F_2, F_3, F_4]^\top \text{ acting on wheelsets (see Figure 2).}
\]

Displacements presented in Figure 2. Then, after an appropriate transformation of equations (7), the vibrations of the whole train can be described by two matrix equations written in the following compact form:

\[
\begin{bmatrix} \mathbf{B}_{TT} & 0 \\ 0 & \mathbf{B}_{RR} \end{bmatrix} \begin{bmatrix} \mathbf{w}_T \\ \mathbf{w}_R \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{TT} & \mathbf{C}_{TR} \\ \mathbf{C}_{RT} & \mathbf{C}_{RR} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{w}}_T \\ \dot{\mathbf{w}}_R \end{bmatrix} = \begin{bmatrix} \mathbf{F}_T \\ \mathbf{F}_R \end{bmatrix}
\]

\[
\begin{bmatrix} \mathbf{K}_{TT} \\ \mathbf{K}_{RT} \end{bmatrix} \begin{bmatrix} \mathbf{w}_T \\ \dot{\mathbf{w}}_R \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{F}_R \end{bmatrix}
\]

where \( \mathbf{B}_{TT}, \mathbf{C}_{TT} \) and \( \mathbf{K}_{TT} \) are diagonal matrices.

Numbering \( i_t = 1, 2, \ldots, N_t \) determines the sequence of cars, counting from the head of the train. It means that successive vertical displacements of wheelsets, included in the vector \( \mathbf{w}_R \), as well as dynamic impact forces \( F_j \) listed in the vector \( \mathbf{F} \) can be new numbered by \( j = 1, \ldots, N \) where \( N = 4N_t \). Moreover, it should be emphasized that dynamic impact forces \( F_j \) acting on wheelsets and on the rail are the same, but the total point loads \( P_j \) acting on the rail (see Figure 2) comprise also static loads \( G_j = (0.25M_c + 0.5M_b + M_u)g \), that is: \( P_j(t) = G_j + F_j(t) \).

Finally, the first equation extracted from (9) is of the form:

\[
\mathbf{B}_{TT}\ddot{\mathbf{w}}_T + \mathbf{C}_{TT}\dot{\mathbf{w}}_T + \mathbf{K}_{TT}\mathbf{w}_T = -\mathbf{C}_{TR}\ddot{\mathbf{w}}_R + \mathbf{K}_{TR}\mathbf{w}_R = \mathbf{F}_T \tag{10}
\]

which describes vibrations of the train, induced kinematically by the rail motion. The second matrix equation of the set (9) defines time dependent dynamic loads acting on rails, as follows:

\[
\mathbf{F} = -\mathbf{C}_{TR}\ddot{\mathbf{w}}_T + \mathbf{K}_{TR}\dot{\mathbf{w}}_T - (\mathbf{B}_R\ddot{\mathbf{w}}_R + \mathbf{C}_{RR}\ddot{\mathbf{w}}_R + \mathbf{K}_{RR}\mathbf{w}_R \mathbf{F}) \tag{11}
\]

4 EQUATIONS OF MOTION OF COUPLED SYSTEM: BRIDGE GIRDER - BALLASTED TRACK - TRAIN

In paper [4] authors have presented a two-dimensional, dynamic model of a system: bridge girder - ballasted track, which is loaded by a set of moving unsprung masses modelling the train. This model has been used in this paper to formulate the equations of motion of a system consisting of three coupled dynamic subsystems: a bridge girder, ballasted track and the train.

Following the assumptions described in detail in [4], the ballasted track has been divided into three sections: the central one lying on the bridge of the length \( L_e \) and two approaching sections of the lengths \( L_1 \) and \( L_2 \), resting on a rigid subgrade, on the left and right side of the bridge, respectively. Taking into account long enough approaches allows to consider the propagation effect of rail vibration, caused by the train entering and exiting the bridge. It has been assumed that the layer of ballast is a continuous, viscoelastic foundation of linear characteristics \( k(x) \) and \( \epsilon(x) \), dependent on \( x \) coordinate measured along the track. This key assumption allows to include the stochastic model of the ballast stiffness, defined in Section 2. Equations of motion of bridge girder and three track sections mentioned above have been derived by using Galerkin's Finite Element Method where both the bridge girder and a couple of two rails are modelled as prismatic Euler-Bernoulli beams with continuous mass distribution. To assembly equations of motion of subsystems (bridge girder and three track sections), the substructure technique similar to the one described in Reference [8] has been used.
The final equations of motion, derived in [4], contain time dependent matrix blocks appearing due to the load in the form of moving point masses modelling the train. After omitting these matrix blocks, a system of two matrix equations is obtained in the following compact form:

\[
\begin{bmatrix}
  B & 0 \\
  0 & B
\end{bmatrix}
\begin{bmatrix}
  \ddot{q}_i \\
  \ddot{q}_j
\end{bmatrix}
+ \begin{bmatrix}
  C_{ii} & -C_{ij} \\
  -C_{ji} & C_{jj}
\end{bmatrix}
\begin{bmatrix}
  q_i \\
  q_j
\end{bmatrix}
= \begin{bmatrix}
  \ddot{F}_i \\
  \ddot{F}_j
\end{bmatrix}
\] 

(12)

which constitutes a start point for further considerations. Equations (12) describe coupled vibrations of the system: bridge girder (“g”) – ballasted track (“r”), whereas the vector \( \mathbf{F}_i \) representing the load on track is not yet defined. Nevertheless, the overall aggregation formula is known:

\[
\mathbf{F}_i = \sum_{j=0}^{N} \mathbf{A}_j \mathbf{q}_j, \quad \mathbf{q}_j = \mathbf{A}_j \mathbf{q}_i
\] 

(13)

which enables to calculate the vector \( \mathbf{F}_i \), on the basis of nodal equivalents \( \mathbf{F}_i \), defined for the \( k \)-th finite element of the \( k \)-th rail section. Index \( i = 0 \) denotes the central section of the track, while indices \( i = 1 \), \( i = 2 \) denote the section before and after the bridge, respectively. Symbol \( n_{ji} \) stands for the number of finite elements used to discretize the \( j \)-th rail section. Matrices \( A_j \) and \( A_i \) realize displacement transformations:

\[
\mathbf{W}_{ji} = \mathbf{A}_j \mathbf{q}_i, \quad \mathbf{q}_j = \mathbf{A}_i \mathbf{q}_i
\] 

(14)

where the vector \( \mathbf{W}_{ji} \) lists element nodal displacements, \( \mathbf{q}_j \) assembles nodal displacements of the \( j \)-th rail section, \( \mathbf{q}_i \) is a collective set of nodal displacements, covering all three rail sections.

Vertical rail displacements within the element length are approximated as: \( w_j(\xi, t) = N_j^0(\xi)W_{ji}(t) \), by typical cubic Hermite polynomials which form the vector of local shape functions \( N_j(\xi) \), where \( \xi = (x/L_i), 0 \leq \xi \leq 1 \). All remaining denotations used in equation (12) are explained in [4].

4.1 Railway track load

As it is shown in Figure 2, the track is loaded by a sequence of moving time-dependent forces \( P_j(t) \) consisting of static \( G \) and dynamic \( F \) axle pressures of train wheelsets. It is assumed that the train passes with constant velocity \( v \), so the location of the \( j \)-th force on the track, in a time \( t \) is described by a function \( s_j(t) = vt - d_j \), where \( d_j \) stands for a distance between the force and the train head. In the initial moment \( t = 0 \), the train head is at the left edge of the track section \( L_i \), preceding the bridge.

Element load distributed on the element length can be expressed using the Dirac delta function \( \delta(\cdot) \), by a formula:

\[
p_{ji}(x_i, t) = \sum_{j=0}^{N} P_j(t) \delta(x_i - s_j^p)
\] 

(15)

where \( s_j^p(t) = s_j(t) - (k-1)L_i + a_j \). Distances \( a_0 = L_i, a_1 = 0, \) \( a_2 = L_i + L_0 \) are the spans of preceding track sections (see [4]). In this case, vector of nodal equivalents of the element load (15) is expressed as:

\[
\mathbf{F}_i = \int_0^1 \mathbf{N}_{ji} P_j(\xi, t) d\xi = \sum_{j=0}^{N} \bar{N}_{ji}^p P_j(t)
\] 

(16)

\[
\bar{N}_{ji}^p = \bar{N}_{ji}^p(t) = \begin{cases} 
N_{ji}(\xi_j^p(t)) & \text{for } \xi_j^p \in (0;1] \\
0 & \text{for } \xi_j^p \notin (0;1]
\end{cases}
\] 

(17)

where \( \xi_j^p = s_j^p(t)/L_i \). Setting axle loads \( P_j = G_j + F_j \) in vectors \( \mathbf{P} = \mathbf{G} + \mathbf{F} \), from \( j = 1 \) to \( j = N \), and organizing the vectors \( \mathbf{N}_j^p \) as columns of matrix:

\[
\mathbf{N}_j = [\bar{N}_j^p, \bar{N}_j^{p+1}, \ldots]
\] 

(18)

we can rewrite the formula (16) into the form:

\[
\mathbf{F}_i = \mathbf{N}_i \mathbf{P} = \mathbf{N}_i (\mathbf{G} + \mathbf{F})
\] 

(19)

Vector \( \mathbf{F} \) gathering dynamic axle pressures of the train is described by the equation (11) which contains wheelsets displacements \( W_j(t) \) collected in the vector \( \mathbf{w}_r \), their speeds \( \mathbf{w}_g \), and accelerations \( \mathbf{w}_a \). Displacements of wheelsets are equal to dynamic rail deflections at contact points which determine the positions of \( P_j \) forces at a time \( t \). Hence, if the force \( P_j \) is placed on the \( j \)-th element in the \( k \)-th track section, then we have \( W_j(t) = W_j(\xi_j^p(t), t) = N_{ji}(\xi_j^p(t))W_{ji}(t) \).

So, considering the definitions (17) and (18) the following results are obtained:

\[
\mathbf{w}_r = \mathbf{N}_i \mathbf{W}_i, \quad \mathbf{w}_g = \mathbf{N}_i \mathbf{W}_i + \mathbf{N}_i \mathbf{W}_i
\] 

(20)

\[
\mathbf{w}_a = \mathbf{N}_i \mathbf{W}_i + \mathbf{N}_i \mathbf{W}_i + \mathbf{N}_i \mathbf{W}_i
\] 

(21)

where \( j = d/d\xi \). Substituting expressions (20) and (21) into equation (11), and then (11) into (19) gives the result:

\[
\mathbf{F}_i = \bar{F}_i - \mathbf{C}_r \mathbf{w}_r - \mathbf{K}_r \mathbf{w}_r - \mathbf{B}_r \mathbf{q}_r - \mathbf{C}_q \mathbf{q}_r - \mathbf{K}_q \mathbf{q}_r
\] 

(22)

where

\[
\mathbf{F}_i = \sum_{i=0}^{N} A_i^T A_i^T \mathbf{N}_i \mathbf{G}, \quad \mathbf{C}_r = \sum_{i=0}^{N} A_i^T A_i^T \mathbf{N}_i \mathbf{C}_k \mathbf{K}_k \mathbf{C}_k \mathbf{A}_i \]

(23)
\[ \tilde{K}_n = \sum_{i=0}^{n} \sum_{k=1}^{m} A_k^T A_i N_{ij} \left( \frac{v^2}{l_{ij}} C_{RR} \tilde{N}_k^T + \frac{v}{l_{ij}} C_{RR} \tilde{N}_k + K_{RR} \tilde{N}_k \right) A_i A_i. \]

4.2 Kinematic excitation of train vibrations

Kinematically induced train vibrations are described by equation (10) where the excitation vector takes now the form:

\[ F_T = -\hat{C}_T \tilde{q}_T - \tilde{K}_T \tilde{q}_T \]

\[ (25) \]

which is obtained by substituting formulae (20) into (13) and summing contributions of all rail elements. Taking into account displacement transformations (14), the following result is obtained:

\[ F_T = -\hat{C}_T \tilde{q}_T - \tilde{K}_T \tilde{q}_T \]

\[ (26) \]

\[ (27) \]

where the first one is directly the equation (10) while the second and third equations are created on the basis of (12).

4.3 Final equation of motion of the considered system

Now, vibrations of the system: bridge girder - ballasted track - train are described by a three coupled matrix equations:

\[ B_T \ddot{w}_T + C_T \dot{w}_T + K_T w_T = F_T \]

\[ B_T \tilde{q}_T + C_T \dot{q}_T + K_T \tilde{q}_T = F_T \]

\[ B_T \dot{q}_T + C_T \ddot{q}_T + K_T q_T = C_T \dot{q}_T - K_T q_T = 0 \]

where the first one is directly the equation (10) while the second and third equations are created on the basis of (12). After substituting the excitation vectors (23), (24) into (26), and transferring the terms dependent on displacements, velocities and accelerations to the left side, we can write equations (26) together, as follows:

\[ \begin{bmatrix} B_T & 0 & 0 \\ 0 & B_T & 0 \\ 0 & 0 & B_T \end{bmatrix} \begin{bmatrix} \ddot{w}_T \\ \dot{w}_T \\ w_T \end{bmatrix} + \begin{bmatrix} C_T & 0 & 0 \\ 0 & C_T & 0 \\ 0 & 0 & C_T \end{bmatrix} \begin{bmatrix} \dot{q}_T \\ \dot{q}_T \\ q_T \end{bmatrix} = \begin{bmatrix} F_T \\ 0 \\ 0 \end{bmatrix} \]

Note that the couplings between train and track vibrations result from the assumption that vertical displacements of wheelsets are equal to rail deflections at contact points, while the rail and girder vibrations are coupled due to connecting ballast layer.

It is also worth noting that the assumption \( F = 0 \) in definition (19) leads to the simplified load model in the form of a sequence of moving forces representing static axle loads. Then, the system of equations (27) becomes de facto a system of two matrix equations with constant coefficients.

In turn, the simplified load model being a series moving unsprung masses can be procured by the assumption that masses and rotational mass moments of the bogies and car bodies are equal to zero. Instead, the mass of each wheelset should be increased by a half of the bogie mass and a quarter of car body mass. Such procedure leads directly to governing equations presented in [4].

5 NUMERICAL ANALYSIS

Numerical vibration analysis of the railway bridge with random ballast stiffness has been performed on an example of a simply supported, single track reinforced bridge with a span of \( l_0 = 30.0 \) m. Flexural rigidity of the bridge girder equals \( E_I = 1.0733558 \times 10^9 \) Nm². Mass of the ballast and sleepers has been included into uniformly distributed mass of the girder \( m_g = 3.3147 \times 10^4 \) kg/m. It is assumed that damping characteristic of the ballast is constant and equals \( c(\nu) = c = 2.8667 \times 10^5 \) Ns/m², whereas the ballast stiffness varies randomly along the track, fluctuating around the mean value \( k = 1.1 \times 10^9 \) N/m² with the variation coefficient \( \sigma_k/k = 0.3 \) %.

Lengths of approaching track sections are taken as: \( l_1 = l_2 = 100 \) m, basing on [8] where it has been stated that this length is sufficient to model a track of infinite length in numerical simulations of bridge vibrations. Flexural rigidity of the beam representing two rails equals \( E_I = 1.2831 \times 10^9 \) Nm², and intensity of mass distribution is \( m_1 = 1.21 \times 10^5 \) kg/m. In calculations, material damping in rails is taken into account, with the retardation time \( \kappa_1 = 2.1 \times 10^3 \) s, as well as structural damping of the bridge girder, with the dimensional parameter \( \mu_1 = 0.984 \) s⁻¹. Vibrations of the bridge and rails are excited by a passage of Shinkansen train moving with velocity \( \nu = 60 \) m/s (216 km/h). Train consists of eight repetitive cars, each of 25 m length, supported by two twin-axle bogies (Figure 3).

![Figure 3. Arrangement of a train.](image-url)
expected value $E[R(t)]$ and standard deviation $D[R(t)]$ of each examined structural response $R(t)$ have been calculated using generally known formulas.

In the first stage of research, the correlation function (2) decaying with harmonic pulsation has been adopted, assuming the following coefficients: $\alpha = 2.0$ and $\theta = 10\pi/3$. The last coefficient corresponds to dominant wavelength of ballast stiffness fluctuation $T_\theta = 0.6$ m, being a sleeper spacing. Probabilistic characteristics of dynamic response of the structure have been calculated on the basis of 100 simulations.

As it is shown in Figure 4, time histories of expected values and standard deviations of both bridge girder and railway track dynamic displacements, obtained for the simplified moving force model and for the fully sprung train model are very similar. Extreme values selected from these solutions and listed in Table 1 differ quite slightly.

Table 1. Extreme expected values and standard deviations: dynamic deflections [mm].

<table>
<thead>
<tr>
<th>Force model</th>
<th>E[w]</th>
<th>D[w]</th>
<th>E[w_r]</th>
<th>D[w_r]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass model</td>
<td>3.46</td>
<td>6.15E-3</td>
<td>3.96</td>
<td>6.81E-2</td>
</tr>
<tr>
<td>Train model</td>
<td>3.36</td>
<td>1.35E-3</td>
<td>3.91</td>
<td>2.79E-2</td>
</tr>
</tbody>
</table>

Moving mass model gives slightly larger amplitudes of expected values and much (several fold) larger values of standard deviations when compared with the two other models. Thus a conclusion follows that the unsprung mass model of the train is not appropriate for an analysis of stochastic vibrations of bridge-track system. This model leads to overestimation of real inertia forces acting on the rail and therefore degenerates the results of standard deviations. However, it should be emphasized that the dispersion of both girder and railway track displacements caused by stochastic fluctuation of ballast stiffness is negligible. Despite the fact that standard deviations of dynamic rail displacements are about tenfold larger than these of the girder, they constitute only a fraction of percent when compared to the expected values. Therefore, from engineering point of view, the proper choice of the train model is not important in this displacement analysis.

In turn, analysis of girder and track accelerations undoubtedly shows significant differences in expected values and standard deviations for the three models of the vehicle. Presented in Figure 5, time histories of expected values calculated for bridge girder accelerations are characterized mostly by low-frequency oscillations when obtained from the simplest moving force model. In analogical solutions obtained for the two other models, high-frequency oscillations with significant amplitudes are imposed on low-frequency solution. In a case of expected values of rail accelerations, high-frequency oscillations are dominating for all considered load models. Expected values of accelerations grow along the degree of vehicle model complication, extreme peaks of accelerations rise, especially for rail vibrations. Data presented in Table 2 shows that absolute of extreme expected values of rail vibration accelerations obtained for the force model are almost two times smaller than values for the sprung model (“the train model”). This decrease is slightly smaller for the girder.

![Figure 4](image-url) Expected values $E[\cdot]$ and standard deviations $D[\cdot]$ of girder “g” and rail “r” displacements at the bridge mid-span, computed for three railroad vehicle models.
Figure 5. Expected values $E[\cdot]$ and standard deviations $D[\cdot]$ of girder “g” and rail “r” accelerations at the bridge mid-span, computed for the three railroad vehicle models.
Neglecting of inertia forces in the moving force model leads to almost complete elimination of random ballast stiffness effects on accelerations of girder vibration, because standard deviations are close to zero (see Figure 5 and Table 2). In this case, such simplified force model is inadequate. However, it can be used to analyze standard deviations of stochastic vibrations of railway track alone, what results from Figure 4. On the other hand, the model consisting of unsprung masses overrates standard deviations of both girder and rail accelerations, thus causing groundless assessment that the ballast stiffness fluctuation affects them very significantly. Therefore, it is recommended to use most realistic, fully sprung model of the train when analyzing stochastic accelerations of girder vibrations. Overrating of standard deviations of bridge accelerations, observed in mass model, results from unacceptable omission of suspension elements of the train in calculating dynamic axle loads acting on the track.

5.2 Influence of the assumed correlation type on accelerations of girder vibration

Figure 6 presents time histories of standard deviations of girder accelerations, calculated assuming three correlation types described in Section 2: exponentially decaying function with harmonic pulsation (2), exponentially decaying function without pulsation (4) and “white noise” process (5). It follows from presented graphs that an omission of the pulsation, equivalent to an assumption that the dominant wavelength of ballast stiffness fluctuation does not exist, causes more than double increase in standard deviations. Standard deviations obtained for the most chaotic “white noise” process are more than quadruple higher than these calculated for function of the type (2). It means that the dispersion of accelerations is highly dependent on the type of correlation function used to approximate the correlation between ballast stiffness fluctuation appearing at two different points on the track.

Table 2. Extreme expected values and standard deviations: accelerations [m/s²].

<table>
<thead>
<tr>
<th></th>
<th>E[\bar{\omega}_g]</th>
<th>D[\bar{\omega}_g]</th>
<th>E[\bar{\omega}_r]</th>
<th>D[\bar{\omega}_r]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force model</td>
<td>0.31</td>
<td>4.83E-3</td>
<td>6.99</td>
<td>0.43</td>
</tr>
<tr>
<td>Mass model</td>
<td>0.49</td>
<td>0.12</td>
<td>-9.70</td>
<td>1.37</td>
</tr>
<tr>
<td>Train model</td>
<td>-0.51</td>
<td>2.72E-2</td>
<td>-12.69</td>
<td>0.41</td>
</tr>
</tbody>
</table>

Therefore, it would be better to determine the real PSD function of the stiffness fluctuation process, by using in situ measurements, instead of its calculation on the basis of theoretically assumed correlation function.

6 GENERAL CONCLUSIONS

An efficient method for analysing effects of randomly varying ballast stiffness on railway bridge vibrations has been presented. An influence of railroad vehicle modelling on probabilistic characteristics of bridge vibrations has been analysed. The following main conclusions have been derived:

- The simplified train model in the form of a sequence of moving forces may be applied only in stochastic analysis of dynamic displacements.
- To analyse both stochastic and deterministic accelerations of bridge vibrations, the train should be modelled as a set of railroad vehicles being MDOF dynamic systems consisting of sprung masses modelling vehicle elements (car body, bogies, wheelsets), connected by viscoelastic constraints.
- The train model in the form of a set of unsprung masses significantly overrates standard deviations of displacements and accelerations of both bridge girder and rails.
- An influence of random stiffness fluctuation of the track ballast on the bridge displacements is negligible while on bridge accelerations is significant and grows when the fluctuation process becomes more chaotic.
- In order to precisely assess considered random effects on dynamic response of the bridge, it is necessary to perform in situ measurements in order to determine the real PSD function of the ballast stiffness fluctuation.

REFERENCES