ABSTRACT: In this paper the shock induced failure of printed circuit boards (PCB) is investigated by modelling the shock effects in terms of the so-called Shock Transmissibility (ST). The ST is defined as the ratio of the acceleration peak responses measured at two points of interest of a structure due to the same loading condition. In order to account for (i) manufacturing imperfections and (ii) defining a failure condition by using the maximum value that the ST can take, a ST measure based on magnitude only frequency domain results is devised. This ST is combined with the hybrid Finite Element/Statistical Energy Analysis (FE/SEA) approach, which models a system as a combination of SEA subsystems (which have a high degree of randomness) and FE components (with deterministic properties) in order to efficiently account for manufacturing variability. The proposed approach is illustrated by application to a single resonator mounted on a PCB.

KEY WORDS: Shock failure; Uncertainty model; Shock Transmissibility; Hybrid FE/SEA approach.

1 INTRODUCTION

Many engineering structures rely on electronic equipment mounted on Printed Circuit Boards (PCBs). Very often PCBs have to withstand severe vibro-acoustic environments which may lead to damage and hinder performance. Shocks loadings are of great interest because they are characterized by short durations (of the order of microseconds), large amplitude vibrations (up to 2000g) and high frequencies content (up to several MHz) [1]. Shock induced failures are strongly influenced by the local PCB vibration response and occur because stress at a particular location exceeds a limit level [1,2]. Vibration failure of PCB can be described by using a macroscopic parameter [2], such as local board acceleration, local board bending moment, local board curvature and local board displacement. It has been recently found that the most representative parameter is the local board curvature [2], but the local board acceleration is still used the most because: (i) random vibrations and shock loading are based on acceleration specification; (ii) vibration response is often monitored via accelerometers; and (iii) usually the maximum acceleration that an electronic component can withstand is specified.

Shocks effects are commonly described in terms of the Shock Response Spectrum (SRS) [1] in order to: (i) compare shocks severities and frequency contents; (ii) understanding the shock effect on components. The SRS is a representation of the peak acceleration response experienced by a Single Degree of Freedom System (SDOF) for various natural frequencies, having specified the damping ratio of the SDOF and the transient input acting on the system. The SRS is extensively used in aerospace industry to specify shock levels and to qualify electronic components. However, its application may produce misleading results because [1]: (i) it uses a SDOF to capture the behavior of a Multi Degree of Freedom (MDOF); (ii) the damping of a component is not easily quantifiable. Another representation of the Shock effect on a complex structure is the Shock Transmissibility (ST). The ST is the ratio of the maximum accelerations experienced in the time domain by two different points of a structure due to the same loading condition. This parameter is strongly dependent on the loading acting on the system, as well as on the local properties and structure details. Given its sensitivity to local properties, the ST can be employed for characterizing shock induced failure of PCB. If a point on a structure (for instance a satellite’ panel) and a point on the PCB are simultaneously monitored by using accelerometers, a variation in the ST can potentially indicate damage on the PCB. However, it is not always possible to experimentally measure the acceleration at the points of interest, because they may be inaccessible. As an example, even during the testing of a complex built-up structure it is extremely difficult to monitor the vibration level experienced by an electronic component mounted on a PCB which is expected to fail. Predicting the value that the ST may take between two points $j$ and $k$, and measuring the response at the point $j$ would therefore be useful to predict and monitor the maximum value that the local acceleration may take at $k$ in order to avoid failures. Nonetheless, a single accelerometer measurement point is inaccurate to represent the entire component’ shock levels and structures with slightly different properties (because of manufacturing imperfections) would have different ST. Therefore (i) predicting the range of possible value the ST may take for an ensemble of manufactured structure by using virtual prototyping, and (ii) defining a failure metric by using the maximum value that the ST can take, are of interest and they are the topics addressed in this paper.
The local PCB vibration response is commonly predicted by building a Finite Element (FE) model which can be [3-5]: (i) very detailed (by including 3D meshes of the components); (ii) based on smeared properties (with stiffness and density increased locally at the component’s location or with stiffness and density of the components averaged over the area of the board); or (iii) based on simple block 3D elements at the components’ location. PCB FE models yield reliable results when [3]: (i) stiffness, density, damping and boundary conditions of the electronic components are correctly specified; (ii) manufacturing variability is accounted for; and (iii) an accurate model is employed. However, often limited input data can be acquired from direct measurements (due to cost or time constraints) to capture inherent variability of the input parameters (aleatory uncertainty). Moreover, some input parameters, whose value is fixed, cannot be directly measured or easily quantified (epistemic uncertainty) because, for instance, they are capturing the effects of several factors rather than representing a specific physical quantity. Choosing an appropriate description of these parameters based on the available information is not always straightforward, and propagating these uncertainties through the FE model can be very computationally intensive, to the point of being infeasible when applied to detailed models having many degrees of freedom and many uncertainties.

In this paper, the hybrid Finite Element/Statistical Energy Analysis (FE/SEA) method [6-9] is proposed to analyse shock induced failure of PCB. The hybrid FE/SEA approach models a system as a combination of SEA subsystems (which have a high degree of randomness) and FE components (with fully deterministic properties). This partition leads to a large reduction in the number of degrees of freedom employed in the model (potentially thousands of finite element nodes are substituted with a single degree of freedom SEA subsystem). Moreover, the method employs a non-parametric model of the uncertainties in the SEA subsystems which avoids the use of Monte Carlo Simulations and direct characterisation of the uncertain parameters. However, the hybrid FE/SEA method is a frequency domain technique and the output of the hybrid method are magnitude only. The lack of the frequency domain phase information does not allow directly transforming the frequency domain results into the equivalent time domain results to yield the ST.

A ST measure in the frequency domain which requires magnitude only acceleration response is first introduced and is combined with the hybrid FE/SEA method in order to yield: (i) the average and variance ST to be expected between a reference point and a point monitored on a structure for a specified loading condition, and (ii) the 95% confidence level on the ST. By measuring during test the maximum acceleration response at a reference point, the 95% confidence level on the ST can then be used to yield the 95% confidence level on the maximum local acceleration at the point of interest. This value can be then compared to the prescribed maximum value acceleration in order to assess possible failures. This would allow identifying design modifications focused on (i) reducing the local acceleration response at the reference point, or on (ii) reducing the ST between the two points.

The paper is organized as follows: The ST measure defined in the time domain and the equivalent one based on magnitude only frequency domain results are described in Section 2. Section 3 provides a brief description of the hybrid FE/SEA approach. The combination of the proposed ST measure with the hybrid FE/SEA model to yield a failure condition by using the 95% confidence level on the ST is presented in Section 4. The proposed approach is illustrated by application to a single resonator mounted on a PCB in Section 5.

2 SHOCK TRANSMISSIBILITY (ST)

2.1 ST in the time domain

A spacecraft is subject to various types of shocks, such as launcher induced shocks, spacecraft release shocks and appendage release shocks [1,10], as well as random loads of mechanical and acoustic nature. Because of these loadings, the structural panels will be vibrating mostly at their resonant frequency. This vibration will be transmitted to a PCB mounted on a panel through the PCB mounting or because of internal impacts. The board will then transmit some of this vibration to the electronic components (such as quartz resonators, resistors, transistors, etc.) mounted on the PCB, leading to possible fatigue or overstress failure of these components when the vibrational energy transmitted coincides with the resonant frequency of those components. The Shock Transmissibility (ST) is defined as the ratio of the maximum accelerations experienced in the time domain by two different points, say \( j \) and \( k \), of a structure due to the same loading condition:

\[
ST_{jk} = \frac{\max_j \{a_j(t)\}}{\max_k \{a_k(t)\}}, \tag{1}
\]

where \( a_j(t) \) and \( a_k(t) \) are the acceleration time-histories measured, respectively at points \( j \) and \( k \), \( \text{abs} \) is the absolute value and \( \max \) indicates the maximum value (peak value) of the acceleration.

It is worth to stress at this point that the ST is inherently different from a transfer function (TF). The TF is defined as the frequency ratio between the response measured at point \( j \) and the loading applied at point \( i \). While the TF is causal (there is a causal relation between the input and output), the ST is not: the response at point \( k \) is not a consequence of the response measured at point \( j \), they both depend on the loading condition.

For a fixed loading condition, the ST between points \( j \) and \( k \) of a system is a single value. This ST value may change because of degradation and/or damage of the structure or when non-linear effects within the transmission path take place (i.e. internal impacts). The ST cannot provide information of the failure mode, but it can be used as a monitoring parameter, i.e. when the ST exceeds certain thresholds a shock induced failure could be expected. However, the points of interest for establishing the ST value may be inaccessible even during the testing of a structure. Predicting the value that the ST may take between points \( j \) and \( k \) by virtual prototyping, and measuring the response at a reference point \( j \) would therefore be useful to establish the vibration level at \( k \), and lead to design decisions which...
would avoid the failure of sensitive components (either by reducing the acceleration level at \( j \) or by reducing the ST between \( j \) and \( k \)).

The loading experienced by an engineering structure in real-life operating condition are not entirely known and cannot be fully reproduced. Commonly a structure is tested by using an electromagnetic shaker and sweeping a range of frequency for identifying and eliminating by resonance the electronic components resonant frequencies which may be excited during typical operational condition or by performing a tap test. Nonetheless, these techniques lead to results valid only for the structure under investigation and depend on the position of the applied loading. Virtual prototyping can lead to a rapid and low cost evolution of the design, and the effects of manufacturing imperfections and various loading conditions can be investigated. The approach proposed in this paper is to use a frequency domain technique, the hybrid FE/SEA method, to account efficiently for manufacturing imperfections and uncertainties of the PCB. In order to model the shock induced failure of PCBs a ST measure based on frequency domain magnitude results is combined with the hybrid FE/SEA method. The ST defined in the frequency domain is presented in the next section.

### 2.2 ST in the frequency domain

In this section the ST is estimated starting from frequency domain results. In principle this could be achieved by considering a complex TF yielded by a linear FE model (or a measured one) and reconstructing the time domain response without performing a full time domain simulation. However, the results provided by the hybrid FE/SEA method, which will be used to model the system, are magnitude only. Therefore, an ST in terms of magnitude only frequency acceleration responses is devised. In particular, the root-mean-square value of the acceleration response in the frequency domain is first recovered to yield the acceleration peak response. The time domain and frequency domain representation of a signal are related to each other by means of Parseval’s theorem. This theorem states that

\[
\int_{-\infty}^{\infty} a_j^2(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |A_j(\omega)|^2 \, d\omega.
\]

where \( A_j(\omega) \) is the Fourier transform of \( a_j(t) \). The r.h.s of Eq. (2) correspond to the mean square value of \( a_j(t) \). The root-mean-square (rms) value of the acceleration, \( a_{j,\text{rms}} \), can then be written in terms of the stationary response:

\[
a_{j,\text{rms}} = \sqrt{\int_{-\infty}^{\infty} a_j^2(t) \, dt} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |A_j(\omega)|^2 \, d\omega.
\]

The rms and the peak of the response provide different information on the vibrational signal: while the rms takes into account the effective energy content of the signal, the peak represents the maximum amplitude of vibration. When fatigue damage is of interest, the rms is generally more damaging then a similar magnitude of peak. When the failure is dominated by over stress at a particular location instead, the peak amplitude is more representative. The ST in terms of the peaks ratio is therefore more suitable to deal with shock induced failure. Moreover, a ST expressed in terms of rms ratio may yield larger or smaller values of that expressed in terms of peak ratios. This can be explained by considering the acceleration responses plotted in Figure 1 and Figure 2. The difference in the two responses is due to the number of cycles subsequent to the first peak, which are dominated by the damping of the structure. The response depicted in Figure 1 dies out after the first peak because of a high damping value. This lead to a large difference between the rms and peak values (indicated with \( a_{\text{rms}} \) and \( a_{\text{max}} \), respectively). In Figure 2 it is possible to observe that the difference between the rms and the peak value is lower.

![Figure 1. Highly damped acceleration response. The rms response, \( a_{\text{rms}} \), is much lower than the peak response, \( a_{\text{max}} \).](image)

In order to be able to use frequency domain results to get the ST, the rms response has to be converted into peak response. This is achieved by modeling the response decay as an exponentially decaying function (as shown in Figure 3) and assuming that the response is dominated by a single mode:

\[
a(t) = \overline{A} \exp\left(-\frac{t}{2\omega_d}\right) \sin(\omega_d t),
\]

where \( \overline{A} \) is the amplitude of acceleration, \( \eta \) is the loss factor of the system (\( \eta = 2\zeta \), where \( \zeta \) is the damping ratio), \( \omega_d \) is the natural frequency of the dominating mode and \( \omega_d \) is the damped natural frequency \( \omega_n = \omega_d \sqrt{1-\zeta^2} \). The natural frequency of the dominating mode can be obtained from:

\[
\omega_n = \frac{\int |A(\omega)|^2 \, d\omega}{\int |A(\omega)|^2 \, d\omega}.
\]
Substituting Eq. (4) into Eq. (3) yields:

\[ a_{j,max} = \sqrt{\int A_j^2 \exp\left[-\eta_j \omega_{j,n} t^* \right] \sin^2(\omega_{j,n} t) \, dt} = \sqrt{A_j^2 \left(\frac{4 - \eta_j^2}{8 \eta_j \omega_{j,n}}\right)} \]  

Eq. (6) can be re-arranged to yield:

\[ \bar{A}_j = a_{j,max} \sqrt{\frac{8 \eta_j \omega_{j,n}}{4 - \eta_j^2}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| A_j(\omega) \right|^2 d\omega \sqrt{\frac{8 \eta_j \omega_{j,n}}{4 - \eta_j^2}} \]  

The \( a_{max,j} \) can be approximated with \( \bar{A}_j \), or it can be derived as:

\[ a_{max,j} = \bar{A}_j \exp\left[-\frac{\eta_j}{2} \omega_{j,n} t^* \right] \sin(\omega_{j,n} t^*), \]  

where

\[ t^* = \frac{1}{\omega_{j,n}\sqrt{4 - \eta_j^2}} 2 \arccos \left[ \frac{\eta_j}{2} \right]. \]  

By using Eq. (7), \( a_{max,j} \) can be expressed as

\[ a_{max,j} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| A_j(\omega) \right|^2 d\omega \frac{8 \eta_j \omega_{j,n}}{4 - \eta_j^2} \exp\left[-\frac{\eta_j}{2} \omega_{j,n} t^* \right] \sin(\omega_{j,n} t^*)} \]  

Finally, the ST (as defined Eq. (1)) can be expressed as

\[ ST_{j,k} = \frac{\int \left| A_j(\omega) \right|^2 d\omega \exp\left[-\eta_j \omega_{j,k} t^* \right] \sin(\omega_{j,k} t^*)}{\int \left| A_k^2(\omega) \right| d\omega \exp\left[-\eta_k \omega_{j,k} t^* \right] \sin(\omega_{j,k} t^*)} \]  

This equation can be approximated (having considered that \( \eta_j^2 \ll 4 \)) as:

\[ ST_{j,k} = \frac{\int \left| A_j(\omega) \right|^2 d\omega \exp\left[-\frac{\eta_j}{2} \omega_{j,n} t^* \right] \sin(\omega_{j,n} t^*)}{\int \left| A_k^2(\omega) \right| d\omega \exp\left[-\frac{\eta_k}{2} \omega_{j,k} t^* \right] \sin(\omega_{j,k} t^*)}. \]
analytically (without using MCS) leading to a large reduction of the computational costs.

The hybrid FE/SEA method has been recently extended to account for the uncertainty in the FE components by using a probabilistic parametric uncertainty model which can be dealt with MCS or with more efficient techniques [14,15]. However for the subsequent analysis the FE components would be considered as entirely deterministic.

3.2 Hybrid FE/SEA equations

The hybrid FE/SEA ensemble average equations at the excitation frequency \( \omega \) are [6]:

a) Subsystem energy balance equations

\[
\omega (\eta_j + \eta_{d,j}) E_j + \sum_k \omega \eta_{ik} n_j (E_j / n_j - E_k / n_k) = P_{in,j}^{\text{ext}} + P_{in,j}^{\text{in}} (14)
\]

where \( \eta_j \) is the damping loss factor of the subsystem \( j \), \( \eta_{d,j} \) is an additional loss factor on the subsystem \( j \) due to the energy dissipated in the FE components, \( \eta_k \) is the coupling loss factor between subsystem \( j \) and subsystem \( k \), \( n_j \) is the modal density of subsystem \( j \) (which is defined as the average number of natural frequencies within a unit frequency band), \( E_j \) is the ensemble average vibrational energy of subsystem \( j \), \( P_{in,j}^{\text{ext}} \) is the external power input to the subsystem arising from the loads acting on the master system and \( P_{in,j}^{\text{in}} \) is the power input arising from external loads directly applied to the subsystem \( j \).

Eq. (13) states that the power dissipated through damping (\( \omega (\eta_j + \eta_{d,j}) E_j \)) plus the net power transmitted to other subsystems (\( \sum_k \omega \eta_{ik} n_j (E_j / n_j - E_k / n_k) \)) is balanced by the power input to the subsystem (\( P_{in,j}^{\text{ext}} + P_{in,j}^{\text{in}} \)), and it is based on the assumption that the power transmitted is proportional to the difference of the average modal energies (defined as \( E_j / n_j \)) of the coupled subsystems. Eq. (13) has the same form as the standard SEA equations [12], but contains two additional terms relating to: (i) the contribution of the master system to the power input \( P_{in,j}^{\text{in}} \), and (ii) the power dissipated in the master system, \( \omega \eta_{d,j} E_j \). These two terms can be expressed in terms of: (i) the total dynamic stiffness matrix \( D_{tot} = \sum_k D_{tot,k} + D_{a} \), where \( D_{a} \) is the dynamic stiffness matrix associated with the FE component \( \omega^2 M + i \omega C + K \), where \( M \) and \( C \) and \( K \) are respectively the FE component mass, damping and stiffness matrices), and \( D_{tot,k} \) is the so-called direct field dynamic stiffness matrix for subsystem \( k \); (ii) the cross-spectral matrix of the loading applied directly to the master system \( S_{\text{ff}} = \text{diag}(\text{Var}\{E_j\}) \), so that

\[
\omega \eta_{d,j} = \frac{2 \alpha \omega}{\pi n_j} \sum_r \ln \left[ D_{a,rr} \left(D_{tot,rr} + D_{a,rr}^{\ast}\right) \right],
\]

where the superscript * indicates the complex conjugate, the superscript T denotes the transpose, \( \Im \) represents the imaginary part of the matrix, and \( \alpha \) is a factor which takes into account the fact that the subsystem wave field may not be perfectly diffuse [7].

Three of the remaining terms in Eq. (14), specifically \( \eta_j, n_j, \) and \( P_{in,j}^{\text{in}} \) are evaluated by using standard SEA procedures [12], while the coupling loss factors are expressed analytically as a function of the total dynamic stiffness matrix in the form

\[
\omega \eta_{d,j} = \frac{2 \alpha \omega}{\pi n_j} \sum_r \ln \left[ D_{a,rr} \left(D_{tot,rr} + D_{a,rr}^{\ast}\right) \right],
\]

Writing Eq. (14) for each subsystem leads to a set of equations that can be solved to yield the ensemble average vibrational energy \( E_j \) of each subsystem. This set of \( E_j \) is then used to calculate the average response of the master system.

b) Master system response equation

\[
S_{\text{ff}} = D_{tot} \left(S_{\text{ff}} + \sum_r \frac{4 \alpha \omega}{n_j} \ln \left[ D_{a,rr} \right] D_{a,rr}^{\ast} \right) D_{a,rr}^{\ast T}.
\]

where \( S_{\text{ff}} \) is the cross-spectrum of the response of the master system (averaged over the non-parametric ensemble), and the two terms on the right-hand side correspond to the forcing arising from external excitation (expressed in terms of the cross spectrum of the forces, \( S_{\text{ff}} \)) and the forcing arising from the subsystems, as yielded by the diffuse field reciprocity relation [8,9].

By using the hybrid FE/SEA variance theory [7] it is also possible to estimate the covariance of the subsystem energies (\( \text{Var}\{E_j\} \), where \( E_j = E_j / n_j \)) and the variance of the cross-spectral matrix of the response of the master system (\( \text{Var}\{S_{\text{ff}}\} \)) over the non-parametric ensemble. For brevity, these equations will not be included in this paper. The reader is referred to the paper by Langley and Cotoni [7] where their full derivations can also be found.

4 SHOCK TRANSMISSIBILITY INCLUDING UNCERTAINTY

In this section the ST (defined in section 2) is combined with a hybrid FE/SEA model of a complex system in order to yield the non-parametric ensemble average and variance ST, as well as the 95% confidence level on the ST. The ST formula (Eq. (11)) requires establishing the modulus square acceleration at two points of the system. Within a hybrid FE/SEA model, these two points can belong to: i) two FE components; ii) two SEA subsystems; or iii) a FE component and a SEA subsystem.

As discussed in the previous section, the hybrid FE/SEA method yields the non-parametric ensemble average and variance of the response. In particular, the results are expressed in terms of the average energy \( E_j \) of a SEA subsystem or as the mean squared amplitude \( \{S_{\text{ff}}\}_{ik} \) of the finite element degrees of freedom.
If a point is located on the deterministic part of the hybrid FE/SEA model, the average modulus square acceleration associated with the \( k \)th degree of freedom of the FE part can be obtained from [16] \[ \mathbb{E}\left[ \left| A_{\text{sub}}(\omega) \right|^2 \right] = \omega^2 \mathbb{E}\left\{ S_{\omega jk} \right\} \] (where the term \( \mathbb{E}\left[ \right] \) indicates the average taken over nominally identical structures). The average modulus square acceleration associated to a point located on the SEA subsystem is calculated from the average energy response \( E_j \) of the \( j \)th SEA subsystem (therefore the points belonging to the same SEA subsystems are characterized by the same average modulus square acceleration) as [16] \[ \mathbb{E}\left[ \left| A_{\text{sub}}(\omega) \right|^2 \right] = 2\omega^2 E_j / M_j, \] where \( M_j \) is the mass of the \( j \)th SEA subsystem.

The ensemble average point-to-point ST (calculated over an ensemble of nominally identical structures) between the points \( j \) and \( k \) is defined as:

\[
\text{ST}_{jk} = \mathbb{E}\left[ \max\left\{ a_j(t) \right\} \right] / \mathbb{E}\left[ \max\left\{ a_k(t) \right\} \right] = \sqrt{\frac{\int_{-\infty}^{\infty} \mathbb{E}\left[ \left| A_j(\omega) \right|^2 \right] d\omega}{\int_{-\infty}^{\infty} \mathbb{E}\left[ \left| A_k(\omega) \right|^2 \right] d\omega}} \frac{\eta_{\omega,\omega_{jk}}^{c_j} p_{\omega,jk}}{\eta_{\omega,\omega_{jk}}^{c_k} p_{\omega,jk}},
\] (19)

where the term \( \mathbb{E}\left[ \left| A_j(\omega) \right|^2 \right] \) is used to indicate the average response of a subsystem, \( \mathbb{E}\left[ \left| A_{\text{sub}}(\omega) \right|^2 \right] \), or of a deterministic component, \( \mathbb{E}\left[ \left| A_j(\omega) \right|^2 \right] \), according to the measurement point location.

If a spatial average is also taken over the points belonging to the deterministic component \( c \) (\( \mathbb{E}\left[ \right] \)), the ST between component \( c_j \) and the component \( c_k \) is defined as:

\[
\text{ST}_{c_j,c_k} = \sqrt{\frac{\int_{-\infty}^{\infty} \mathbb{E}_c\left[ \left| A_j(\omega) \right|^2 \right] d\omega}{\int_{-\infty}^{\infty} \mathbb{E}_c\left[ \left| A_k(\omega) \right|^2 \right] d\omega}} \frac{\eta_{\omega,\omega_{jk}}^{c_j} p_{\omega,jk}}{\eta_{\omega,\omega_{jk}}^{c_k} p_{\omega,jk}},
\] (20)

where the term \( \mathbb{E}_c\left[ \left| A_j(\omega) \right|^2 \right] \) is used to indicate spatial average over the non-parametric ensemble average results of a subsystem, \( \mathbb{E}\left[ \left| A_{\text{sub}}(\omega) \right|^2 \right] \), or of the degrees of freedom belonging to the FE component \( \mathbb{E}_c\left[ \left| A_j(\omega) \right|^2 \right] \).

The variance of the ST can be derived in a similar fashion. Moreover, the probability density function of the ST can be estimate starting from the ensemble average mean and variance of the maximum accelerations responses calculated at \( j \) and \( k \), and assuming a probability distributions (for instance a lognormal distribution) of the accelerations. For brevity, these further developments of the theory are not included in the present paper. Knowing the complete pdf, the 95% confidence level on the ST (\( \text{ST}_{jk,95\%} \)) can be estimate as

\[
\text{Prob}[\text{ST}_{jk} < \text{ST}_{jk,\text{max}}] = \int_0^{\text{ST}_{jk,\text{max}}} p(\text{ST}_{jk}) d\text{ST}_{jk} = 0.95
\] (21)

The \( \text{ST}_{jk,95\%} \) is the value for which there is a 95% probability that the response variable (the ST in this case) would not exceed the maximum value of the ST, \( \text{ST}_{jk,\text{max}} \).

This \( \text{ST}_{jk,95\%} \) can be used to assess possible failure due to excessive acceleration levels at a point \( k \), by measuring the maximum acceleration at a reference point \( j \). Specifically, the 95% confidence level on the acceleration at \( k \) can be estimated as

\[
\text{max}(a_k(t))_{95\%} = \frac{\text{max}(a_j(t))}{\text{ST}_{jk,95\%}}
\] (22)

if the value so derived is larger than certain acceptable levels, then design modifications should be explored.

5 NUMERICAL APPLICATION

The example system investigated in this section is composed by a simply supported plate coupled via a translational spring to a beam with a free-roller support in order to represent, with the simplest possible dynamic model, a generic class of systems in which a thin PCB is coupled to a quartz resonator. The aim of the analysis is to compute the average Shock Transmissibility (ST) between a point on the PCB and a point on the resonator and between the two components and to compare this value to that yielded by a single deterministic analysis in order to show the effects of including uncertainty in the shock transmissibility calculations.

The system properties are described in the following section. The results obtained by using the hybrid FE/SEA method in combination with the proposed ST are discussed in subsection 5.2.

5.1 Description of the system

Quartz resonators are typically used for keeping track of timing, providing a stable clock signal for digital integrated circuits and for stabilizing frequency for radio transmitters and receivers. These resonators are composed by a rectangular plate of quartz connected to a pairs of conductive electrodes which are enclosed in a ceramic package. They use the resonance of the quartz blank to generate an electrical signal with a very precise frequency (by using the piezoelectric properties of the quartz crystal). The most common type of this resonator is the AT-cut one, which uses the thickness shear mode of vibration. Quartz resonators are sensitive to shocks because they may lead to damage of the mounting system or cracking of the quartz plate itself, which can significantly affect the quartz resonator oscillatory frequency.

In this numerical application, a single quartz component mounted on a PCB is considered. Although this component is design to work at the thickness shear mode frequency of 12 MHz, it is assumed that a shock induced failure is experienced because of large amplitude vibrations at the flexural mode, which are damaging the mounting system. Because the flexural mode is the one of interest, the quartz resonator is simply modelled as a beam mounted on a plate representing a
the second moment of area of the beam and \( h \) is the thickness of the quartz plate expressed in \( \mu \text{m} \).

![Figure 4](image-url)

**Figure 4.** System under investigation: beam with a free-roller support coupled via a translational spring \( k \) to a simply supported plate. \( P1 \) and \( P2 \) are the observation points on the beam and on the plate, respectively. \( F \) is the point force applied to the system.

The quartz resonator behaves like a cantilever beam. Its properties are specified in Table 2 and are such that the frequencies corresponding to the flexural and shear modes, \( f_b \) and \( f_s \), respectively, are

\[
\begin{align*}
    f_b &= 0.56 \sqrt{YI/\rho AL^3} = 3.46\text{KHz} \\
    f_s &= \frac{1}{2h} \sqrt{G/\rho} = 12\text{MHz}
\end{align*}
\]

(23)

(24)

where \( I \) is the second moment of area of the beam and \( \hat{h} \) is the thickness of the quartz plate expressed in \( \mu \text{m} \).

![Table 1](image-url)

**Table 1.** Plate properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length ( L_x )</td>
<td>250 mm</td>
</tr>
<tr>
<td>Length ( L_y )</td>
<td>350 mm</td>
</tr>
<tr>
<td>Thickness ( h )</td>
<td>1.6 mm</td>
</tr>
<tr>
<td>Density ( \rho )</td>
<td>2481 kg/m(^3)</td>
</tr>
<tr>
<td>Young’s modulus ( Y )</td>
<td>( 4 \times 10^9 ) N/m(^2)</td>
</tr>
<tr>
<td>Shear modulus ( G )</td>
<td>( 1 \times 10^9 ) N/m(^2)</td>
</tr>
<tr>
<td>Poisson ratio ( \nu )</td>
<td>0.3</td>
</tr>
<tr>
<td>Loss factor ( \eta )</td>
<td>1%</td>
</tr>
</tbody>
</table>

The quartz resonator is modelled as a beam with a free-roller support and it is coupled to the plate via a translational spring whose value is \( 9 \times 10^6 \) N/m. The coupling spring is attached at a fixed location within the interior of the plate, specifically at point \((0.15, 0.25)\) measured in meters along the \( x \) and \( y \) axes. The other extreme of the spring is attached at the end of the beam, as shown in Figure 4.

### Table 2. Quartz resonator properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length ( L )</td>
<td>6 mm</td>
</tr>
<tr>
<td>Cross section ( b \times h )</td>
<td>( 1.82 ) mm ( \times 0.143 ) mm</td>
</tr>
<tr>
<td>Density ( \rho )</td>
<td>2650 kg/m(^3)</td>
</tr>
<tr>
<td>Young’s modulus ( Y )</td>
<td>( 7.7 \times 10^9 ) N/m(^2)</td>
</tr>
<tr>
<td>Shear modulus ( G )</td>
<td>( 3.1 \times 10^9 ) N/m(^2)</td>
</tr>
<tr>
<td>Poisson ratio ( \nu )</td>
<td>0.17</td>
</tr>
<tr>
<td>Loss factor ( \eta )</td>
<td>( 6.7 \times 10^{-4} ) %</td>
</tr>
</tbody>
</table>

The system is driven by a unit force applied on the PCB at point \((0.12, 0.09)\) measured in meters along the \( x \) and \( y \) axes. Two points are monitored: a point on the plate where the springs are attached (indicated as \( P2 \) in Figure 4) and a point at the tip of the beam (indicated as \( P1 \) in Figure 4).

The aim of the analysis is to estimate the average ST calculated between the two points of interest and the one between the two components in the frequency range 100-5000 Hz and to compare this value to the deterministic ST. The hybrid FE/SEA model of the system comprises a SEA subsystem, the plate, and a FE component, the beam. In order to provide a benchmark for the hybrid FE/SEA model, an ensemble of random systems has been analyzed in detail by using the Lagrange-Rayleigh-Ritz method. In this model, the out-of-plane motion of the plate and of the beam is expanded as a modal summation, and the full set of degrees of freedom consists of the modal amplitudes of the plate and of the beam (this model has been validated against a full FE model). An ensemble of 350 random systems is generated by adding 20 point masses, each mass having 1.0\% of the mass of the bare plate, at random locations within the plate.

The ensemble average vibrational energy of the plate and the average auto-spectra of displacement response computed at the tip of the beam yielded by the hybrid FE/SEA method and by the benchmark Monte Carlo Simulations (MCS) are shown in Figure 5, together with the cloud of ensemble results yielded by the MCS study (cloud of grey curves). The mean energy response of the plate is similar of that of a single uncoupled SEA subsystem. The peak observed in the FE component response is due to the modal dynamics of the beam, and in particular it correspond to the resonance of the first bending mode.

It is worth emphasizing that the computational time of the benchmark MCS was about 12 hours, while that required by the hybrid FE/SEA method was about 1 minute.

#### 5.2 Average ST versus deterministic ST

The results obtained with the hybrid FE/SEA method are converted into the equivalent acceleration descriptions, as described in section 4. The ensemble average acceleration of the SEA subsystem and the ensemble average acceleration measured at the tip of the beam are shown in Figure 6. By using Eq. (19) it has been found that the average ST between the two points of interest is 2.26 (the ST yielded by the FE MCS is 1.7).
The accelerations ensemble and spatial average were also computed, leading to a ST between the components of 0.84 (obtained by using Eq. (20), while the ST yielded by the FE MCS is 0.6). As expected the component-to-component ST yields ST values lower than the point-to-point ST.

The point-to-point ST calculated by considering a member of the deterministic ensemble is 1.4 while the component-to-component ST is 0.63. It is therefore clear that manufacturing variability of the board may significantly affect the prediction of the ST. It is worth emphasizing that the results yielded by the proposed approach, which represent the ensemble average response, should be augmented by considering the variance prediction, as well as confidence interval, in order to fully taken into account the uncertainty effects. Although the theory exists these results have not been included in the present study.

6 CONCLUSIONS

The application of the Shock Transmissibility (ST) to the shock-induced failure analysis of Printed Circuit Boards (PCBs) has been addressed in this paper. A ST measure in the frequency domain has been devised and coupled with the hybrid Finite Element/Statistical Energy Analysis (FE/SEA) method in order to derive the average and variance ST over an ensemble of manufactured systems. These results can used to predict the 95% confidence level on the ST, and ultimately the 95% confidence level on the maximum acceleration which would be experienced at one point of a structure. The effects of uncertainty on the ST have been investigated by analyzing a simplified model of a resonator mounted on a PCB.

ACKNOWLEDGMENTS

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REFERENCES